Republic of Iraq
Ministry of Higher Education
and Scientific Research,
Al-Nahrain University,
College of Sciences
Department of Mathematics and Computer Applications

About the Compactness of Fuzzy Cone Metric Spaces

A Thesis
Submitted to Council of the college of Science of Al-Nahrain University in Partial Fulfillment of the Requirements For the Degree of Master of Science in Mathematics

By

Amer Abdul Karim Abdulla

B.Sc. Mathematics and Computer Applications / College of Science / Al-Nahrain University, 2014

Supervised by

Dr. Fadhel Subhi Fadhel

Assistant Professor

September 2016 Dhu al-Hijjah 1437

بسم الله الرحمن الرحيم

قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ
الْعَلِيمُ الْحَكِيمُ

صدق الله العظيم

سورة البقرة

الاية (32)

Supervisor Certification

I certify that this thesis entitled **(About the Compactness of Fuzzy Cone Metric Spaces)** was prepared (**Amer Abdul Karim Abdulla**) under my supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of master of science in mathematics.

Signature:

Name: **Dr. Fadhel Subhi Fadhel**

Title: Assistant Professor

Data: / / 2016

In view of the available recommendations, I forward this thesis for debate by the examining committee

Signature:

Name: **Dr. Osama H. Mohammed**

Title: Assistant Pofessor

Address: Head of Mathematics and Computer

Applications Department

Data: / / 2016

Committee Cerification

We, the examining committee certify that we have read this thesis entitled **"About the compactness of Fuzzy Cone Metric Spaces"** and examined the student "**Amer Abdul Karim Abdulla**" that in our opinion, it is accepted for the degree of Master of Science in Mathematics.

Signature:
Name: **Dr. Radhi A. Zaboon**
Scientific Degree: **Asst. Professor**
Address: Collage of Science, Al-Mustansiriya University
Date: / / 2016
(The head of committee)

Signature:
Name: **Dr. Osama H. Mohammed**
Scientific Degree: **Asst. Professor**
Address: Collage of Science, Al-Nahrain University
Date: / / 2016
(Member)

Signature:
Name: **Dr. Alaa A. Elawi**
Scientific Degree: **Asst. Professor**
Adderss: Collage of Science, University of Baghdad
Date: / / 2016
(Member)

Signature:
Name: **Dr. Fadhel S. Fadhel**
Scientific Degree: **Ass. Professor**
Address: Collage of Science, Al-Nahrain University
Date: / / 2016
(Member/ Supervisor)

I, hereby certify upon the decision of the examining committee.

Signature:
Name: **Dr. Hadi M. A. Abood**
Scientific Degree: **Professor**
Title: Dean of the college of Science, Al-Nahrain University
Data: / / 2016

Dedication

To my Mother and Father

With all Love and Respect

Acknowledgements

Thanks to Allah for his help and giving me the ability to complete this work.

I would like to express my sincere appreciation and gratitude to my supervisor Asst. Prof. Dr. Fadhel Subhi Fadhel for his continuous help and advice during the preparation of this thesis.

I am deeply indebted to the College of Science and to the staff members of the Department of Mathematics and Computer Application, of Al-Nahrain University for allowing me to be one of their students.

Special thanks go to Asst. Prof. Dr. Osama H. Mohammed head of the department of Mathematics and Computer applications for his advice throughout my study.

I also would like to present the most beautiful words of thanks to my parents for their continuous help and encouragement.

My sincere thanks goes also to my friends and my fellow students for their supporting during my study.

Amer Abdul Karim Abdulla

2016

Summary

The generalization of metric spaces from ordinary sets to fuzzy set theory and then to the so called cone metric spaces is a promising topics of theoretical mathematics.

Therefore, this thesis has two objectives. The first objective is to study cone metric spaces and then constructing the so called fuzzy cone metric spaces using a new direction which is based on fuzzy point. The second objective is to study the compactness of fuzzy sets in fuzzy cone metric spaces and then give the relationship among different types of compactness, such as compact fuzzy sets, pre-compact fuzzy sets, sequentially compact fuzzy sets, countable compact fuzzy sets and locally compact fuzzy set.

Contents

List of Symbols

Symbols	Definition
$\tilde{A}, \tilde{B}, \ldots$	Fuzzy sets
$\mu_{\tilde{A}}(x)$	Membership function
X	Universal set
I^X	Family of all fuzzy subsets of X.
$\tilde{\varnothing}$	Empty fuzzy set
$\tilde{A}^c$	The complement of a fuzzy set $\tilde{A}$
$\odot$	Algebraic product of two fuzzy sets
$\mathbb{R}$	Set of real number
A_α	α-level sets of a fuzzy set $\tilde{A}$
$A_{\alpha+}$	Strong α-level sets of a fuzzy set $\tilde{A}$
$\tilde{p}_x^\lambda$	Fuzzy point
$\mu_{\tilde{p}_x^\lambda}(y)$	Membership function of a fuzzy point
${}^c\tilde{p}_x^\lambda$	Complement of a fuzzy point
$\{\tilde{q}_{x_n}^{\lambda_n}\}, n \in \mathbb{N}$	Sequence of fuzzy points
X^*	Set of all fuzzy points
$\tilde{\in}$	Belong in fuzzy sets
$\mathrm{int}(P)$	Interior set of P
$U_\varepsilon(\tilde{q}_x^\lambda)$	Neighborhood of fuzzy point $\tilde{q}_x^\lambda$
$\odot$	Algebraic product
$\oplus$	Probabilistic sum

List of Figures

Chapter One

Introduction and Literature Review

1.1 Introduction

In real life problems, we use many properties which cannot be dealt with satisfactory on the simple belong or not belong basis. These properties perhaps best indicated for example by shade of gray, rather than by black or white. Assigning to each individual in a population on a “belong” or “not belong” values, as is done ordinary set theory, is not adequate way of dealing properties of this type, [36].

Historically, the accepted birth date of the theory of fuzzy sets returns to 1965, when the first article entitled “fuzzy sets” submitted by Zadeh L. appeared in the journal of information and control. Also, the term “fuzzy” was introduced and coined by Zadeh for the first time [15]. In which original definition of fuzzy set is to consider a class of object with continuum grade of membership, such a set is characterized by membership function which assigns to each object a grade of membership value ranging between zero and one.

Zike D. in 1982 [37], studied the fuzzy point, and discussed the fuzzy metric spaces with the metric defined between two fuzzy points.

Huang L., Zhang X. [14], introduced Cone metric space in 2007, as a generalization of metric spaces by replacing the set of real numbers is by an ordered Banach space. They introduced the basic definitions and discuss some properties.

Hazim M. [13], study the Fuzzy Metric Spaces with respect to fuzzy point definition, as well as, study the many types of compactness in fuzzy metric spaces.

Fuzzy cone metric spaces introduced in 2013 by Bag T. depended on t-norm definition, and gave some basic results and fixed point theorems in such spaces [30].

1.2 Literature Review

1.2.1 Basic Concepts of Fuzzy Sets

In this section, some fundamental definitions and elemantray concepts related to fuzzy set theory are given including some basic algebraic operations, as well as, some illustrative example for completeness purpose. Additional concepts may be found in any text book concerning fuzzy sets (see [4], [13], [17]).

We start this section by the definition of ordinary or nonfuzzy sets in order to give a comparison with fuzzy sets, and to give the reason for the introduction of fuzzy sets.

Definition (1.2.1.1), [33]:

Let X be a classical set of objects of finite dimension, called the universal set, whose generic elements are denoted by x. The membership in a classical subset A of X is often viewed as a characteristic function χ from X into {0, 1}, such that:

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

{0, 1} is called a valuation set.

Definition (1.2.1.2), [33]:

Let X be the universal set and $\tilde{A}$ be any subset of X, then $\tilde{A}$ is called fuzzy subset of X, which is characterized by a membership function $\mu^*_{\tilde{A}} : X \longrightarrow [a, b]$, where a, b $\in \mathbb{R}$ and in a special case $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$, i.e.,

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}.$$

Remark (1.2.1.3), [34]:

For simplicity, the collection of all fuzzy subsets of X will be denoted by X* or I^X, where I = [0, 1], i.e.,

$$I^X = \{\tilde{A} : \tilde{A} \text{ is a fuzzy subsets of } X\}.$$

The basic concepts with logical and algebraic operations defined on fuzzy sets may be summarized in the next remark:

Remarks (1.2.1.4), [4], [26], [33]:

Let X be the universal set, and $\tilde{A}$, $\tilde{B} \in I^X$ with membership function $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$, respectively; then:

1. X may be considered as a fuzzy set with membership function $\mu_{\tilde{X}}(x) = 1$, for all $x \in X$, which is denoted by 1_X; while the empty fuzzy set $\tilde{\varnothing}$ is a fuzzy set with membership function $\mu_{\tilde{\varnothing}}(x) = 0$, $\forall\, x \in X$, which is denoted by 0_X.
2. The height of $\tilde{A}$ is the greatest membership value, i.e.,

$$\text{hgt}(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$$

3. The elements of X, such that $\mu_{\tilde{A}}(x) = \frac{1}{2}$ are called the crossover points of $\tilde{A}$.

4. $\tilde{A}$ is said to be normal if there exists $x_0 \in X$, such that $\mu_{\tilde{A}}(x_0) = 1$, otherwise $\tilde{A}$ is subnormal. Also, if a fuzzy set $\tilde{A}$ is subnormal, then it may be normalized by dividing $\mu_{\tilde{A}}$ on hgt($\tilde{A}$) $\neq 0$.

5. The ordinary or nonfuzzy set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$ is called the support of $\tilde{A}$ and is denoted by Supp($\tilde{A}$).

6. $\tilde{A}^c$ is the complement of $\tilde{A}$ which is also a fuzzy set with membership function:

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x), \ \forall\, x \in X.$$

7. $\tilde{A} = \tilde{B}$ if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $\forall\, x \in X$.

8. $\tilde{A} \subseteq \tilde{B}$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, $\forall\, x \in X$.

9. The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{D}$ and may be defined with the following membership function:

$$\mu_{\tilde{D}}(x) = \text{Min}\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}, \ \forall\, x \in X$$

10. The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ and may be associated with the following membership function:

$$\mu_{\tilde{C}}(x) = \text{Max}\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}, \ \forall\, x \in X$$

11. The addition of $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ with membership function:

$$\mu_{\tilde{C}}(x) = \text{Min}\{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x), 1\}, \ \forall\, x \in X$$

12. The subtraction of $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ with membership function:

$$\mu_{\tilde{C}}(x) = \text{Max}\{0, \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)\}, \ \forall\, x \in X$$

13. The algebraic product of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$ with membership function:

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x)\,\mu_{\tilde{B}}(x),\ \forall\ x \in X.$$

14. The probabilistic sum of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$ with membership function:

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x),\ \forall\ x \in X.$$

15. $\tilde{A}$ and $\tilde{B}$ are said to be separated if $\tilde{A} \cap \tilde{B} = \tilde{\varnothing}$.

16. A fuzzy subset $\tilde{A}$ of $\mathbb{R}$ is said to be convex fuzzy set, if:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min}\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\}$$

for all x_1, $x_2 \in \mathbb{R}$, and all $\lambda \in [0, 1]$, where $\mu_{\tilde{A}}(x)$ is standing for a suitable membership function.

Example (1.2.1.7), [6]:

Let X = (0,150] be the set of possible ages for a human being, then the fuzzy set:

$$\tilde{A} = \text{"About 50 years old"}$$

may be expressed as:

$$\tilde{A} = \{(x,\ \mu_{\tilde{A}}(x)) \mid x \in X\}$$

with membership function:

$$\mu_{\tilde{A}}(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

Example (1.2.1.8), [31]:

The membership function of the fuzzy set $\tilde{A}$ of real numbers "close to 1", is can be defined by

$$\mu_{\tilde{A}}(x) = \exp(-\beta(x-1)^2)$$

where β is a positive real number.

1.2.2 α-Level Sets

In this section, an important notion in fuzzy sets will be discussed, which is the so called the α-cut or α-level sets, which corresponds to any fuzzy set $\tilde{A}$. α-Level sets are nonfuzzy sets and may be considered as an intermediate set that connect between fuzzy sets and ordinary or nonfuzzy sets, that may be used to prove most of the theoretical results that are satisfied in nonfuzzy sets are also satisfied here for fuzzy sets.

