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Variational Formulations of Some Variable Delay Differential Systems

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بسم الله الرحمن الرحيم

وَقُل رَّبِّ أدخِلني مُدخَلَ صِدقٍ

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الآية (٨٠)

Dedication

إلى شهداءِ المعرفة وكل من كان سائراً فى طريق العلم وشاءَ القدر وحالت الظروف بينهُ وبين انهائه للدرب ... أُهدي جُهدي اجلالاً واحتراماً.

To the martyrs of knowledge and all those who walked the path of science, but fate and circumstances cut short their endeavors ...
I dedicate my effort with much honor and respect.

Supervisor Certification

I certify that this thesis was prepared under my supervision at the department of Mathematics, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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We certify that we have read this thesis entitled "*Variational Formulations of Some Variable Delay Differential Systems*" and as examining committee examined the student *(Sara Alaudeen Abdul Kadir)* in its contents and in what it connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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Abstract

The main theme of this work is to introduce the general form and fundamental concepts in ordinary and partial delay-differential equations with variable delays and then to find the variational formulation of delay-differential equations with variable delays in both cases, ordinary and partial and to provide the rules of minimizing the obtained functional in the subject of calculus of variation. Finally, to minimize the variational formulation using the direct-Ritz method and finding the approximate solution of delay-differential equations with variable delays.

Contents

Introduction

Delays (also called hereditary, memories, retarded arguments, past actions, dead times, or time lags), [Niculescu, 2001] are inherent in many physical and engineering systems. In particular, pure delays are often used to ideally represent the effect of transmission, transportation and initial phenomena, [Asl, 2003], such that incorporating time delays into mathematical models can be a challenged aspect. The type of delay incorporated into the system is also very important; therefore, various authors have made extensive work on systems of single delays, [Mori, 1981], [Mori, 1989], [Wang, 1992], [Su, 1994] and [Su, 1995]. Another type of delay is commensurate delay. These are delays where there exists a delay value, τ, such that all delays τ_i, (i = 1, 2, …, m) are rational multiples of τ. It is noted that there are some similarities between the commensurate and the single delay case, [Niculescu, 1998]. Multiple delays involve more computations, but the result provides a great insight into complicated systems. A mathematical model may incorporate constant time delays or delays that are vary with time. The delay may be discrete or continuous, depending upon the dynamics of the problem to be modeled, [Whitaker, 2000], and it can also be distinguished among monotonic, autonomous and vanishing lags, [Baker, 1994].

Time lags might for instance occur if some non negligible transportation time involved in the system or if the system needs a certain amount of time to sense information and react on it. The characteristic feature of a system with time-lags is that the dynamics of a certain time does not only depend on the instantaneous state of the system, but also on past values. The dependence on the past can take various shapes; the simplest type is that of constant retardation, also called for first instance, the reaction time of a system, [Gluesing-Luerssen, 2002]. Modeling such a system leads to functional-differential equations (also called differential equations with deviating arguments), [Driver, 1977], where this system can be classified as a system of linear equation or a system of nonlinear functional-differential equations, depending on the differential operator that appeared, [Whitaker, 2000].

A functional equation is an equation invites unknown function for different argument values. The difference between the argument values of an unknown function and the argument t in the functional equation is called argument deviations. If all arguments deviations are constants, the functional equation is called a difference equation, [Kolmanovskii, 1992].

The functional-differential equation is a differential equation which is also functional. Where delay-differential equations (DDE's, for short), sometimes called differential equations with retarded arguments, [Driver, 1977] and [El'sgol'c, 1964], (delayed arguments or time lags, [Shampine, 2000]); are special class of functional-differential equations in which the unknown function and its derivatives occur with their respective values at previous time. A completely different form of past dependence arises if the process under consideration depends on the full history of the system over a certain time interval. In this case, the mathematical formulation leads to delay-differential equations, [Hale, 1993].

Historically, DDE's occurred as long as in the work of L. Euler, [El'sgol'c, 1964], but these equations appeared in literatures in the second half of the eighteenth century by Kandorse in 1771, as models for simulating more real life phenomenon, [El’sgol’ts, 1973]. In the eighteenth century, mathematicians encountered DDE's because they were trying to attend their knowledge of mechanics of discrete particles to the mechanics of the continuum particles, which later came to be studied in terms of DDE's, [Pinney, 1959]. In the late thirties and early forties of the eighteenth century, Minorsky, in his study of ship stabilization and automatic steering, had many applications in the theory of self-oscillating systems, the study of problems connected with rocket motion, the problems of long range planning in economics, a series of biological problems, and in many other areas of science and technology, in which the study was expanded, [El’sgol’ts, 1973]. But the systematic study of these equations was first undertaken in the twentieth century (especially in the last forty years of this century by Myshkis A. in the Soviet Union, Wright E. and Bellman R. in other countries), to meet the demands of applied science, in particular of control theory, [El'sgol'c, 1964].