In fuzzy set theory, if one wants to exhibit an element $x \in X$ that is typically belong to a fuzzy set $\tilde{A}$, then its membership value must to be greater than some threshold level $\alpha \in (0, 1]$. The ordinary set of such elements is called the α-level sets of $\tilde{A}$ and is denoted by A_α [10], i.e.,

$$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in (0, 1]\}$$

Also, the strong α-level set is defined by:

$$A_{\alpha+} = \{x \in X : \mu_{\tilde{A}}(x) > \alpha, \alpha \in (0, 1]\}$$

Remarks (1.2.2.1), [16], [13]:

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of a universal set X, then the following properties are satisfied for all $\alpha \in (0, 1]$:

1. $\tilde{A} = \tilde{B}$ if and only if $A_\alpha = B_\alpha$, $\forall \alpha \in (0, 1]$

2. $\tilde{A} \subseteq \tilde{B}$ if and only if $A_\alpha \subseteq B_\alpha$, $\forall\ \alpha \in (0, 1]$.

3. If $\alpha \le \beta$, then $A_\alpha \supseteq A_\beta$, $\forall\ \alpha, \beta \in (0, 1]$.

4. $A_\alpha \cap A_\beta = A_\beta$ and $A_\alpha \cup A_\beta = A_\alpha$, if $\alpha \le \beta$, $\forall\ \alpha, \beta \in (0, 1]$.

5. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
6. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.

Remarks (1.2.2.2), [35]:

Let $\tilde{A}$ be any fuzzy subset of the universal set X, then:

1. The set of all $\alpha \in (0, 1]$ that represent distinct α-levels of $\tilde{A}$ is called the image of $\tilde{A}$ and is denoted by Im($\tilde{A}$), i.e.,

   $$\text{Im}(\tilde{A}) = \{\alpha : \mu_{\tilde{A}}(x) = \alpha, \text{ for some } x \in X\}$$

2. The support of $\tilde{A}$ is exactly the same as the strong α-level of $\tilde{A}$ for $\alpha = 0$, i.e., $A_{0+} = \text{Supp}(\tilde{A})$.

3. The core of $\tilde{A}$ is the α-level set of $\tilde{A}$ for $\alpha = 1$, i.e., $A_1 = \text{cor}(\tilde{A})$.

4. The height of $\tilde{A}$ may also be viewed as the supremum value of α's of the α-levels for which $A_\alpha \neq \emptyset$.

1.2.3 The Membership Function

An important notion of the theory of fuzzy sets is the definition and construction of the membership functions, which admits certain properties of fuzzy sets. Therefore, in this section, the construction of such functions will be discussed in details. The characteristic function assigns to each element x of X a number, $\mu_{\tilde{A}}(x)$, in the closed unit interval [0, 1] that characterizes the degree of membership of x in $\tilde{A}$. In defining the membership function, the universal set X always assumed to be classical set (either discrete or continuous).

Two approaches may be used to define the membership function of a given fuzzy set; namely numerical or tabulated approach and theoretical or functional approach.

A numerical approach expresses the degree of membership function of a fuzzy set as a vector of numbers whose dimension depends on the level of discretization, i.e., the number of discrete elements in the universal set X. This method has some advantages and disadvantages, which are in advantage case its simplicity of construction and the disadvantage encountered in its very consuming and long definition, especially with those sets of so many elements.

Functional definition defines the membership function of a fuzzy set in an analytic expression, which allows the membership grade for each element in the defined universe of discourse to be calculated. Certain standard families or 'shapes' of membership functions are commonly used for fuzzy sets based on the universe of real numbers and on the definition of the fuzzy set.

Among the most common membership functions, which are often used in practice, include the following types:

1. Any symmetric, triangular shaped membership function used to define fuzzy numbers, which is characterized by the three parameters a, b and s, where $a, s \in \mathbb{R}$ $(s \neq 0)$ and $0 < b \leq 1$, as shown in Fig.(1.1), is represented by the generic form:

$$\mu_{\tilde{A}}(x) = \begin{cases} b\left(1 - \dfrac{|x-a|}{s}\right), & \text{when } a - s \leq x \leq a + s \\ 0, & \text{otherwise} \end{cases}$$

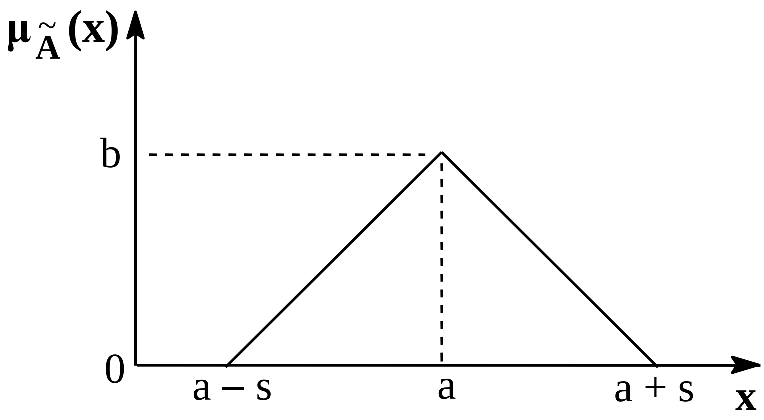

Fig.(1.1) The triangular membership function.

2. Another important class of membership functions used to define fuzzy intervals is of trapezoidal shaped, which is captured by the generic graphical representation in Fig.(1.2). Each function in this class is fully characterized by the five parameters a, b, c, d and e, where a, b, c, d $\in \mathbb{R}$ and $0 < e \le 1$, via the general form:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(a-x)e}{a-b}, & \text{when } a \le x < b, a \ne b \\ e, & \text{when } b \le x < c \\ \dfrac{(d-x)e}{d-c}, & \text{when } c \le x < d, d \ne c \\ 0, & \text{otherwise} \end{cases}$$

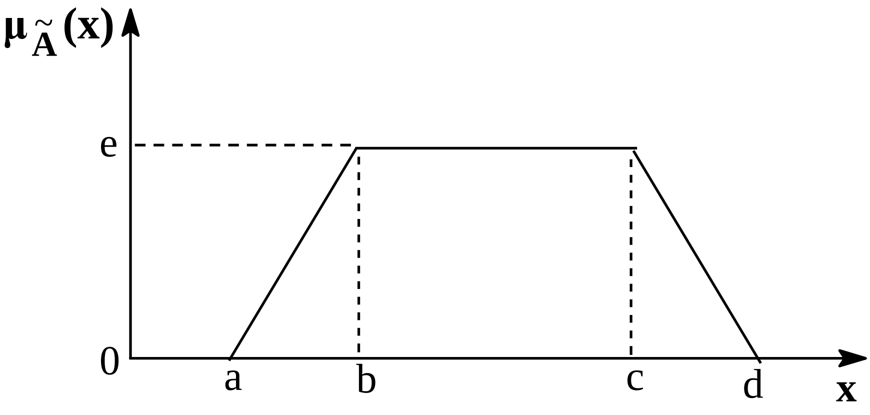

Fig.(1.2) The trapezoidal membership function.

3. Bell-shaped membership functions are also quite common in defining fuzzy numbers. A generic graph is shown in Fig.(1.3). These functions are presented by the formula:

$$\mu_{\tilde{A}}(x) = c\,e^{\frac{-(x-a)^2}{b}}, \; b \neq 0$$

which involves three parameters a, b, and c, where $a \in \mathbb{R}$, $b \in \mathbb{R}\backslash\{0\}$ and $0 \le c \le 1$, whose rules are indicated in Fig.(1.4).

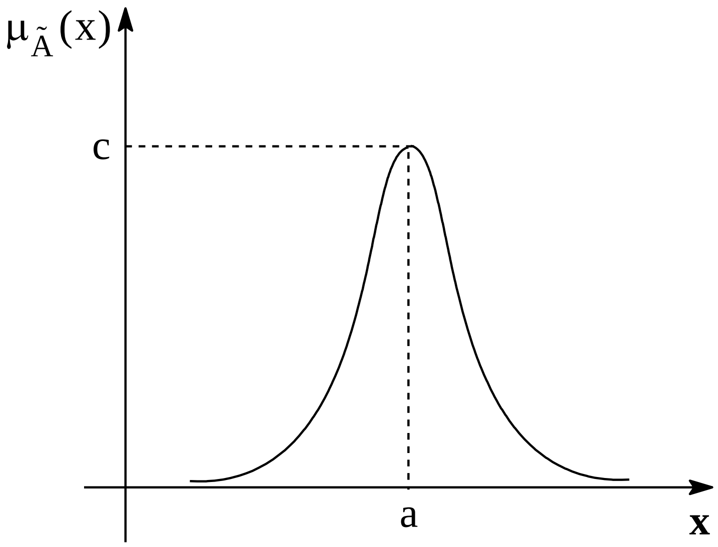

Fig.(1.3) Bell-shaped fuzzy set.

4. The S-membership function is defined as follows:

$$S(x; a, b, c) = \begin{cases} 0 & \text{for } x < a \\ 2\left[(x-a)/(c-a)\right]^2 & \text{for } a \le x < b \\ 1-2\left[(x-c)/(c-a)\right]^2 & \text{for } b \le x < c \\ 1 & \text{for } x > c \end{cases}$$

Functions in this family have an "S" shape whose precise appearance is determined by the value of the parameters of a, b, c, as illustrated in Fig.(1.4). Note that the S-function is flat with constant value 0 for x < a; and constant value 1 for x > c. In between of a and c, the S-function is a quadratic function of x, b = (a + c)/2.

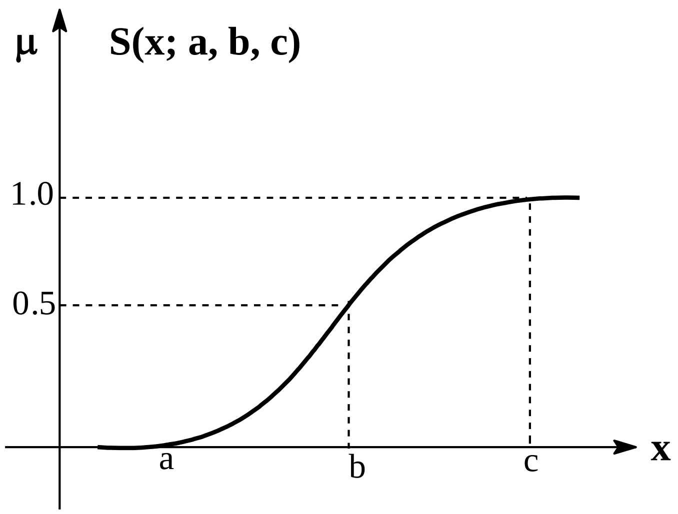

Fig.(1.4) The S-Function.

5. The π-function which may be used also to define fuzzy numbers is defined by:

$$\pi(x; b, c) = \begin{cases} S(x; c-b, c-b/2, c) & \text{for } x \le c \\ 1-S(x; c, c+b/2, c+b) & \text{for } x > c \end{cases}$$

Functions in this family are also of bell- shaped, with the sides of the bell being generated from the S-functions. Functions of this type may be used as an alternative to the triangular-functions as they give a membership value, which approaches 0 in a more gradual manner, as illustrated in Fig.(1.5). Note that the b parameter is now the bandwidth at the crossover points. The π-function goes to zero at the points $x = c \pm b$, while the crossover points are at $x = c \pm b/2$.

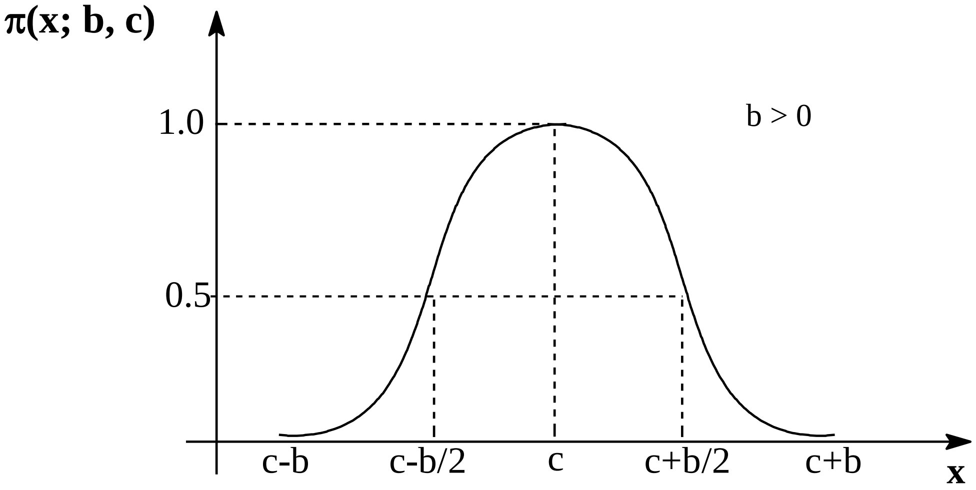

Fig.(1.5) The π-function.

1.2.4 Admissible and Nonadmissible Membership Functions

The assignment of the membership function of a fuzzy set is subjective in nature and, in general, reflects the context in which the problem is viewed. Although, the assignment of the membership function of a fuzzy set $\tilde{A}$ is "subjective", it can not be assigned arbitrarily, as the following example illustrate:

Example (1.2.4.1), [7], [15]:

In this example, we consider the universal set as the class of real numbers, and the fuzzy set $\tilde{A}$ of a all real numbers that are much greater than one. The fuzzy subset $\tilde{A}$ of the universal set $\mathbb{R}$ may be defined mathematically using the membership function, such as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-1}{x}, & \text{if} \quad x > 1 \\ 0 \quad , & \text{if} \quad x \leq 1 \end{cases}$$

While the function:

$$\mu_{\tilde{A}}(x) = \begin{cases} e^{-(x-1)}, & \text{if} \quad x > 1 \\ 0 \quad , & \text{if} \quad x \leq 1 \end{cases}$$

is monotonically decreases as x increases, and:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-e^{-1000(x-1)}, & \text{if} \quad x > 1 \\ 0 \quad , & \text{if} \quad x \leq 1 \end{cases}$$

that increases monotonically, but is approximately equal to 1 for x = 1.1, which is not much greater than 1, hence it is not an adequate characteristic function. Functions like those are called nonadmissible membership functions related to the fuzzy set $\tilde{A}$. The function $\mu_{\tilde{A}}(x)$ as defined in this example and other functions, such as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-e^{-0.1(x-1)}, & \text{if} \quad x > 1 \\ 0 \quad , & \text{if} \quad x \leq 1 \end{cases}$$

or:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1-\frac{1}{\cosh(x-1)}, & \text{if} \quad x > 1 \\ 0 \quad , & \text{if} \quad x \leq 1 \end{cases}$$

which satisfy the condition that $\mu_{\tilde{A}}(x) \in [0, 1]$, $\forall\ x \in X$ and are consistent with the specification of the considered fuzzy set, will be an admissible membership functions for $\tilde{A}$.

1.2.5 Fuzzy Points

In this section the basic definition and properties of fuzzy points are introduced

Definition (1.2.5.1), [36]:

A fuzzy point $\tilde{p}_x^{\lambda}$ (or fuzzy singleton) of a fuzzy set $\tilde{A}$ is also a fuzzy subset of X, where $x \in X$ is the support of the fuzzy point, and $\lambda \in (0, 1]$ is the grade of this fuzzy point, with membership function:

$$\mu_{\tilde{p}_x^{\lambda}}(y) = \begin{cases} \lambda & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x \end{cases}$$

and $\tilde{p}_x^{1-\lambda}$ is the complement fuzzy point of $\tilde{p}_x^{\lambda}$, which is also denoted by ${}^{c}\tilde{p}_x^{\lambda}$.

Definition (1.2.5.2), [36]:

A fuzzy point $\tilde{q}_{x_0}^{\lambda_0}$ belongs to a fuzzy set $\tilde{A}$ (written as $\tilde{q}_{x_0}^{\lambda_0}\ \tilde{\in}\ \tilde{A}$) if and only if $\mu_{\tilde{A}}(x_0) > \lambda_0$, and $\tilde{q}_{x_0}^{\lambda_0}$ does not belong to a fuzzy set $\tilde{A}$ (written as $\tilde{q}_{x_0}^{\lambda_0}\ \tilde{\notin}\ \tilde{A}$) if and only if $\mu_{\tilde{A}}(x_0) \leq \lambda_0$; where we use the symbol $\tilde{\in}$ in order to distinguish from the ordinary belong $\in$ in ordinary sets.

Now, for completeness, we state the following concluding proposition, which give another approach for characterizing the inclusion in fuzzy sets:

Proposition (1.2.5.3), [36]:

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of the universal set X, such that $\tilde{A} \subseteq \tilde{B}$ and $\lambda \in (0, 1]$. If $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}$, then $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{B}$, for all $\tilde{q}_x^{\lambda}$.

Additional properties related to fuzzy points are given in the next propositions:

Proposition (1.2.5.4), [36], [37]:

Let $\tilde{A}_i$, for all $i \in I$, where I is any index set; be fuzzy subsets of the universal set X and $\alpha, \beta \in (0, 1]$, then:

1. If $\tilde{q}_x^{\alpha} \tilde{\in} \bigcup\limits_{i\in I} \tilde{A}_i$ then there exists $i_0 \in I$, such that $\tilde{q}_x^{\alpha} \tilde{\in} \tilde{A}_{i_0}$.
2. If $\tilde{q}_x^{\alpha} \tilde{\in} \bigcap\limits_{i\in I} \tilde{A}_i$, then $\tilde{q}_x^{\alpha} \tilde{\in} \tilde{A}_i$, for all $i \in I$.

Proof:

1. Since $\bigcup\limits_{i\in I} \tilde{A}_i$ is a fuzzy set with membership function $\underset{x\in X}{\text{Max}}\, \mu_{\tilde{A}_i}(x)$ since $\tilde{q}_x^{\alpha} \tilde{\in} \bigcup\limits_{i\in I} \tilde{A}_i$, then $\mu_{\bigcup\limits_{i\in I} \tilde{A}_i}(x) \geq \alpha$. Hence, there exist at least one $i_0 \in I$, such that $\mu_{\tilde{A}_{i_0}}(x) \geq \alpha$. Therefore, from definition (1.2.5.1), $\tilde{q}_x^{\alpha} \tilde{\in} \tilde{A}_{i_0}$
2. Since $\bigcap\limits_{i\in I} \tilde{A}_i$ is a fuzzy set with membership function $\underset{x\in X}{\text{Min}}\, \mu_{\tilde{A}_i}(x)$, and since $\tilde{q}_x^{\alpha} \tilde{\in} \bigcap\limits_{i\in I} \tilde{A}_i$, then:

$$\mu_{\bigcap_{i\in I}\tilde{A}_i}(x) \geq \alpha$$

Hence for all $i \in I$, $\mu_{\tilde{A}_i}(x) \geq \alpha$

Therefore, from definition (1.2.5.2), $\tilde{q}_x^{\alpha} \,\tilde{\in}\, \tilde{A}_i$, for all $i \in I$.

Proposition (1.2.5.5), [36], [37]:

Let $\tilde{A}$ be fuzzy subsets of the universal set X and $\lambda, \lambda_0, \lambda_1, \lambda_2 \in (0, 1]$, then:

1. $\tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{q}_{x_2}^{\lambda_2}$ if and only if $\lambda_1 < \lambda_2$ and $x_1 = x_2$.

2. $\tilde{q}_{x_0}^{\lambda_0} \,\tilde{\in}\, \tilde{A}^c$ if and only if ${}^c\tilde{q}_{x_0}^{\lambda_0} \,\tilde{\notin}\, \tilde{A}$.

3. If $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}$, then $\tilde{q}_x^{\lambda'} \,\tilde{\in}\, \tilde{A}$ if there exists $\lambda' < \lambda$, where $\lambda' \in (0, 1]$.

Proof:

1. If $\tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{q}_{x_2}^{\lambda_2}$. Hence: $\mu_{\tilde{q}_{x_2}^{\lambda_2}}(x) > \mu_{\tilde{q}_{x_1}^{\lambda_1}}(x)$, for the same $x \in X$, i.e., $\lambda_2 > \lambda_1$.

Since $\mu_{\tilde{q}_{x_2}^{\lambda_2}}(x) = \lambda_2$ if $x = x_2$ and $\mu_{\tilde{q}_{x_1}^{\lambda_1}}(x) = \lambda_1$ if $x = x_1$, which is true for the same value of x, i.e., if $x_1 = x_2$.

Conversely, if $x_1 = x_2$, $\mu_{\tilde{q}_{x_1}^{\lambda_1}}(x) = \lambda_1$, $\mu_{\tilde{q}_{x_2}^{\lambda_2}}(x) = \lambda_2$, where $\lambda_2 > \lambda_1$, which implies that: $\tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{q}_{x_2}^{\lambda_2}$

2. If $\tilde{q}_{x_0}^{\lambda_0} \,\tilde{\in}\, \tilde{A}^c$, then $\mu_{\tilde{A}^c}(x_0) \geq \lambda_0$. Which is equivalent to $1 - \mu_{\tilde{A}}(x_0) \geq \lambda_0$,

i.e., $\mu_{\tilde{A}}(x_0) \leq 1 - \lambda_0$. Therefore, $\tilde{q}_{x_0}^{1-\lambda_0} \,\tilde{\notin}\, \tilde{A}$, i.e., ${}^c\tilde{q}_{x_0}^{\lambda_0} \,\tilde{\notin}\, \tilde{A}$.

Conversely, if ${}^{c}\tilde{q}_{x_0}^{\lambda_0} \;\tilde{\notin}\; \tilde{A}$, implies to: $\mu_{{}^{c}\tilde{q}_{x_0}^{\lambda_0}}(x) \geq \mu_{\tilde{A}}(x)$, $\forall$ x $\in$ X.

Hence, $1 - \lambda_0 \geq \mu_{\tilde{A}}(x)$, i.e., $1 - \mu_{\tilde{A}}(x) \geq \lambda_0$. Therefore, $\mu_{\tilde{A}^c}(x) \geq \lambda_0$, i.e., $\tilde{q}_{x_0}^{\lambda_0} \;\tilde{\in}\; \tilde{A}^c$.

3. From part (1) above, suppose that there exists $\lambda' < \lambda$ and for $x_1 = x_2$, then $\tilde{q}_x^{\lambda'} \;\tilde{\in}\; \tilde{q}_x^{\lambda}$. Since $\tilde{q}_x^{\lambda} \;\tilde{\in}\; \tilde{A}$, then $\mu_{\tilde{A}}(x) > \lambda$. Hence, $\mu_{\tilde{A}}(x) > \lambda > \lambda'$, $\lambda' \in (0, 1]$. Therefore, $\mu_{\tilde{A}}(x) > \lambda'$, for some λ', i.e., $\tilde{q}_x^{\lambda'} \;\tilde{\in}\; \tilde{A}$.

Proposition (1.2.5.6), [36], [37]:

Let $\tilde{A}$ be a fuzzy subsets of the universal set X, then $\tilde{A} \neq \tilde{\varnothing}$ if and only if there exists at least one fuzzy point $\tilde{q}_x^{\lambda}$, where x $\in$ X, $\lambda \in (0, 1]$, such that $\tilde{q}_x^{\lambda} \;\tilde{\in}\; \tilde{A}$.

Proof:

$\Rightarrow$ Since $\tilde{\varnothing}$ is a fuzzy set with membership function $\mu_{\tilde{\varnothing}}(y) = 0$, $\forall$ y $\in$ X and since $\tilde{A} \neq \tilde{\varnothing}$. Hence, there exists at least one y = x, such that $\mu_{\tilde{A}}(x) \neq 0$. Let $\mu_{\tilde{A}}(x) = \lambda > 0$, therefore $\tilde{q}_x^{\lambda} \;\tilde{\in}\; \tilde{A}$

$\Leftarrow$ If there exists at least one fuzzy point $\tilde{q}_x^{\lambda}$, such that $\tilde{q}_x^{\lambda} \;\tilde{\in}\; \tilde{A}$ then $\mu_{\tilde{A}}(x) \geq \lambda > 0$, i.e., $\mu_{\tilde{A}}(x) \neq 0$, for some x $\in$ X

Hence, $\tilde{A} \neq \tilde{\varnothing}$.

Chapter Two

Fuzzy Cone Metric Spaces

The Fuzzy metric spaces and cone metric spaces represents a generalizing for the ordinary metric spaces, and in addition fuzzy cone metric spaces are the generalized for the fuzzy metric spaces and cone metric spaces.

2.1 Fuzzy Metric Spaces, [37]

In this section, we shall study the fuzzy metric spaces with respect to the fuzzy point, which is defined by Zike [37], in which in own reference he gave some basic properties with an illustrate example.

Definition (2.1.1):

A function $d^*: I^X \times I^X \longrightarrow [0, \infty)$ is called fuzzy distance function if d* satisfies the following conditions:

1- $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = 0$ if and only if $\lambda_1 \leq \lambda_2$ and $x_1 = x_2$.

2- $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = d^*({}^c\tilde{q}_{x_2}^{\lambda_2}, {}^c\tilde{q}_{x_1}^{\lambda_1})$.

3- $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) \leq d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3})$.

4- If $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) < r$, where $r > 0$, then there exist $\lambda' > \lambda_1 > \lambda_2$, such that $d^*(\tilde{q}_{x_1}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) < r$.

Also, (I^X, d^*) is called fuzzy metric space.

The next example is an application of definition (2.1.1), which is very important

Example (2.1.2):

Let (X, d) be the universal metric space, and let $\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2} \in I^X$; and suppose that d* be defined as follows:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} + d(x_1, x_2) \qquad \ldots(2.1)$$

where $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in (0, 1]$, then (I^X, d^*) is a fuzzy metric space.