The topic of DDE's was in a rapid state of development. It was the Russian mathematician, Krasovskii who found an accommodation for differential equations with deviating arguments as operators in function spaces. It is worth noting that the theory of differential equations with deviating arguments is not just a simple extension of the theory of ordinary differential equations, but it has a more complicated theory, [El’sgol’ts, 1973].

In recent decades, DDE’s have become a powerful tool for the modelization of spatially distributed systems. In these systems, the geometry is often such that one can replace a propagated effect by a time delayed version of this effect and the study of DDE has been devoted to describe many processes with delayed effects or time lags, [Guillouzie, 1999]. Some modelers or researchers ignore the lag effect which think are small, Whilst Kuang comments under the heading “small delays can have large effects”, [Baker, 1994].

Delay differential equations are integrable in closed form only under very specialized circumstances and therefore qualitative and approximate methods are of the almost importance in studying them, [El'sgol'c, 1964].

Many authors and researchers studied DDE's such as; [Wright, 1946] studied the analysis of the existence of the solution and its properties for the nonlinear DDE's, [Wright, 1948] studied the solution properties for the linear DDE's with asymptotically constant coefficients, [Smith, 1957] studied the uniqueness of the solution and its properties for the linear DDE's with varying coefficients, [Pinney, 1959] described the basic theory concerning the stability of systems of DDE's, [Halany, 1966] presents the methods of solution of DDE's including the integral transformation method and power series method, etc., [Cook, 1967] pointed out that the study of DDE contains equations in which the lag of the argument is a function of the dependent variable, [Myshkis, 1972] and [Hale, 1977] referred to the classical monographs of the theory of ordinary and partial DDE's, [Kolmanovskii, 1986] and [Stépán, 1989] summarized the most important theorems of DDE's, [Mastinsek, 1994] discussed the semigroups of operators associated with DDE's, [Falbo, 1995] studied the analytic solution of linear DDE's, [Mao, 1997] discussed the exponential stability of DDE's, [Guglielmi, 1999] used Runge-Kutta methods to obtain a $\tau(0)$-stable numerical algorithm to solve homogeneous linear DDE's, [Guillouzic, 1999] considered the effect of external noise on DDE's involving one variable, thus leading to univarianet stochastic DDE's, [Baker, 2000] considered the main issues that DDE's be addressed when constructing robust numerical codes of their solutions, [Al-Saady, 2000] used the cubic spline interpolation functions to solve the DDE's, [Marie, 2001] introduced the variational formulation of DDE’s with constant delays, and solved such type of problems using the direct Ritz method, [Gluesing-Luerssen, 2002] described the linear time-invarients DDE's with commensurated point delays which is used on control-theoretic context, [Al-Daynee, 2002] evaluated the variational formulation of delay BVP’s using two approaches, namely the variational problems with constraint and variational problem using Rayleigh quotient formula, [Bica, 2003] obtained the existence and uniqueness of the positive periodic solution of neutral delay integro-differential equations, [Verheyden, 2003] presented the collocation method with an iterative linear system solver to compute the periodic solutions of a system of autonomous DDE's. [Insperger, 2004] presented an updated version of semi-discretization method for periodic systems with a single discrete time delays, [Salih, 2005] studied and modified some numerical and approximate methods for solving the n^{th} order linear DDE's with constant coefficients, [Al-Kubeisy, 2004] solved the DDE's numerically by using the linear multistep methods and improved the accuracy of the results by using the variable step size method, [Buite 2004] generalized the ordinary DDE's to partial DDE's and formulated the variational formulation of the special types of the partial DDE's and their solutions using the direct Ritz method, [Forde, 2005] used the ordinary and partial DDE's to model the biological systems, [Al-Defae'e, 2005] modified the numerical methods of solution to solve ordinary DDE's with variable delay using the direct Ritz method, [Bica, 2006] obtained the smooth dependence by lag of the positive periodic solution of a neutral delay integro-differential equations, [Al-Esawi, 2006] derived an estimate the magnitude of the solutions for special types of ordinary and partial DDE's in order to find the solution by any suitable method, such as the Laplace transformation method and [Lue, 2007] investigated the exponential stability of p-th mean of solutions of stochastic DDE's.