Remarks (2.1.3):

1. In particular, if $X = \mathbb{R}$ and $d(x_1, x_2) = |x_1 - x_2|$, then we get the fuzzy distance function given by [37], as:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} + |x_1 - x_2|$$

2. If $X = \mathbb{R}^2$ and $d(x_1, x_2) = \sqrt{(x_1^1 - x_2^1)^2 + (x_1^2 - x_2^2)^2}$, then the fuzzy distance function takes the form:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} + \sqrt{(x_1^1 - x_2^1)^2 + (x_1^2 - x_2^2)^2}$$

where $x_1 = (x_1^1, x_1^2)$, $x_2 = (x_2^1, x_2^2)$.

3. If $X = C[a, b]$ and $d(f, g) = \left(\int_a^b |f - g|^2 \, dx\right)^{1/2}$, $\forall f, g \in C[a, b]$, and hence the fuzzy distance function takes the form:

$$d^*(\tilde{q}_f^{\lambda_1}, \tilde{q}_g^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} + \left(\int_a^b |f - g|^2 \, dx\right)^{1/2}$$

Definition (2.1.4):

Let (I^X, d^*) be fuzzy metric space, $\widetilde{A}$ be s subset of I^X, and $\{\tilde{q}_{x_n}^{\lambda_n}\}$ be a sequence of fuzzy points in $\widetilde{A}$. Then:

1. $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is said to be converge to $\tilde{q}_x^{\lambda}$ if for all $\alpha > 0$, there exist $N \in \mathbb{N}$, such that:

$$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) < \alpha. \text{ For all } n > N.$$

2. $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is said to be Cauchy sequence if for all $\alpha > 0$, there exist $N \in \mathbb{N}$, such that:

$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) < \alpha$. For all n,m > N.

3. $\widetilde{A}$ is said to be complete if every Cauchy sequence is convergent.

Some further results concerning the fuzzy distance function (2.1) are given as an additional property without proof.

Theorem (2.1.5):

Let (I^X, d^*) be a fuzzy metric space and $\tilde{q}_{x_1}^{\lambda_1}$, $\tilde{q}_{x_2}^{\lambda_2}$ be two fuzzy points in I^X, then:

$$d^*(\tilde{q}_{x_1}^{\lambda}, \tilde{q}_{x_2}^{\lambda_2}) \le d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}), \text{ if } \lambda < \lambda_1.$$

Theorem (2.2.5):

Let (I^X, d^*) be a fuzzy metric space and $\tilde{q}_{x_1}^{\lambda_1}$, $\tilde{q}_{x_2}^{\lambda_2}$ be two fuzzy points in I^X, and if $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) < r$, then there exists $\lambda' \in (0, 1]$, such that $\lambda' < \lambda_2$, and $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda'}) < r$.

Theorem (2.1.7):

Let (I^X, d^*) be a fuzzy metric space and $\tilde{q}_{x_1}^{\lambda_1}$, $\tilde{q}_{x_2}^{\lambda_2}$ be two fuzzy points in I^X, then:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda}) \le d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}), \text{ whenever } \lambda > \lambda_2$$

2.2 Cone Metric Spaces

Huan Long - Guang and Zhang Xian introduced Cone metric space [14], as a generalization of metric spaces by replacing the set of real numbers by an ordered Banach space. They introduced the basic definitions and discuss some properties.

In this section we shall define cone metric spaces and prove some properties.

Definition (2.2.1), [14]:

Let E be a Banach space and P a subset of E, then P is called a cone if:

1. P is nonempty closed and $P \neq \{0\}$;
2. If $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies that $ax + by \in P$;
3. If $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subseteq E$, they defined the partial ordering according to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ that but $x \neq y$, while $x \ll y$ will represent $y - x \in \text{int}(P)$, where $\text{int}(P)$ denotes the interior of P, [14].

The cone P is called normal if there is a real number $k > 0$, such that for any $x, y \in P$, in which $0 \leq x \leq y$ implies that

$$\|x\| \leq k\|y\| \qquad \ldots(2.2)$$

and the least positive number k satisfying (2.2) is called the normal constant of P.

Cone P is called regular if every increasing sequence is convergent. That is, if $\{x_n\}$ is a sequence, such that:

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots \leq y$$

for some $y \in E$, then there is $x \in E$, such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every below bounded decreasing sequence is convergent, [14].

The next lemma gives the relationship between regular and normal cones.

Lemma (2.2.2), [14]:

Every regular cone is normal.

In the following suppose that E is Banach space, P is a cone in E with $\mathrm{int}(P) \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition (2.2.3), [14]:

Let X be a set and suppose that the mapping $d: X \times X \to E$ satisfies:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$, for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example (2.2.4), [31]:

Let $E = \mathbb{R}^2$, $P = \{(x.y) \in E: x, y \geq 0\}$, $X = \mathbb{R}$ and $d: X \times X \to E$ defined by

$$d(x, y) = (|x - y|, \alpha|x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition (2.2.5), [26]:

Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$, $n \in \mathbb{N}$ a sequence in X. Then:

1. $\{x_n\}$, $n \in \mathbb{N}$ converges to $x \in X$ if for every $c \in E$ with $c \gg 0$, there exist $N \in \mathbb{N}$, such that $d(x_n, x) \ll c$, for all $n \geq N$. We denoted this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$, $n \in \mathbb{N}$ is Cauchy sequence if for every $c \in E$ with $c \gg 0$, there exist $N \in \mathbb{N}$, such that $d(x_n, x_m) \ll c$, for all $n, m \geq N$.

3. The cone metric space (X, d) is said to be complete cone metric space if every Cauchy sequence is convergent.

2.3 Fuzzy Cone Metric Spaces with Respect to Fuzzy Points

In this section, will be introduced for the first time based on fuzzy point approach fuzzy cone metric with illustrate example, as well as, give the basic properties.

Definition (2.3.1):

Let (X, d) be the universal metric space and let E be a Banach space and $P \subseteq E$ is a cone. Then a function $d^*: I^X \times I^X \longrightarrow E$ is called fuzzy cone distance function if d* satisfies the following conditions:

1. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = 0$ if and only if $\lambda_1 \leq \lambda_2$ and $x_1 = x_2$.
2. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = d^*({}^c\tilde{q}_{x_2}^{\lambda_2}, {}^c\tilde{q}_{x_1}^{\lambda_1})$.
3. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) \leq d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3})$.
4. If $r - d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \in int(P)$, where $r \in int(P)$, then there exist $\lambda' > \lambda_1 > \lambda_2$, such that $r - d^*(\tilde{q}_{x_1}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) \in int(P)$.

The pair (I^X, d^*) is called fuzzy cone metric space.

The next theorem is of great importance for this chapter, since in this example we give the definition and the proof of the fuzzy cone distance function in terms of definition (2.3.1):

Theorem (2.3.2):

Let (X, d) be the universal metric space, let $\tilde{q}_{x_1}^{\lambda_1}$, $\tilde{q}_{x_2}^{\lambda_2} \in I^X$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E: x, y \geq 0\}$; and suppose d* be defined as follows:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = (\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2)) \qquad \dots (2.3)$$

where $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in (0, 1]$, then (I^X, d^*) is a fuzzy cone metric space.

As one can see the satisfaction of the conditions of the definition (2.3.1) for the fuzzy function (2.3) may be assumed as follows:

1. Since by taking $\lambda_1 \leq \lambda_2$, i.e., $\lambda_1 - \lambda_2 \leq 0$, then $\max\{\lambda_1 - \lambda_2, 0\} = 0$ and $x_1 = x_2$, which implies to $d(x_1, x_2) = 0$. Hence:

$$d^*(\tilde{q}_{x_0}^{\lambda_1}, \tilde{q}_{x_0}^{\lambda_2}) = (\max\{\lambda_1 - \lambda_2, 0\}, d(x_0, x_0)) = (0, 0) = 0$$

Therefore, $d^*(\tilde{q}_{x_0}^{\lambda_1}, \tilde{q}_{x_0}^{\lambda_2}) = 0$.

If $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = 0$, then from eq. (2.3)

$$(\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2)) = (0, 0)$$

and hence $\max\{\lambda_1 - \lambda_2, 0\} = 0$ and $d(x_1, x_2) = 0$, which is satisfied only if $\lambda_1 - \lambda_2 \leq 0$ and $x_1 = x_2$.

2. Since $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = (\max\{\lambda_1 - \lambda_2, 0\} + d(x_1, x_2))$. Hence:

$$\begin{aligned} d^*({}^c\tilde{q}_{x_2}^{\lambda_2}, {}^c\tilde{q}_{x_1}^{\lambda_1}) &= d^*(\tilde{q}_{x_2}^{1-\lambda_2}, \tilde{q}_{x_1}^{1-\lambda_1}) \\ &= (\max\{1 - \lambda_2 - 1 + \lambda_1, 0\}, d(x_2, x_1)) \\ &= (\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2)) \\ &= d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \end{aligned}$$

Therefore $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = d^*({}^c\tilde{q}_{x_2}^{\lambda_2}, {}^c\tilde{q}_{x_1}^{\lambda_1})$.

3. Now, for all $\tilde{q}_{x_1}^{\lambda_1}$, $\tilde{q}_{x_2}^{\lambda_2}$ and $\tilde{q}_{x_3}^{\lambda_3} \in I^X$; to prove the satisfaction of the triangle inequality $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) \leq d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3})$, which is asserted as follows:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) = (\max\{\lambda_1 - \lambda_3, 0\}, d(x_1, x_3)), \text{ where } \lambda_2 \in (0, 1].$$

$$= (\max\{\lambda_1 - \lambda_2 + \lambda_2 - \lambda_3, 0\}, d(x_1, x_3))$$

$$\leq (\max\{\lambda_1 - \lambda_2, 0\} + \max\{\lambda_2 - \lambda_3, 0\}, d(x_1, x_2) + d(x_1, x_3))$$

$$= d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3}).$$

4. If $r - d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \in int(P)$, where $r = (r_1, r_2) \in int(P)$.

Since $\lambda_1 > \lambda_2$, i.e., $\lambda_1 - \lambda_2 > 0$, and hence $\max\{\lambda_1 - \lambda_2, 0\} = \lambda_1 - \lambda_2$, which implies to:

$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \ll (r_1, r_2)$$

which means that:

$$(r_1, r_2) - d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \in int(P)$$

$$(r_1, r_2) - (\max\{\lambda_1 - \lambda_2, 0\}, d(x_1, x_2)) \in int(P)$$

$$(r_1, r_2) - (\lambda_1 - \lambda_2, d(x_1, x_2)) \in int(P)$$

$$(r_1 - (\lambda_1 - \lambda_2), r_2 - d(x_1, x_2)) \in int(P)$$

therefore,

$$r_1 - (\lambda_1 - \lambda_2) > 0$$

$$r_1 > (\lambda_1 - \lambda_2)$$

Hence, $\lambda_1 - \lambda_2 < r_1$, i.e., $\lambda_1 < r_1 + \lambda_2$.

Let $\lambda' \in (0, 1]$ be chosen so that $\lambda_2 < \lambda_1 < \lambda' < \min\{1, r_1 + \lambda_2\}$, which implies to $\lambda' \leq r_1 + \lambda_2$, and so $0 < \lambda' - \lambda_2 < r_1$, i.e., $r_1 - (\lambda' - \lambda_2) > 0$.

Then:

$$(r_1 - (\lambda' - \lambda_2), r_2 - d(x_1, x_2)) \in \text{int}(P)$$

$$(r_1, r_2) - (\max\{\lambda' - \lambda_2, 0\}, d(x_1, x_2)) \in \text{int}(P)$$

$$r - d^*(\tilde{q}_{x_1}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) \in \text{int}(P).$$

Therefore, (I^X, d^*) is a fuzzy cone metric space.

2.4 Fundamentals of Fuzzy Cone Metric Spaces

Now, we are in a position to give the basic properties of fuzzy cone metric spaces. These definitions are given also in terms of the fuzzy points in order to make a compatibility between the definition of the fuzzy distance function (see definition (2.3.1) and eq. (2.3)) and those definitions.

Definition (2.4.1):

Let (I^X, d^*) be a fuzzy cone metric space and let $\tilde{q}_x^{\lambda}$ be a fuzzy point in I^X, then the fuzzy neighborhood of a point $\tilde{q}_x^{\lambda}$ is the fuzzy set $U_\varepsilon(\tilde{q}_x^{\lambda})$ consisting all points $\tilde{q}_{x'}^{\lambda'} \tilde{\in} I^X$, such that $\varepsilon - d^*(\tilde{q}_{x'}^{\lambda'}, \tilde{q}_x^{\lambda}) \in \text{int}(P)$, where the number ε is called the radius of $U_\varepsilon(\tilde{q}_x^{\lambda})$ and $\tilde{q}_x^{\lambda}$ is the center of the neighborhood, i.e.,

$$U_\varepsilon(\tilde{q}_x^{\lambda}) = \{\tilde{q}_{x'}^{\lambda'} \tilde{\in} I^X \mid \varepsilon - d^*(\tilde{q}_{x'}^{\lambda'}, \tilde{q}_x^{\lambda}) \in \text{int}(P), \text{ where } \varepsilon \in \text{int}(P)\}$$

Definition (2.4.2):

Let $\tilde{A}$ be a fuzzy subset of the fuzzy cone metric space (I^X, d*). A fuzzy point $\tilde{q}_x^{\lambda}$ is called a fuzzy limit point of the set $\tilde{A}$ if every neighborhood of $\tilde{q}_x^{\lambda}$ contains a point $\tilde{q}_{x'}^{\lambda'} \neq \tilde{q}_x^{\lambda}$, such that $\tilde{q}_{x'}^{\lambda'} \tilde{\in} \tilde{A}$.

Definition (2.4.3):

Let (I^X, d*) be a fuzzy cone metric space. A fuzzy point $\tilde{q}_x^{\lambda}$ is called an interior fuzzy point of $\tilde{G} \tilde{\in} I^X$ if and only if there exists a fuzzy neighborhood $\tilde{q}_x^{\lambda} \tilde{\in} U_{\tilde{q}_x^{\lambda}} \subseteq \tilde{G}$.

Definition (2.4.4):

A fuzzy set $\tilde{A}$ in a fuzzy cone metric space (I^X, d*) is called open fuzzy set if for all $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}$, there exists $\varepsilon \in$ int(P), such that $U_\varepsilon(\tilde{q}_x^{\lambda}) \subseteq \tilde{A}$.

Definition (2.4.5):

A fuzzy set $\tilde{A}$ in the fuzzy cone metric space (I^X, d*) is said to be fuzzy closed set if $\tilde{A}^c$ is a open fuzzy set in I^X, or every point of $\tilde{A}$ is a fuzzy limit point of $\tilde{A}$.

Definition (2.4.6):

A fuzzy set $\tilde{A}$ in the fuzzy cone metric space (I^X, d*) is bounded if there exists h $\in$ int(P), and $\tilde{q}_{x_0}^{\lambda_0}$, such that $h - d^*(\tilde{q}_{x_0}^{\lambda_0}, \tilde{q}_x^{\lambda}) \in$ int(P), for all $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}$.

Definition (2.4.7):

Let (X, d) be the universal metric space and (I^X, d^*) be fuzzy cone metric space. A sequence $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ of fuzzy points in $\tilde{A}$ is said to be converge to $\tilde{q}_x^{\lambda}$ (termed as $\tilde{q}_{x_n}^{\lambda_n} \longrightarrow \tilde{q}_x^{\lambda}$) if for all $\varepsilon \in \mathrm{int}(P)$, there exist $N \in \mathbb{N}$, such that: $\varepsilon - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) \in \mathrm{int}(P)$, for all $n \geq N$, where $x, x_n \in X$, $\lambda, \lambda_n \in (0, 1]$, $\forall\, n \in \mathbb{N}$.

Definition (2.4.8):

A fuzzy point $\tilde{q}_x^{\lambda}$ is an accumulation fuzzy point of $\tilde{A}$ if for all $\varepsilon \in \mathrm{int}(P)$, there exists $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\in} \tilde{A}$, such that $\varepsilon - d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x_n}^{\lambda_n}) \in \mathrm{int}(P)$, for infinitely many n.

Remark (2.4.9):

Let (I^X, d^*) be a fuzzy cone metric space and $\{\tilde{q}_{x_i}^{\lambda_i}\}$, $i = 1, 2, \ldots$; be a sequence of fuzzy points, which is converge to $\tilde{q}_x^{\lambda} \tilde{\in} I^X$, then every fuzzy neighborhood $\tilde{U}_\varepsilon(\tilde{q}_x^{\lambda})$, $\varepsilon \in \mathrm{int}(P)$ contains all but (or except) infinitely many terms of $\{\tilde{q}_{x_i}^{\lambda_i}\}$.

Definition (2.4.10):

Let (X, d) be the universal metric space and (I^X, d^*) be fuzzy cone metric space. A sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$ in $\tilde{A}$ is said to be a Cauchy sequence if for all $\varepsilon \in \mathrm{int}(P)$, there exists $N \in \mathbb{N}$, such that:

$\varepsilon - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) \in \text{int}(P)$, for all n, m ≥ N.

where $x_n, x_m \in X$, $\lambda_n, \lambda_m \in (0, 1]$, $\forall$ n, m $\in \mathbb{N}$.

An important characterization result which may be considered as the main result of this chapter is the next theorem, which relates between the convergence of a sequence of fuzzy points with the convergence of two sequences in ordinary sense. This theorem is of great importance, which will be used later.

<u>Theorem (2.5.11):</u>

Let (X, d) be the universal metric space and (I^X, d^*) be fuzzy cone metric space. A sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ is converge to $\tilde{q}_x^{\lambda}$ if and only if there exists two non-fuzzy sequences, namely sequence of supports $\{x_n\} \subset X$ and monotonic sequence of images $\{\lambda_n\} \subseteq (0, 1]$, $n \in N$, such that $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$, $x \in X$, $\lambda \in (0, 1]$.

Proof:

To prove the first condition. If $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is converge to $\tilde{q}_x^{\lambda}$, so for all $\varepsilon \in \text{int}(P)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2)$ there exists $N \in \mathbb{N}$, such that $\varepsilon - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) \in \text{int}(P)$, for all n ≥ N. Hence:

$$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) = (\max\{\lambda_n - \lambda, 0\}, d(x_n, x))$$

$$\ll (\varepsilon_1, \varepsilon_2), \text{ for all } n \geq N$$

Implies to

$$(\varepsilon_1, \varepsilon_2) - (\max\{\lambda_n - \lambda, 0\}, d(x_n, x)) \in \text{int}(P), \text{ for all } n \geq N$$

Therefore

$$(\varepsilon_1 - \max\{\lambda_n - \lambda, 0\}, \varepsilon_2 - d(x_n, x)) \in int(P), \text{ for all } n \geq N$$

i.e.,

$$\varepsilon_1 - \max\{\lambda_n - \lambda, 0\} > 0, \text{ and } \varepsilon_2 - d(x_n, x) > 0.$$

Which implies to:

$$\max\{\lambda_n - \lambda, 0\} < \varepsilon_1, \text{ for all } n \geq N$$

Therefore

$$d(x_n, x) < \varepsilon_2, \text{ for all } n \geq N, \text{ i.e., } x_n \longrightarrow x, \text{ and } \lambda_n \longrightarrow \lambda.$$

Conversely. If $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$

Hence, for all $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$, such that:

$$d(x_n, x) < \varepsilon_1, \max\{\lambda_n - \lambda, 0\} < \varepsilon_2, n \geq N = \max\{N_1, N_2\}$$

i.e.,

$$\varepsilon_1 - d(x_n, x) > 0, \text{ and } \varepsilon_2 - \max\{\lambda_n - \lambda, 0\} > 0, \text{ for all } n \geq N.$$

or equivalently

$$(\varepsilon_2 - \max\{\lambda_n - \lambda, 0\}, \varepsilon_1 - d(x_n, x)) \in int(P)$$

which implies to

$$(\varepsilon_2, \varepsilon_1) - (\max\{\lambda_n - \lambda, 0\}, d(x_n, x)) \in int(P)$$

Therefore

$$(\max\{\lambda_n - \lambda, 0\}, d(x_n, x)) \ll (\varepsilon_2, \varepsilon_1)$$

Hence:

$$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) = \max\{\lambda_n - \lambda, 0\} + d(x_n, x) \ll (\varepsilon_2, \varepsilon_1)$$

i.e.,

$$(\varepsilon_2, \varepsilon_1) - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) \in int(P), \text{ for all } n \geq N.$$

Hence, $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is converge sequence of fuzzy points.

Proposition (2.4.12):

Let (X, d) be the universal metric space and (I^X, d^*) be fuzzy cone metric space. A sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ is Cauchy sequence if and only if there exists two nonfuzzy Cauchy sequences, namely the Cauchy sequence of supports $\{x_n\} \subset X$ and monotonic Cauchy sequence of images $\{\lambda_n\} \subseteq (0, 1]$, $n \in N$.

Proof:

Suppose that $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is Cauchy sequence, so for all $\varepsilon \in int(P)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2)$there exists $N \in \mathbb{N}$, such that $\varepsilon - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) \in int(P)$, for all n, $m \geq N$. Hence:

$$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) = (\max\{\lambda_n - \lambda_m, 0\}, d(x_n, x_m))$$

$$\ll (\varepsilon_1, \varepsilon_2), \text{ for all } n, m \geq N$$

Implies to

$$(\varepsilon_1, \varepsilon_2) - (\max\{\lambda_n - \lambda_m, 0\}, d(x_n, x_m)) \in int(P), \text{ for all } n, m \geq N$$

Therefore

$$(\varepsilon_1 - \max\{\lambda_n - \lambda_m, 0\}, \varepsilon_2 - d(x_n, x_m)) \in int(P), \text{ for all } n, m \geq N$$

i.e.,

$$\varepsilon_1 - \max\{\lambda_n - \lambda_m, 0\} > 0, \text{ and } \varepsilon_2 - d(x_n, x_m) > 0, \text{ for all } n, m \geq N$$

Which implies to:

$$\max\{\lambda_n - \lambda_m, 0\} < \varepsilon_1, \text{ for all } n, m \geq N$$

Also

$d(x_n, x_m) < \varepsilon_2$, for all $n, m \geq N$, i.e., $\{x_n\}$ and $\{\lambda_n\}$ are Cauchy sequence.

$\Leftarrow$ If $\{x_n\}$ and $\{\lambda_n\}$ are Cauchy sequence.

Hence, for all $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$, such that:

$$d(x_n, x_m) < \varepsilon_1,\ \max\{\lambda_n - \lambda_m, 0\} < \varepsilon_2,\ n, m \geq N = \max\{N_1, N_2\}$$

i.e.,

$$\varepsilon_1 - d(x_n, x_m) > 0, \text{ and } \varepsilon_2 - \max\{\lambda_n - \lambda_m, 0\} > 0, \text{ for all } n, m \geq N.$$

or equivalently

$$(\varepsilon_2 - \max\{\lambda_n - \lambda_m, 0\},\ \varepsilon_1 - d(x_n, x_m)) \in \mathrm{int}(P), \text{ for all } n, m \geq N.$$

which implies to

$$(\varepsilon_2, \varepsilon_1) - (\max\{\lambda_n - \lambda_m, 0\},\ d(x_n, x_m)) \in \mathrm{int}(P), \text{ for all } n, m \geq N.$$

Therefore

$$(\max\{\lambda_n - \lambda_m, 0\},\ d(x_n, x_m)) \ll (\varepsilon_2, \varepsilon_1), \text{ for all } n, m \geq N.$$

Hence:

$$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) = \max\{\lambda_n - \lambda_m, 0\} + d(x_n, x_m) \ll (\varepsilon_2, \varepsilon_1), \text{ for all}$$

$n, m \geq N$, i.e.,

$$(\varepsilon_2, \varepsilon_1) - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) \in \mathrm{int}(P), \text{ for all } n, m \geq N.$$

Hence, $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is Cauchy sequence.

Proposition (2.4.13):

Let (X, d) be the universal metric space and (I^X, d^*) be fuzzy cone metric space. If a sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ is converge to $\tilde{q}_x^{\lambda}$, then $\{{}^c\tilde{q}_{x_n}^{\lambda_n}\}$ is convergent to ${}^c\tilde{q}_x^{\lambda}$.

Proof:

If a sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ is converging to $\tilde{q}_x^{\lambda}$. Then using theorem (2.4.11), there exist sequence of supports $\{x_n\} \subset X$ and monotonic sequence of images $\{\lambda_n\} \subseteq (0, 1]$, $n \in \mathbb{N}$, such that:

$$x_n \longrightarrow x \text{ and } \lambda_n \longrightarrow \lambda,\ x \in X,\ \lambda \in (0, 1].$$

Therefoere:

$$x_n \longrightarrow x, \text{ and } 1 - \lambda_n \longrightarrow 1 - \lambda$$

Which implies to

$$\{{}^c\tilde{q}_{x_n}^{\lambda_n}\} \longrightarrow {}^c\tilde{q}_x^{\lambda}.$$

<u>Theorem (2.4.14):</u>

Every convergent sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ in a fuzzy cone metric space (I^X, d^*) is a Cauchy sequence.