This thesis consists of three chapters. In chapter one, the basic concepts of delay differential equations are discussed in which the study includes its classification and some analytical methods of solution.

In chapter two, the variational formulation of ordinary delay-differential equations were studied in details and for different types of functionals and the necessary and sufficient conditions for an extremum are found for different cases of functionals. Also, the approximate solution of the variational problems is discussed and illustrated by examples using the direct-Ritz method.

In chapter three, the variational formulation of partial delay differential equations is also discussed as a generalization to those formulations given in chapter two, are the necessary and sufficient conditions for an extremum were also given for different cases of functionals and illustrate by examples. In addition, this chapter presents an application of the subject in real life problems that is of solving the simple food web problem.

Finally, the results are given either in tabulated form or illustrated in figures in order to give a good comparison between the approximate and exact solutions or by using the residue error depending on the problem under consideration.

1 Basic Concepts of Delay-Differential Equations

Delay-differential equations are of sufficient importance in many applications, such as mixing of liquids, population growth and automatic control systems, [Driver, 1977], that constitute the basic mathematical models with time delays for real life phenomena, for instance in mechanics, physics, engineering, economics, biology and technology, [Asl, 2003].

Therefore, in this chapter, we give some basic concepts of delay-differential equations, where this chapter consists of two sections.

In section one, the main aspect of differential-difference equations are introduced, as well as, its classification and basic properties, followed by a brief review of some analytical methods that may be used to solve differential-difference equations.

In section two, the more general case of delay-differential equations with variable delays are introduced for the two types, ordinary and partial delay-differential equations.

1.1 Differential-Difference Equations

Differential-difference equations, where differential equations without delays are a special case of it; however, in some simple cases differential-difference equations may be related to an infinite system of ordinary differential equations, [Baker, 1994], are DDE's in which their time lags are constant (sometimes called scalars, [Drager, 1997] or point-delays, [Kolmanovskii, 1992]).

The general form of the n-th order differential-difference equation with multiple delays is given by:

$$F(t; x(t), x(t-\tau_{01}), x(t-\tau_{02}), \ldots, x(t-\tau_{0m}), x'(t), x'(t-\tau_{11}), x'(t-\tau_{12}), \ldots, x'(t-\tau_{1m}), \ldots, x^{(n)}(t), x^{(n)}(t-\tau_{n1}), x^{(n)}(t-\tau_{n2}), \ldots, x^{(n)}(t-\tau_{nm})) = g(t) \quad \ldots\ldots(1.1)$$

where F is a given function and τ_{ij} (for $i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, m$) are constants called delays, where i refers to the order of the derivative with respect to the dependent variable for each, $j = 1, 2, \ldots, m$.

The first order linear differential-difference equation may be classified into three types. The first type, which is the simplest type of differential-difference equations is that; in which the delay terms is through the state variable and not through the derivative of the state variable and is called retarded differential-difference equations (RDDE's, for short).

These types of equations occurred in a number of applications such as, in the physical applications, for example:

$$x'(t) = F(t; x(t), x(t-\tau)),$$

where $x(t) \in \mathbb{R}^n$, $F : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and $\tau > 0$ is a single constant fixed time delay, and in control problems, for example:

$$x'(t) = K(x(t) - x(t-\tau)),$$

where K is the feedback gain function and τ is the time delay, and also in the study of distribution of primes, for example:

$$x'(t) = -\alpha x(t-1)[1 + x(t)].$$

The second type of differential-difference equations is that; in which the delay terms is through the derivative of the state variable and not through the state variable itself and is called neutral differential-difference equations (NDDE's, for short), [Hale, 1993], for example:

$$x''(t) = -x'(t) - x'(t-1) - 3\sin x(t) + \cos(t).$$

Also variants of NDDE's have also been used as a model in the history of growth of single species, for example:

$$x'(t) = -\alpha\left\{\int_{-1}^{0} x'(s-\tau)\, ds\right\}(1 + x(t));$$

and in the describing the spread of disease taking into account age dependence, for example:

$$x'(t) = -\int_{t-\tau}^{t} a(t-u)g(x'(t-u))\,du.$$

The third type is a combination between the two obvious types and is called the advanced differential-difference equations (ADDE’s, for short), [Hale, 1993]. These types of equations occur in the theory of epidemics and models in the biomedical science, for example:

$$x''(t) = f(x(t-\tau))\, x'(t-\tau) - \alpha x'(t) - x(t).$$

The main difference between differential equations and differential-difference equations is the kind of initial conditions that should be used in the differential-difference equation, which are different from the differential equations; so that one should be specified in differential-difference equations an initial function on some interval of length τ, say $[t_0 - \tau, t_0]$ to find the solution of DDE, for all $t \geq t_0$, [Ladde, 1987].

For intention, to consider the different types of differential-difference equations as a model of real life problems, the existence of solutions is required.

Therefore, depending on the initial history, the value of x(t) on some interval means the following: if for some $t_0 \in \mathbb{R}$ and $\beta > t_0$, the function $x : [t_0 - \tau, \beta] \longrightarrow \mathbb{R}^n$, where $\tau = \underset{i,j}{\operatorname{Max}} \{\tau_{ij}\}$, for (i = 0, 1, …, n and j = 1, 2, …, m) satisfies eq.(1.1) for $t \in [t_0, \beta]$, then we may say that x is a solution of eq.(1.1) on $[t_0 -\tau, \beta]$. If $\Phi : [t_0 - \tau, t_0] \longrightarrow \mathbb{R}^n$ and x are a solution that coincides with Φ on $[t_0 -\tau, t_0]$, we say that x is a solution through (t_0, Φ).

Let $C[t_0 - \tau, t_0]$ be the space of all continuous functions from $[t_0 - \tau, t_0]$ onto $\mathbb{R}^n$, then if $\Phi \in C[t_0 - \tau, t_0]$ and $x : [t_0-\tau, \beta] \longrightarrow \mathbb{R}^n$, it will be said that x is a solution of eq.(1.1) with the initial function Φ, or simply a solution through Φ, if x is a solution through (t_0, Φ), [Taylor, 2004].

Now, the existence and uniqueness theory for differential-difference equations which may be derived from the more general theory of DDE's and also benefits from an analogy with similar results in the theory of ODE, are presented below:

Theorem (1.1) (Local Existence), [Hale, 1993]:

Suppose $\Omega \subset \mathbb{R} \times C[t_0 - \tau, t_0]$ is open, and $F : \Omega \longrightarrow \mathbb{R}^n$ is continuous. If $(t_0, \Phi) \in \Omega$, then there is a solution of eq.(1.1) through (t_0, Φ).

Corollary (1.2), [Taylor, 2004]:

If $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous, then for any $\Phi \in C[t_0 - \tau, t_0]$ there is a solution of eq.(1.1) through Φ.

Theorem (1.3) (Uniqueness), [Hale, 1993]:

Suppose $\Omega \subset \mathbb{R} \times C[t_0 - \tau, t_0]$ is open, and $F : \Omega \longrightarrow \mathbb{R}^n$ is continuous and $F(t, \Phi)$ is Lipschitzian in Φ on every compact set in Ω. If $(t_0, \Phi) \in \Omega$, then there is a unique solution for eq.(1.1) through (t_0, Φ).

Corollary (1.4), [Taylor, 2004]:

If $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is Lipschitizian, then for any $\Phi \in C[t_0 - \tau, t_0]$ there is a unique solution of eq.(1.1) through Φ.

Many theoretical and numerical methods are presented in literatures for solving differential-difference equations, and among the most common used methods are; the method of steps (or the method of successive integration) and the Laplace transformation method.

Method of steps; is the best well known theoretical method for solving differential-difference equations occurred and because of the initial condition, which is given for a time step interval with length equals to τ; one must find the solution for $t \geq t_0$, which is divided into steps with length τ, and the solution for each next time step for $t \geq t_0$, which are respectively $x_1(t)$, $x_2(t)$, … on $[t_0, t_0 + \tau]$, $[t_0 + \tau, t_0 + 2\tau]$, … also furnishes a method of finding explicit solutions, [El'sgol'c, 1964]. As an illustration to this approach, consider the DDE:

$$x'(t) = F(t; x(t), x(t - \tau_{01}), x(t - \tau_{02}), \ldots, x(t - \tau_{0m}), x'(t), x'(t - \tau_{11}), x'(t - \tau_{12}), \ldots, x'(t - \tau_{1m})), \; t \geq t_0 \qquad (1.2)$$

with initial condition:

$$x_0(t) = \varphi_0(t), \; t \in [t_0 - \tau, t_0];$$

when restricting the eq.(1.2) to the interval $[t_0, t_0 + \tau]$, then eq.(1.2) becomes;

$$x'(t) = F(t; x(t), \varphi_0(t - \tau_{01}), \varphi_0(t - \tau_{02}), \ldots, \varphi_0(t - \tau_{0m}), x'(t), \varphi_0'(t - \tau_{11}), \varphi_0'(t - \tau_{12}), \ldots, \varphi_0'(t - \tau_{1m}))$$