Proof:

Let $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ be a convergent sequence in (I^X, d^*). Then using theorem (2.4.11), there exist sequence of supports $\{x_n\} \subset X$ and monotonic sequence of images $\{\lambda_n\} \subseteq (0, 1]$, $n \in \mathbb{N}$, such that $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$, $x \in X$, $\lambda \in (0, 1]$. Since $\{x_n\}$ is convergent nonfuzzy sequence, hence it is a Cauchy sequence in X. Also, since $\{\lambda_n\}$ is convergent sequence of images in $(0, 1] \subset \mathbb{R}$.

Therefore, using proposition (2.4.12), the sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in \mathbb{N}$ is a Cauchy sequence.

Definition (2.4.15):

Let (X, d) be the universal metric space and (I^X, d^*) be a fuzzy metric space, then (I^X, d^*) is said to be complete fuzzy metric space if every Cauchy sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\}$ in I^X is converge to a fuzzy point $\tilde{q}_x^{\lambda}$ in I^X.

Theorem (2.4.15):

Let (X, d) be a complete universal metric space, then (I^X, d^*) is a complete fuzzy cone metric space.

Proof:

Let $\{\tilde{q}_{x_n}^{\lambda_n}\}$, $n \in N$ be a Cauchy sequence in (I^X, d^*). Hence, from definition (2.4.10), for all $\varepsilon \in int(P)$, there exists $N \in \mathbb{N}$, such that:

$$\varepsilon - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) \in int(P), \text{ for all } n, m > N$$

using proposition (2.4.12), implies to:

$$d(x_n, x_m) < \varepsilon_1 \quad \text{and} \quad \max\{\lambda_n - \lambda_m, 0\} < \varepsilon_1, \text{ for all } n, m > N.$$

Since $\{x_n\}$ is a Cauchy sequence in (X, d) which is a complete metric space, hence there exist $x \in X$, such that $x_n \longrightarrow x \in X$.

Also, $\{\lambda_n\}$ is a Cauchy sequence of real numbers which is complete, hence there exist $\lambda \in (0, 1]$, such that $\{\lambda_n\}$ is converge to $\lambda \in (0, 1]$.

Hence, from theorem (2.4.11), implies that: $\tilde{q}_{x_n}^{\lambda_n} \longrightarrow \tilde{q}_x^{\lambda}$.

Then (I^X, d^*) is a complete fuzzy cone metric space.

Chapter Three

Compactness of Fuzzy Cone Metric Spaces

In this chapter we will study some different types of compactness fuzzy sets such as pre-compact, sequentially compact, countably compact and locally compact, then study the relationship between them, and their properties.

3.1 Compact Fuzzy Cone Metric Spaces

Compactness comes to be one of the most important and useful notions in theoretical mathematics.

Definition (3.1.1):

Let (I^X, d^*) be a fuzzy cone metric space, a fuzzy set $\tilde{A}$ in I^X is said to be compact fuzzy set if every open cover for $\tilde{A}$ has a finite subcover. i.e., if $\tilde{U}_i$ is an open cover for $\tilde{A}$, then $\tilde{A} \subseteq \bigcup_{i=1}^{n} \tilde{U}_i$.

Theorem (3.1.2):

Let (I^X, d^*) be a fuzzy cone metric space, and let $\tilde{A}$ be a compact fuzzy set in I^X, then $\tilde{A}$ is bounded.

Proof

To prove $\tilde{A}$ is bounded, let $\tilde{U}_i$ be an open cover for $\tilde{A}$ with center $\tilde{q}_{x_i}^{\lambda_i}$ and radius $\varepsilon_i \in \mathrm{int}(P)$. Let $\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{q}_{x_0}^{\lambda_0}$, and $\varepsilon_i \leq \varepsilon$, for all i. Since $\tilde{A}$ is compact, then $\tilde{U}_i$ has finite subcover which covering $\tilde{A}$, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in} I^X$ implies $\tilde{q}_x^{\lambda} \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in} X^*$, we have $\varepsilon - d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x_0}^{\lambda_0}) \in \mathrm{int}(P)$, for each $\tilde{q}_x^{\lambda} \tilde{\in} I^X$. Hence $\tilde{A}$ is bounded. ■

Remarks (3.1.3):

A class $\{\tilde{A}_i\}$ of fuzzy subsets of a fuzzy cone metric space (I^X, d^*) is said to have the finite intersection property if every finite subclass $\{\tilde{A}_{i_1}, \tilde{A}_{i_2}, \ldots, \tilde{A}_{i_n}\}$ has a nonempty intersection, i.e.,

$$\tilde{A}_{i_1} \cap \tilde{A}_{i_2} \cap \ldots \cap \tilde{A}_{i_n} \neq \tilde{\varnothing}$$

Theorem (3.1.4):

The fuzzy cone metric space (I^X, d^*) is compact if and only if every family of closed subsets of (I^X, d^*) satisfies the finite intersection property.

Proof:

Suppose that (I^X, d^*) is compact and $\{\tilde{F}_n\}$, n = 1, 2, …; is a family of closed fuzzy sets which is satisfy finite intersection property. Now suppose that $\bigcap_n \tilde{F}_n = \tilde{\varnothing}$, i.e., has a zero membership function. By using De-Morgan's law of fuzzy sets (see [37]), we obtain:

$$I^X = \tilde{\varnothing}^c \subseteq \left(\bigcap_n \tilde{F}_n\right)^c \subseteq \bigcup_n \tilde{F}_n^c \text{, and each } \tilde{F}_n^c \text{ is open fuzzy set}$$

Which implies that $\{\tilde{F}_n^c\}$ is an open cover of fuzzy compact space I^X.

Hence, there exists a finite subcover of the space I^X, which belongs to $\{\tilde{F}_n^c\}$, if $I^X \subseteq \bigcup_{i=1}^{N} \tilde{F}_i^c$, and therefore:

$$\tilde{\varnothing} = (I^X)^c \subseteq \left[\bigcup_{i=1}^{N} \tilde{F}_i^c\right]^c \subseteq \bigcap_{i=1}^{N} \tilde{F}_i \text{, } N \in \mathbb{N}$$

Thus, $\{\tilde{F}_n\}$ does not satisfy the property of finite intersection which is a contradiction.

Conversely, suppose that (I^X, d*) is not compact.

Let $\{\tilde{G}_n\}$, n = 1, 2, …; fuzzy open cover of the space I^X which has no finite fuzzy subcover, i.e., $X^* \subseteq \bigcup_n \tilde{G}_n$, which implies $X^* \not\subset \bigcup_j \tilde{G}_j$, for all j of n.

Then, $\bigcap_j \tilde{G}_j^c \neq \tilde{\emptyset}$, which implies $\bigcap_n \tilde{G}_n^c \neq \tilde{\emptyset}$.

Since $\{\tilde{G}_n^c\}$ satisfies the property of finite intersection, we have $I^X \not\subset \bigcup_n \tilde{G}_n$, for all n. Which is Contradiction to assumption. ■

3.2 Pre-Compact Fuzzy Cone Metric Spaces

Definition (3.2.1):

Let $\tilde{A}$ be a fuzzy subset of a fuzzy cone metric space (I^X, d*) and let $\varepsilon \in int(P)$. A finite fuzzy set $\tilde{W}$ of fuzzy points:

$$\tilde{W} = \{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\},\ x_1, x_2, \ldots, x_n \in X \text{ and } \lambda_1, \lambda_2, \ldots, \lambda_n \in (0, 1]$$

is called an ε-fuzzy net for $\tilde{A}$ if for every fuzzy point $\tilde{p}_x^{\lambda} \tilde{\in} \tilde{A}$, there exists $\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{W}$, for some i ∈ {1, 2, …, n}; such that $\varepsilon - d^*(\tilde{p}_x^{\lambda}, \tilde{q}_{x_i}^{\lambda_i}) \in int(P)$.

Definition (3.2.2):

A fuzzy set $\tilde{A}$ of a fuzzy cone metric space (I^X, d*) is said to be fuzzy pre-compact set (or fuzzy totally bounded) if $\tilde{A}$ possess an ε-fuzzy net, for every $\varepsilon \in int(P)$.

Now, we are in a position to give the relationship between fuzzy compact and fuzzy pre-compact sets.

Theorem (3.2.3):

If (I^X, d^*) is compact fuzzy cone metric space, then (I^X, d^*) is pre-compact fuzzy cone metric space.

Proof:

Let (I^X, d^*) be a compact fuzzy cone metric space (I^X, d^*). Assume to contrary (I^X, d^*) is not pre-compact.

Now let $\tilde{U}_i$ be an open cover with center $\tilde{q}_{x_i}^{\lambda_i}$ and radius $\varepsilon \in int(P)$. Since I^X is compact, then there exist a finite subcover of I^X, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in} I^X$ implies $\tilde{q}_x^{\lambda} \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, i.e., for each $\tilde{q}_x^{\lambda} \tilde{\in} I^X$ there exist $i_0 \in \{1, 2, \ldots, n\}$; such that $\varepsilon - d^*(\tilde{p}_x^{\lambda}, \tilde{q}_{x_{i0}}^{\lambda_{i0}}) \in int(P)$. Then the set $\{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$ form an ε-fuzzy net for I^X, which is contradiction. i.e., (I^X, d^*) is pre-compact fuzzy cone metric space. ■

3.3 Sequentially Compact Fuzzy Cone Metric Spaces

In the next definition, a new type of fuzzy compactness will be introduced, which is the fuzzy sequentially compact sets.

Definition (3.3.1):

A fuzzy subset $\tilde{A}$ of a fuzzy cone metric space (I^X, d^*) is said to be sequentially compact fuzzy set if every sequence of fuzzy points $\{\tilde{q}_{x_n}^{\lambda_n}\} \tilde{\in} \tilde{A}$

has a convergent subsequence in $\tilde{A}$, i.e., if $\{\tilde{q}_{x_n}^{\lambda_n}\}$ is a sequence of fuzzy points in $\tilde{A}$, then there exist a subsequence $\{\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}\}$ of $\{\tilde{q}_{x_n}^{\lambda_n}\}$ such that for every $\varepsilon \in$ int(P) there exist $N \in \mathbb{N}$, and $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}$ such that $\varepsilon - d^*(\tilde{q}_{x_{n_i}}^{\lambda_{n_i}}, \tilde{q}_x^{\lambda}) \in$ int(P), for each $n_i \geq N$.

The next theorem gives the relationship between fuzzy sequentially compact and fuzzy pre-compact sets.

Theorem (3.3.2):

Let (X*, d*) be a fuzzy cone metric space, then (I^X, d^*) is a sequentially compact fuzzy cone metric space if and only if (I^X, d^*) is a complete and pre-compact fuzzy cone metric space.

Proof:

If (I^X, d^*) is a sequentially compact fuzzy cone metric space. Assume to contrary that I^X is not fuzzy pre-compact. Then, there exists $\varepsilon \in$ int(P), such that I^X possess no finite ε-fuzzy net.

Take $\tilde{p}_x^{\lambda} \tilde{\in} I^X$, hence there exists $\tilde{q}_{x1_i}^{\lambda 1_i} \tilde{\in} I^X$, such that $d^*(\tilde{p}_x^{\lambda}, \tilde{q}_{x1_i}^{\lambda 1_i}) - \varepsilon \in$ int(P), otherwise $\{\tilde{q}_{x1_i}^{\lambda 1_i}\}$ is an ε-fuzzy net.

Also, there exists $\tilde{q}_{x2_i}^{\lambda 2_i} \tilde{\in} I^X$, such that: $d^*(\tilde{p}_x^{\lambda}, \tilde{q}_{x2_i}^{\lambda 2_i}) - \varepsilon \in$ int(P) and $d^*(\tilde{q}_{x1_i}^{\lambda 1_i}, \tilde{q}_{x2_i}^{\lambda 2_i}) - \varepsilon \in$ int(P), otherwise $\{\tilde{q}_{x1_i}^{\lambda 1_i}, \tilde{q}_{x2_i}^{\lambda 2_i}\}$ is an ε-fuzzy net.

and so on, we get a sequence of fuzzy points $\{\tilde{q}_{x1_i}^{\lambda 1_i}, \tilde{q}_{x2_i}^{\lambda 2_i}, \ldots\}$, such that $d^*(\tilde{q}_{xk_i}^{\lambda k_i}, \tilde{q}_{xm_i}^{\lambda m_i}) - \varepsilon \in$ int(P), for every $k \neq m$ … (3.1)

Now suppose the sequence $\{\tilde{q}_{xn_i}^{\lambda n_i}\}$ has convergent subsequence $\{\tilde{q}_{xn_j}^{\lambda n_j}\}$. Hence $\{\tilde{q}_{xn_j}^{\lambda n_j}\}$ is Cauchy sequence which is contradict eq (3.1), i.e., if (I^X, d^*) is fuzzy sequentially cone compact metric space, which implies that (I^X, d^*) is pre-compact fuzzy cone metric space.