$$= g(t, x(t), x'(t)), \; t \in [t_0, t_0 + \tau].$$

Under suitable hypotheses of g, the existence of a unique solution of this equation (hence a solution of eq.(1.2)) on $[t_0, t_0 + \tau]$ may be established. Denoting this solution by x_1 and restricting eq.(1.2) to the interval $[t_0 + \tau, t_0 + 2\tau]$, one obtains the ordinary-differential equation with initial condition $x_1(t) = \varphi_1(t)$, $t \in [t_0, t_0 + \tau]$:

$$x'(t) = F(t; \varphi_1(t - \tau_{01}), \varphi_1(t - \tau_{02}), \ldots, \varphi_1(t - \varphi_{0m}), x'(t), \varphi_1(t - \tau_{11}), \varphi_1(t - \tau_{12}), \ldots, \varphi_1(t - \tau_{1m}))$$

$$= g(t; x(t), x'(t)),\ t \in [t_0 + \tau, t_0 + 2\tau].$$

For which one can again establish the existence of a unique solution x_2.

Proceeding inductively, considering eq.(1.2) as an ordinary-differential equation on a sequence of intervals $[t_0 + n\tau, t_0 + (n + 1)\tau]$, it is sometimes possible to show the existence of a unique solution of the DDE's on $[0, \infty)$, [El'sgol'ts, 1973].

Although, Laplace transformation method is extremely useful in obtaining the solution of linear DDE's with constant coefficients. As it is known, Laplace transformation method can be used to solve ODE's and we can also use the same approach to solve DDE's. For this approach, suppose that f is a function of t defined on $[0, \infty)$ then the Laplace transform of f(t) denoted by F(s) is defined by:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,\ s > 0.$$

The Laplace transform exists if the integral depends on f(t) and the number s converges for some values of $s \in [0, \infty)$, [Brauer, 1973].

Two approaches may be used in Laplace transformation method for solving DDE's, the first approach is to solve the DDE's by using Laplace transformation method directly without using the method of steps, [Bellman, 1963], while the second approach depends on the method of steps firstly to transform the DDE, to an equivalent ODE and then apply the Laplace transformation method to solve the resulting equation, [Brauer, 1973].

Moreover, the linear multistep method for solving ordinary DDE's, [Al-Kubeisy, 2004] and the expansion methods, [Salih, 2004] have also been used to solve the differential-difference equations.

1.2 Delay-Differential Equations with Variable Delays

Almost all of the relevant papers of DDE's are devoted to the investigation of delay-differential equations with constant delays; in few papers only very special types of DDE's with variable delays (DDEv's, for short), in other literatures called DDE's with time-lag functions (delay functions, [Baker, 1994], continuous lags, [Whitaker, 2000] or time-varying delays, [Michiels, 2005]) when these delays depend on values of the unknown function and its derivatives, that is functions of the solution itself, [Baker, 1994], are studied. The lag functions that arise most frequently in the modeling problems are constant, [El'sgol'c, 1964]. Whereas the effects of these constants functions on the dynamical systems have been largely treated in literatures ([Niculescu, 2001] and [Gu, 2003]). Although DDEv's is one of the most important type of delay-differential equations, since the argument deviations of this type of differential equations arising from particular and concrete in geometrical or mechanical problems, [El'sgo'lc, 1964] and also need a deeper analysis since its presence may induce complex behaviors, for example; the so-called quenching phenomena as suggested and discussed by [Lauisell, 1999].

So, in this section, a simple classification of DDEv's to ordinary and partial delay-differential equations is introduced with some properties for each type.

1.2.1 Ordinary Delay-Differential Equations with Variable Delays:

It is well known that an ordinary delay-differential equation with constant delays (ODDEc's, for short), is a delay-differential equation in which the unknown function occur with fixed arguments. But the generalization of ODDEc is often called an ordinary delay-differential equation with variable delays, is an equation in which the unknown functions occur with various different arguments, [Wiener, 1993]. Over many decades, abstract ODDEv's have been the subject and a proving ground for a wealth of mathematical theories, since ODDEv's related to some population dynamics problems that can be fitted into it by some transformation, [Techvenche, 2006]. The general form for the n^{th} order ODDE with multiple variable delays, takes the form:

$$F(t; x(t), x(t-\varphi_{01}), x(t-\varphi_{02}), \ldots, x(t-\varphi_{0m}), x'(t), x'(t-\varphi_{11}), x'(t-\varphi_{12}), \ldots, x'(t-\varphi_{1m}), \ldots, x^{(n)}(t), x^{(n)}(t-\varphi_{n1}), x^{(n)}(t-\varphi_{n2}), \ldots, x^{(n)}(t-\varphi_{nm})) = g(t) \quad \text{(1.3)}$$

where F is a given function and $\varphi_{ij} = \varphi_{ij}(t; x(t), x'(t), \ldots, x^{(n)}(t))$, for all ($i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, m$) are the delay functions, which are also called state-dependent delays if they depend on the values of the solution x(t) and its derivatives, or state-independent delays if they are constant delays or depend on the argument t only, [Hern'andez, 2008], for example:

$$x'(t) = x(t - \varphi(t)),$$

where x is an unknown function of the real variable t and $\varphi(t)$ is a given continuous function, [Techvenche, 2006].

Also, the general form of the n^{th} order ordinary functional-differential equations is given by:

$$F(t; x(t), x(k_{01}t), x(k_{02}t), \ldots, x(k_{0m}t), x'(t), x'(k_{11}t), x'(k_{12}t), \ldots, x'(k_{1m}t), \ldots, x^{(n)}(t), x^{(n)}(k_{n1}t), x^{(n)}(k_{n2}t), \ldots, x^{(n)}(k_{nm}t)) = g(t) \quad \text{.....(1.4)}$$

where $k_{ij} = k_{ij}(t; x(t), x'(t), \ldots, x^{(n)}(t))$, for each (i = 0, 1, …, n and j = 1, 2, …, m), [El'sgol'ts, 1973].

For the eqs.(1.3) and (1.4), if g(t) = 0, then they are called the homogeneous functional-differential equations, otherwise they are nonhomogeneous,

The simplest possible ODDEv's, namely linear ODDEv's, where the general form of linear ODDE with variable delays and variable coefficients can be stated as:

$$a_{00}(t)x(t) + a_{01}(t)x(t-\varphi_{01}) + a_{02}(t)x(t-\varphi_{02}) + \ldots + a_{0m}(t)x(t-\varphi_{0m}) + a_{10}(t)x'(t) + a_{11}(t)x'(t-\varphi_{11}) + a_{12}(t)x'(t-\varphi_{12}) + \ldots + a_{1m}(t)x'(t-\varphi_{1m}) + \ldots + a_{n0}(t)x^{(n)}(t) + a_{n1}(t)x^{(n)}(t-\varphi_{n1}) + a_{n2}(t)x^{(n)}(t-\varphi_{n2}) + \ldots + a_{nm}(t)x^{(n)}(t-\varphi_{nm}) = g(t),$$

where $a_{ij}(t)$, for all (i = 0, 1, …, n and j = 0, 1, …, m) are functions with respect to t.

Although, the study of nonlinear ODDE's attracts the attention of several researchers and there are some works in this field such as; the work of, [Muroya, 2007] which established the necessary and sufficient conditions of the global asymptotic stability of the zero solution of the following non-linear ODDEv.

$$x'(t) = -a(t)x(t) - \sum_{i=0}^{m} a_i(t) g_i(x(\varphi_i(t))),\ t \geq t_0,$$

with the initial condition:

$$x(t) = \varphi(t),\ t \leq t_0;$$

and assuming that:

(1) $a(t)$ is continuous and bounded on $[t_0, +\infty)$.

(2) $a_i(t)$, $0 \leq i \leq m$, are piecewise continuous and bounded on $[t_0, \infty)$.

(3) $\inf_{t \geq t_0} a(t) > 0$, $a_i(t) \geq 0$, $0 \leq i \leq m$.

(4) $\sum_{i=0}^{m} a_i(t) > 0$, $\int_{t_0}^{\infty} \sum_{i=0}^{m} a_i(t)\,dt = +\infty$.

(5) $\varphi_i(t)$ is piecewise continuous on $[t_0, +\infty)$.

(6) $\varphi_i(t)$, $t \geq t_0$, $\sup_{t \geq t_0} \{t - \varphi_i(t)\} \leq \hat{\tau} < +\infty$, $0 \leq i \leq m$.

(7) $\varphi(t) = \inf_{0 \leq i \leq m} \varphi_i(t)$ is a monotone increasing function on $[t_0, +\infty)$.

(8) $\varphi(t)$ is continuous and bounded on $[t_0 - \hat{\tau}, t_0]$.

(9) $g_i(x)$ is continuous on $(-\infty, +\infty)$ and for $x > 0$.

(10) $g_i(-x) < g_i(0) = 0 < g_i(x)$, $0 \leq i \leq m$.

The work of [Tiryaki, 2007] which established some new sufficient conditions which ensure the solution of the third order nonlinear ODDE that oscillates or converges to zero and the work of, [Drager, 1997] which studied the initial value problems for nonlinear ODDE's with distributed infinite delays.

As in differential-difference equations the first order linear ODDEv may be classified into three types retarded, neutral and advanced ordinary delay-differential equations with variable delays. Also according to the new universally accepted study proposed by Kamenskii G. K. by the following ODDEv with the highest order

$$x^{(m)}(t) = F(t;\ x^{(m_1)}(t),\ x^{(m_2)}(t),\ \ldots,\ x^{(m_k)}(t),\ x^{(m_1)}(t-\varphi_1),$$
$$x^{(m_2)}(t-\varphi_2),\ \ldots,\ x^{(m_k)}(t-\varphi_k)),$$

here $x(t) \in \mathbb{R}^n$, all $m_i \geq 0$, $\varphi_i = \varphi_i(t;\ x^{(m_1)}(t), x^{(m_2)}(t), \ldots, x^{(m_k)}(t)) \geq 0$, for $(i = 1, 2, \ldots, k)$, is called an ordinary delay-differential equation of retarded type , if $\text{Max}\{m_1, m_2, \ldots, m_k\} < m$; an ordinary delay-differential equation of neutral type if $\text{Max}\{m_1, m_2, \ldots, m_k\} = m$ and of advanced type if $\text{Max}\{m_1, m_2, \ldots, m_k\} > m$, [Kolmanovskii, 1992].

1.2.2 Partial Delay -Differential Equations with Variable Delays:

The theory of partial delay-differential equations (PDDE's, for short) is essentially less developed since such equations are of infinite dimensional in both time (as delay equations) and space (as PDE's variables), which makes the analysis more difficult, [Rezounenko, 2008]. The most interesting of the existence theorems are those of Gul′ I. M., which prove the existence of a unique solution to the Cauchy's problem for a system of PDDE's, in the so-called nonspecial case, i.e., when the PDDE can be reduced to PDE without deviating arguments, [El'sgol'c, 1964].

Partial delay-differential equations with variable delays play an important role in different fields of applied mathematics, say in control theory [Kaiser, 2000] and mathematical physics when the retardation generally appears only as a time-lag, [El'sgol'c, 1964], and have been studied for many years using different methods by some researchers from Taves and Webb in (1974), Chucshov in (1992), Chueshov and Rezounenko in (1995), Wu in (1996), Boulet de Monveletal in (1998) and Rezounenko in (2003), [Rezounenko, 2008].

Partial delay-differential equation with variable delays, is a delay-differential equation in which the unknown function is a function of two or more independent variables where the delay terms are functions may occur in one or more of these variables, and involving more than one partial derivative, for example, [El'sgol'ts, 1973]:

$$\frac{\partial}{\partial x}u(x, t) + u(x, t) = f(x, t; u(x, t-1))$$

and

$$\frac{\partial^2}{\partial x^2}u(x, t) + \frac{\partial^2}{\partial t^2}u(x, t) = 2\frac{\partial^2}{\partial t^2}u(x-1, t).$$

In this subsection, we will restrict our discussion to the PDDEv in which the unknown function depends only on two variables and the delay terms may occur in one or both of these variables. The general form of the n^{th} order PDDE with multiple variable delays is:

$$F(x, t; u(x, t), u(x - \varphi_{01}, t - \psi_{01}), u(x - \varphi_{02}, t - \psi_{02}), \ldots, u(x - \varphi_{0m}, t-\psi_{0m}), \frac{\partial}{\partial x}u(x, t), \frac{\partial}{\partial x}u(x - \varphi^{x}_{11}, t - \psi^{x}_{11}), \frac{\partial}{\partial x}u(x - \varphi^{x}_{12}, t - \psi^{x}_{12}),$$

$$\ldots, \frac{\partial}{\partial x}u(x - \varphi^{x}_{1m}, t - \psi^{x}_{1m}), \frac{\partial}{\partial t}u(x, t), \frac{\partial}{\partial t}u(x - \varphi^{t}_{11}, t - \psi^{t}_{11}),$$

$$\frac{\partial}{\partial t}u(x - \varphi^{t}_{12}, t - \psi^{t}_{12}), \ldots, \frac{\partial}{\partial t}u(x - \varphi^{t}_{1m}, t - \psi^{t}_{1m}), \frac{\partial^2}{\partial x^2}u(x, t),$$