Now to prove I^X is complete. Let $\{\tilde{q}_{xn}^{\lambda n}\}$ be a Cauchy sequence in X*, i.e. for all $\varepsilon_1 \in int(P)$, there exists $N_1 \in \mathbb{N}$, such that:

$\varepsilon_1 - d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) \in int(P)$, for all $n, m \geq N_1$, where $x, x_n \in X$, $\lambda, \lambda_n \in (0, 1]$, $\forall\, n \in \mathbb{N}$.

Since I^X is sequentially compact, then $\{\tilde{q}_{xn}^{\lambda n}\}$ has convergent subsequence $\{\tilde{q}_{xn_i}^{\lambda n_i}\}$, i.e., for all $\varepsilon_2 \in int(P)$, there exist $N_2 \in \mathbb{N}$, such that:

$\varepsilon_2 - d^*(\tilde{q}_{xn_i}^{\lambda n_i}, \tilde{q}_{x}^{\lambda}) \in int(P)$, for all $n_i \geq N_2$, where $x, x_n \in X$, $\lambda, \lambda_n \in (0, 1]$, $\forall\, n \in \mathbb{N}$.

Then $\tilde{q}_{x}^{\lambda}$ is also limit point for $\{\tilde{q}_{xn}^{\lambda n}\}$, since

$$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x}^{\lambda}) \leq d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{xn_i}^{\lambda n_i}) + d^*(\tilde{q}_{xn_i}^{\lambda n_i}, \tilde{q}_{x}^{\lambda})$$

$$\ll \varepsilon_1 + \varepsilon_2, \text{ for each } n, n_i \geq N = Max\{N_1, N_2\}.$$

i.e., $\{\tilde{q}_{xn}^{\lambda n}\}$ is convergent, then I^X is complete.

Conversely, if (I^X, d^*) is pre-compact and complete fuzzy cone metric space to prove that X* is sequentially compact fuzzy cone metric spaces.

Since (I^X, d^*) is pre-compact fuzzy cone metric space, then I^X possess an ε-fuzzy net, for every $\varepsilon \in int(P)$. Let $\tilde{W} = \{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_k}^{\lambda_k}\}$ be an ε-fuzzy net for I^X.

Let $\{\tilde{q}_{xn_i}^{\lambda n_i}\}$ be a subsequence of $\{\tilde{q}_{xn}^{\lambda n}\}$, then there exist $\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{W}$, such that

$\varepsilon - d^*(\tilde{q}_{xn}^{\lambda n}, \tilde{q}_{xj}^{\lambda_j}) \in int(P)$, for infinitely many n, j = 1,2, ..., k.

$$d^*(\tilde{q}_{xn_i}^{\lambda n_i}, \tilde{q}_{xm_i}^{\lambda m_i}) \leq d^*(\tilde{q}_{xn_i}^{\lambda n_i}, \tilde{q}_{xj}^{\lambda_j}) + d^*(\tilde{q}_{xj}^{\lambda_j}, \tilde{q}_{xm_i}^{\lambda m_i})$$

$$\ll \varepsilon_1 + \varepsilon_2, \text{ for infinitely many } n_i, m_i, j = 1,2, \ldots, k.$$

Hence $\{\tilde{q}_{xn_i}^{\lambda n_i}\}$ is form a Cauchy sequence, and since X* is complete, then $\{\tilde{q}_{xn_i}^{\lambda n_i}\}$ is convergent.

Therefore, the sequence $\{\tilde{q}_{xn}^{\lambda n}\}$ has a convergent subsequence. Hence (I^X, d^*) is sequentially compact fuzzy cone metric space. ■

Corollary (3.3.3):

Let $\tilde{A}$ be a compact fuzzy set of fuzzy cone metric space (I^X, d^*), then $\tilde{A}$ is sequentially compact fuzzy set.

Proof

Since $\tilde{A}$ is compact and by theorem (3.2.3) and theorem (3.3.2), then $\tilde{A}$ is sequentially compact.

Lemma (3.3.4):

If (I^X, d^*) is sequentially compact fuzzy cone metric space, and $\tilde{G}_\alpha$ for some $\alpha \in A$ is infinitely open cover for I^X. Then every ball of radius $\varepsilon \in int(P)$ is contained in one of the open sets $\tilde{G}_\alpha$.

Proof:

Assume for contrary that for any $n \in \mathbb{N}$ there is an open ball $\tilde{B}_n$ with center $\tilde{q}_x^\lambda$ and radius ε ∈ int(P) which is not contained in $\tilde{G}_\alpha, \forall \alpha \in A$. Since I^X is sequentially compact, then every sequence in I^X has a convergent subsequence.

Therefore, the subsequence $\{\tilde{q}_{xn_i}^{\lambda n_i}\}$ of $\{\tilde{q}_{xn}^{\lambda n}\}$ in $\tilde{B}_n$ is convergent to $\tilde{q}_x^\lambda \tilde{\in} I^X$.

Since $\tilde{G}_\alpha$ is an open cover for I^X there exist an index $\alpha_0 \in A$, and an open set $\tilde{G}_{\alpha_0}$, such that $\tilde{q}_x^\lambda \tilde{\in} \tilde{G}_{\alpha_0}$.

Since $\tilde{G}_{\alpha_0}$ is open and $\tilde{q}_x^\lambda \tilde{\in} \tilde{G}_{\alpha_0}$, then $\tilde{U}_\varepsilon(\tilde{q}_x^\lambda) \tilde{\in} \tilde{G}_{\alpha_0}$, and since $\tilde{q}_x^\lambda$ is a limit point for a subsequence of the sequence $\{\tilde{q}_{xn}^{\lambda n}\}$, then $\{\tilde{q}_{xn}^{\lambda n}\} \tilde{\in} \tilde{U}_\varepsilon(\tilde{q}_x^\lambda)$ for finitely many n. Which implies that $\{\tilde{q}_{xn}^{\lambda n}\} \tilde{\in} \tilde{G}_{\alpha_0}$, which is a contradiction and hence every ball is contained in one of the open sets $\tilde{G}_\alpha$, foe some α. ■

<u>Theorem (3.3.5):</u>

Let (I^X, d*) be a sequentially compact fuzzy cone metric space, then (I^X, d*) is a compact fuzzy cone metric space.

Proof

Let $\tilde{G}_\alpha$ be an infinitely open cover for I^X.

By using above lemma (3.3.4), every open ball is contained in one of the open sets $\tilde{G}_\alpha$.

Since I^X is sequentially compact then I^X is a pre compact by using theorem (3.2.3). Hence I^X has an ε-fuzzy net, for each ε ∈ int(P).

Let $\tilde{W} = \{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$ be an ε-fuzzy net for X*, then each $\tilde{q}_x^{\lambda} \tilde{\in} I^X$ belong to the union of ball $\tilde{U}_\varepsilon(\tilde{q}_{x_i}^{\lambda_i})$, i=1,2,…,n. Now each $\tilde{U}_\varepsilon(\tilde{q}_{x_i}^{\lambda_i})$ is contained in one of $\tilde{G}_\alpha$, say $\tilde{G}_{\alpha_i}$, i=1,2,…,n.

Therefore the collection of $\tilde{G}_{\alpha_i}$, i=1,2,…,n is a finite subcover for I^X. Then I^X is compact fuzzy. ■

Corollary (3.3.6):

Let (I^X, d*) be a pre-compact fuzzy cone metric space and complete, then (I^X, d*) is a compact fuzzy cone metric space.

Proof

If (I^X, d*) be a pre-compact fuzzy cone metric space and complete, then (I^X, d*) is a sequentially compact fuzzy cone metric space by using theorem (3.3.2) and theorem (3.3.5). which implies (I^X, d*) is a compact fuzzy cone metric space. ■

3.4 Countably Compact Fuzzy Cone Metric Spaces

In this section will study a new concept which is countably compact fuzzy cone metric spaces and its relationship with other types of compactness of fuzzy one metric spaces.

Definition (3.4.1):

A fuzzy cone metric space (I^X, d*) is said to be countably compact fuzzy set if every open countably cover has finite subcover.

The relationship between fuzzy compact cone metric spaces and countably compact fuzzy cone metric spaces is given in the next theorem:

Theorem (3.4.2):

Let (I^X, d*) be a compact fuzzy cone metric space, then (I^X, d*) is countably compact fuzzy cone metric space.

Proof:

Let $\tilde{G}_\alpha, \forall\alpha \in A$ be countably open cover for I^X, and since I^X, is compact fuzzy, then $\tilde{G}_\alpha, \forall\alpha \in A$ has finite subcover which is covering I^X. Hence (I^X, d*) is also countably compact fuzzy cone metric space. ■

Theorem (3.4.3):

Let (I^X, d*) be a countably compact fuzzy cone metric space, then (I^X, d*) is also pre-compact fuzzy cone metric space.

Proof:

Let $\tilde{U}_i$ be a countable open cover for I^X, and center $\tilde{q}_{x_i}^{\lambda_i}$ with radius $\varepsilon \in$ int(P). Since I^X is countably compact fuzzy cone metric space, then there exist a finite subcover of I^X.

Hence, for each $\tilde{q}_x^\lambda \,\tilde{\in}\, I^X$ implies $\tilde{q}_x^\lambda \,\tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, i.e., for each $\tilde{q}_x^\lambda \,\tilde{\in}\, I^X$ there exist $i_0 \in \{1, 2, \ldots, n\}$; such that $\varepsilon - d^*(\tilde{q}_x^\lambda, \tilde{q}_{x_{i_0}}^{\lambda_{i_0}}) \in$ int(P).

Therefore, the set $\{\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n}\}$ is form an ε-fuzzy net for I^X, i.e., (I^X, d*) is pre-compact fuzzy cone metric space. ■

Corollary (3.4.4):

If (I^X, d^*) is complete and pre-compact fuzzy countably compact fuzzy cone metric space, then (I^X, d^*) is also countably compact fuzzy cone metric space.

Proof

If (I^X, d^*) is complete and pre-compact fuzzy countably compact fuzzy cone metric space, then by theorem (3.3.6) and theorem (3.4.2), implies (I^X, d^*) is countably compact fuzzy cone metric space. ■

Corollary (3.4.5):

Let (I^X, d^*) be a sequentially compact fuzzy cone metric space, then (I^X, d^*) is a countably compact fuzzy cone metric space.

Proof

Since (I^X, d^*) is sequentially compact fuzzy cone metric space, then by theorem (3.3.5) and theorem (3.4.2), (I^X, d^*) is a countably compact fuzzy cone metric space. ■

3.5 Locally Compact Fuzzy Cone Metric Spaces

Now, we are in a position to introduce the definition of fuzzy locally compactness and its relationship with the other types of fuzzy compactness.

Definition (3.5.1):

A fuzzy set $\tilde{A}$ of I^X is said to be fuzzy locally cone compact if for all $\tilde{q}_x^\lambda \tilde{\in} \tilde{A}$, $x \in X$, $\lambda \in (0, 1]$ there exists a fuzzy neighborhood $\tilde{U}_\varepsilon(\tilde{q}_x^\lambda)$ of $\tilde{q}_x^\lambda$, such that $\tilde{U}_\varepsilon(\tilde{q}_x^\lambda)$ is fuzzy compact set.

Theorem (3.5.2):

Every fuzzy compact cone metric space (I^X, d*) is fuzzy locally compact cone metric space.

Proof:

Since (I^X, d*) is compact fuzzy cone metric space, every cover $\tilde{U}_i$ has finite subcover, i.e., $X^* \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$.

Now for each $\tilde{q}_x^\lambda$ $\tilde{\in}$ I^X implies that $\tilde{U}_\varepsilon(\tilde{q}_x^\lambda) \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i$, i.e., for each $\tilde{q}_x^\lambda$ $\tilde{\in}$ I^X has compact $\tilde{U}_\varepsilon(\tilde{q}_x^\lambda)$. ($I^X$, d*) is locally compact fuzzy cone metric space. ■

Corollary (3.5.3):

If (I^X, d*) is a pre-compact fuzzy cone metric space and complete, then (I^X, d*) is locally compact fuzzy cone metric space

Proof

If (I^X, d*) is pre-compact fuzzy cone metric space, then by using theorem (3.3.6) and theorem (3.5.2), we have (I^X, d*) is locally compact fuzzy cone metric space. ■

Corollary (3.5.4):

If (I^X, d*) is sequentially compact fuzzy cone metric space, then (I^X, d*) is locally compact fuzzy cone metric space.

Proof

If (I^X, d*) is sequentially compact fuzzy cone metric space, then by using theorem (3.3.5) and theorem (3.5.2), we have (I^X, d*) is locally compact fuzzy cone metric space ■

Finally, we end this chapter with diagram (3.1) which illustrates the relationship among the different types of fuzzy compactness:

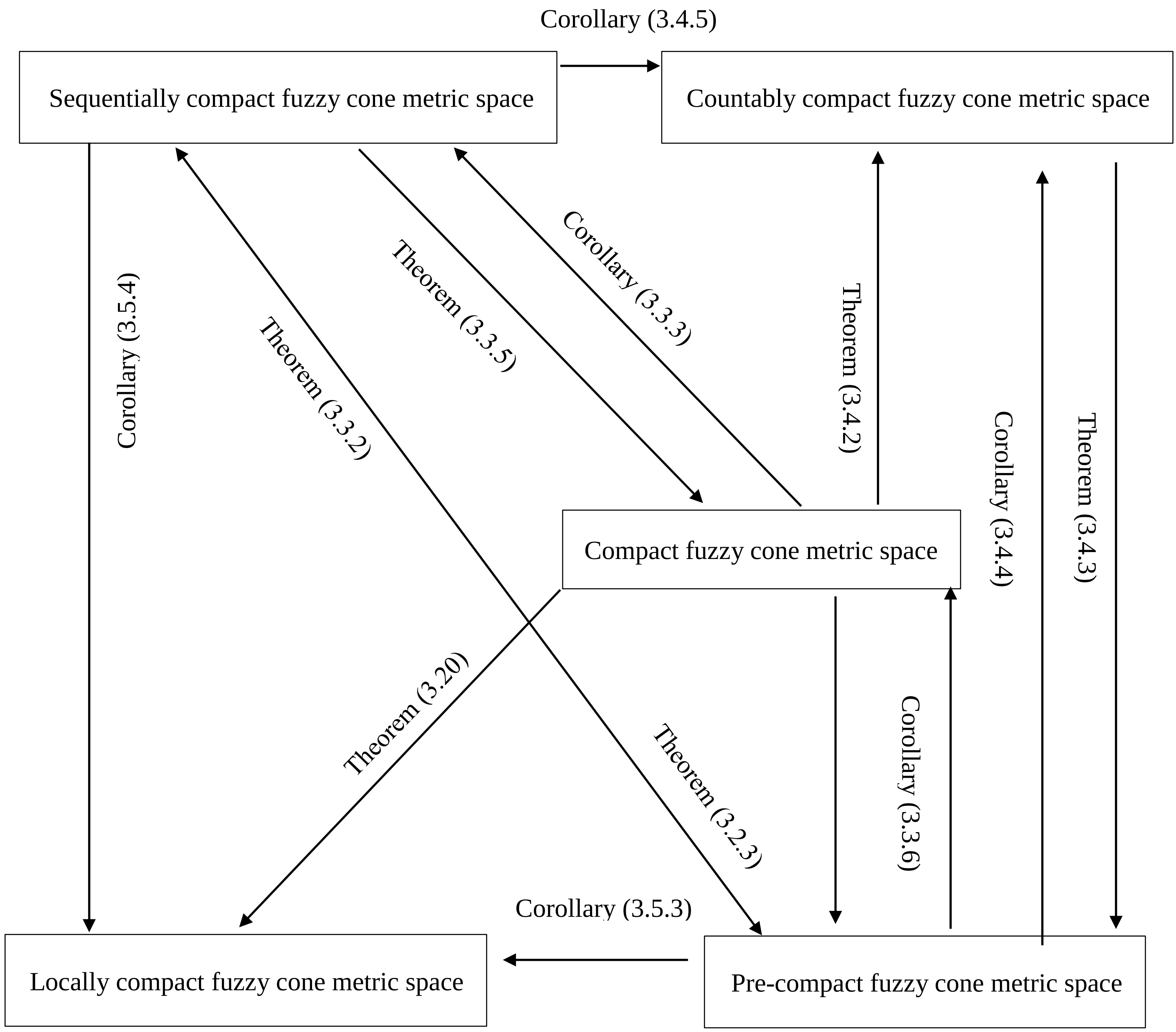

Diagram (3.1) The relationship among different types of fuzzy compactness.

Conclusion and Recommendations

CONCLUSIOS AND FUTURE WORK

From the present study of this thesis we can conclude the compactness of fuzzy cone metric spaces may be considered as a generalized of fuzzy metric spaces and cone metric spaces, in which some relationships with the usual metric spaces had been proved.

Also, the following problems may be recommended as an open problem for future work:

1. Study other types of compactness, such as para-compact, meta compact, pseudo compact, meso compact, ultra compact and then study the relation between them in fuzzy cone metric spaces.
2. Study the fixed point theorem in fuzzy cone metric spaces.
3. Study other types of fuzzy cone metric spaces depending on other approaches for constructing the fuzzy metric spaces, such as α-level sets or the membership functions.

Reference

1. Bhattacharya P., “Fuzzy Subgroups; Some Characterizations”, J. Math. Anal. Appl., Vol. 128, pp. 241-252, 1988.
2. Bijendra S., Chauhanb M. S., “Common fixed points of compatible maps in fuzzy metric spaces”, Fuzzy Sets and Systems Volume 115, Issue 3, 1 November 2000, Pages 471–475.
3. Butnariu D., “Fixed Point for Fuzzy Mappings”, Fuzzy Sets and Systems, Vol.7, No.2, pp.193-205, 1982.
4. Chang C. L., “Fuzzy Topological Spaces”, J. Math. Anal. Appli., Vol 24, pp. 182-190, 1968.
5. Dubois D. and Prade, H., “Fuzzy Sets and Systems: Theory and Applications”, Academic Press, Inc., 1980.
6. Erceg M. A., “Metric Spaces in Fuzzy Set Theory” J. Math. Anal. Appl., Vol. 69, PP 205-230, 1979.
7. Fadhel F. S., “About Fuzzy Fixed Point Theorem”, Ph.D. Thesis, Department of Mathematics and computer Applications, College of Science, Al-Nahrain University, 1998.
8. Gregori A. and Veeramani, P., “On Some Result in Fuzzy Metric Space”, Fuzzy Sets and System, 64, pp.395-399, 1994.
9. Gregori A. and Veeramani, P., “On Some Results of Analysis for Fuzzy Metric Spaces”, Fuzzy Sets and System, 90, pp.365-368, 1997.
10. Gregori A. and Tina A. Folger, “Fuzzy Sets Uncertainty, and Information”, Printed in the United States of America, 1988.
11. Gregori A. and Veeramani, P., “Some properties of fuzzy metric spaces”, Fuzzy Sets and Systems Volume 115, Issue 3, 1 November 2000, Pages 485–489
12. Gerla G. and Scarpati, L., "Extension Principles for Fuzzy Set Theory", Journal of Information Sciences, 106, 49-69, 1998.
13. Hazim M., “On Compactness of Fuzzy Metric Spaces”, Ph.D. Thesis College of Education, Al-Mustansiriyah University, 2010.

14. Huang L., Zhang X., “Cone metric spaces and fixed point theorems of contractive mappings”, J. Math. Anal. Appl. 332 (2007)1468–1476.
15. Kandel A., “Fuzzy Mathematical Techniques with Applications”, Addison Wesley Publishing Company, Inc., 1986.
16. Katsaras A. K. and D. B. Liu, “Fuzzy Vector Space and Fuzzy Topological Vector Space”, J. of Math. Anal. And Appli, 58, pp. 135-146, 1977.
17. Katsaras A. K., “Fuzzy Topological Vector Space (II)”, J. Fuzzy Sets and Systems, 12, pp.143-154, 1984.
18. Klir G. J. and Boyuan, “Fuzzy Sets and Logic; Theory and Applications”, Prentice Hall of India, 2000.
19. Klir G. J. and Yuan, B., “Fuzzy Set Theory: Foundations and Applications”, Prentice Hall PTR, 1997.
20. Kloden P. E., "Fuzzy Dynamical Systems", Fuzzy Sets and Systems, Vol.7, pp.275-296, 1982.
21. Ma Ming, “A Comparison Between Two Definitions of fuzzy Normed Space”, J. of Harbin Inst, of Technology, Supplementary Issue of Mathematics, pp. 43- 47, 1985.
22. Nguyen, H. T., “A Note on the Extension Principle for Fuzzy Sets”, UCB/ERLMEMO M-611, University of California, Berkeley, 1976.
23. (a) Ming P. P. and Ming, L. Y., "Fuzzy Topological 1. Neighborhood Structure of a Fuzzy Point and Moore-Smith Convergence", J. Math. Anal. Appl., Vol.76, 571-599, 1980.
24. (b) Ming, P. P. and Ming, L. Y., "Fuzzy Topological 2. Product and Quotient Spaces", J. Math. Anal. Appl., Vol.77, 20-37, 1980.
25. Pearson D. W., “A Property of Linear Fuzzy Differential Equations”, Appl. Math. Lett., Vol.10, No.3, pp.99-103, 1997.
26. Rezapour H., Hamlbarani R., Some notes on the paer “Cone metric spaces and fixed point theorems of contractive mappings”, J. Math. Anl. Appl., 345 (2008)19–724.

27. Rosenfeld A., “Fuzzy Groups”, J. Math. Anal. Appl, Vol. 35, pp. 512-517, 1971.
28. Scheizerb and Skara, “Statistical Metric Spaces”, Pacific J. Math., Vol.10, pp.314-334, 1960.
29. Sushil S., “Common fixed point theorems in fuzzy metric spaces”, Fuzzy Sets and Systems Volume 127, Issue 3, 1 May 2002, Pages 345–352.
30. Bag T., “Fuzzy cone metric spaces and fixed point theorems of contractive mappings”. Annals of Fuzzy Mathematics and Informatics, Vol. 6(3) (2013) 657–668.
31. Tamara S. A. “Some generalization of Banach ‘s contraction principle in complete cone metric space”, Al- Mustansiriyah J. Sci. Vol. 24, No 5, 2013.
32. Valentin G. and Salvador, R., “Some Properties of Fuzzy Metric Spaces”, Fuzzy Sets and Systems, Vol.115, pp.485-489, 2002.
33. Wu C. M. and Fang G., “Fuzzy Generalization of Kolnogroffs Theorem”, J. of Harbin Inst. of Technology, No. 1, pp.1-7, 1984.
34. Wu C. M., “Continuity and Boundedness for the Mappings Between Fuzzy Normed Space”, J. of Fuzzy Math., Vol.1, No.1, 1993.
35. Yan J., Ryom M. and Power J., “Using Fuzzy Logic Towards Intelligent Systems”, Prentic Hall, 1994.
36. Zadeh L. A., “Fuzzy Sets”, Information and Control, Vol.8, 338- 353, 1965.
37. Zike D., “Fuzzy Pseudo-Metric Space”, J. Math. and Appl., Vol.80, pp.14-95, 1982.
38. Zimmerman H. J., "Fuzzy Set Theory and its Applications", Kluwer-Nijhcff Publishing, USA, 1985.
39. Zun-Q. X., Fang-F. G., "Fuzzy metric spaces", Journal of Applied Mathematics and ComputingMarch 2004, Volume 16, Issue 1, pp 371-381

الخلاصة

أن الفضاءات المترية الاعتيادية الى الفضاءات المترية الضبابية(Fuzzy metric spaces) من ثم الى الفضـــاءات مترية المخروطية (cone) يعتبر من المواضـــيع الواعدة في الرياضـــيات الصرفة.

لذلك لهذا الرسالة هدفيين رئيسيين. الهدف الاول هو دراسة الفضاءات المترية المخروطية و من ثم بناء الفضاءات المترية الضبابية المخروطية باستخدام اسلوب جديد بلاعتماد على النقاط الضبابية.

الهدف الثاني هو الدراسة التراص (compact) الفضاءات المتريية الضبابية المخروطية و من ثم أعطاء العلاقة بين المفاهيم المختلفة للتراص, على سـبيل المثال تراص المجاميع الضـبابية المترية الاعتيادي (compact fuzzy cone metric spaces) , تراص المجاميع الضبابية المترية الاولية (pre-compact fuzzy cone metric spaces) , تراص المجاميع الضـبابية المترية التتابيعة (sequentially compact fuzzy cone metric spaces) , تراص المجاميع الضـــبابية المترية المعدودة(countably compact fuzzy cone metric spaces) تراص المجاميع الضبابية المترية المحلية (locally compact fuzzy cone metric spaces).

جمهورية العراق
وزارة التعليم العالي و البحث العلمي
جـــامعة النهـــرين
كليـــة العلوم
قسم الرياضيات و تطبيقات الحاسوب

حول تراص الفضاءات المترية الضبابية المخروطية

رسالة ماجستير

مقدمة الى مجلس كلية العلوم / جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير علوم / الرياضيات

من قبل

عامر عبد الكريم عبد الله

بكلوريوس/ علوم الرياضيات و تطبيقات الحاسوب (2014)

بأشراف

الأستاذ المساعد د. فاضل صبحـــي فاضـــل

ذو الحجة 1437		2016 أيلول