$$\frac{\partial^2}{\partial x^2}u(x-\varphi^{x}_{21}, t-\psi^{x}_{21}), \frac{\partial^2}{\partial x^2}u(x-\varphi^{x}_{22}, t-\psi^{x}_{22}), \ldots, \frac{\partial^2}{\partial x^2}u(x-\varphi^{x}_{2m}, t-\psi^{x}_{2m}), \frac{\partial^2}{\partial t^2}u(x, t), \frac{\partial^2}{\partial t^2}u(x-\varphi^{t}_{21}, t-\psi^{t}_{21}), \frac{\partial^2}{\partial t^2}u(x-\varphi^{t}_{22}, t-\psi^{t}_{22}),$$

$$\ldots, \frac{\partial^2}{\partial t^2}u(x - \varphi^{t}_{2m}, t - \psi^{t}_{2m}), \frac{\partial^2}{\partial x\partial t}u(x, t), \frac{\partial^2}{\partial x\partial t}u(x - \varphi^{xt}_{2(1)(1)}, t - \psi^{xt}_{211}), \frac{\partial^2}{\partial x\partial t}u(x - \varphi^{xt}_{2(1)(2)}, t - \psi^{xt}_{2(1)(2)}), \ldots, \frac{\partial^2}{\partial x\partial t}u(x - \varphi^{xt}_{2(1)(m)},$$

$$t - \psi^{xt}_{2(1)(m)}), \ldots, \frac{\partial^n}{\partial x^n}u(x, t), \frac{\partial^n}{\partial x^n}u(x - \varphi^{x}_{n1}, t - \psi^{x}_{n1}), \frac{\partial^n}{\partial x^n}u(x - \varphi^{x}_{n2}, t - \psi^{x}_{n2}), \ldots, \frac{\partial^n}{\partial x^n}u(x - \varphi^{x}_{nm}, t - \psi^{x}_{nm}), \frac{\partial^n}{\partial t^n}u(x, t), \frac{\partial^n}{\partial t^n}u(x - \varphi^{t}_{n1}, t-\psi^{t}_{n1}), \frac{\partial^n}{\partial t^n}u(x-\varphi^{t}_{n2}, t-\psi^{t}_{n2}), \ldots, \frac{\partial^n}{\partial t^n}u(x-\varphi^{t}_{nm}, t-\psi^{t}_{nm}),$$

$$\frac{\partial^n}{\partial x^{n-1}\partial t}u(x, t), \frac{\partial^n}{\partial x^{n-1}\partial t}u(x - \varphi^{xt}_{n(1)(1)}, t - \psi^{xt}_{n(1)(1)}), \frac{\partial^n}{\partial x^{n-1}\partial t}u(x - \varphi^{xt}_{n(1)(2)}, t - \psi^{xt}_{n(1)(2)}), \ldots, \frac{\partial^n}{\partial x^{n-1}\partial t}u(x - \varphi^{xt}_{n(1)(m)}, t - \psi^{xt}_{n(1)(m)}),$$

$$\frac{\partial^n}{\partial x^{n-2}\partial t^2}u(x - \varphi^{xt}_{n(2)(1)}, t - \psi^{xt}_{n(2)(1)}), \frac{\partial^n}{\partial x^{n-2}\partial t^2}u(x - \varphi^{xt}_{n(2)(2)},$$

$$t - \psi^{xt}_{n(2)(2)}), \ldots, \frac{\partial^n}{\partial x^{n-2} \partial t^2} u(x - \varphi^{xt}_{n(2)(m)}, t - \psi^{xt}_{n(2)(m)}), \ldots,$$

$$\frac{\partial^n}{\partial x \partial t^{n-1}} u(x, t), \frac{\partial^n}{\partial x \partial t^{n-1}} u(x - \varphi^{xt}_{n(n-1)(1)}, t - \psi^{xt}_{n(n-1)(1)}),$$

$$\frac{\partial^n}{\partial x \partial t^{n-1}} u(x - \varphi^{xt}_{n(n-1)(2)}, t - \psi^{xt}_{n(n-1)(2)}), \ldots, \frac{\partial^n}{\partial x \partial t^{n-1}} u(x -$$

$$\varphi^{xt}_{n(n-1)(m)}, t - \psi^{xt}_{n(n-1)(m)})) = g(x, t)$$

where F is a given function, $\varphi^{x,t}_{(i)(j)k}$ and $\psi^{x,t}_{(i)(j)k}$, for all (i = 0, 1, …, n; j = 1, 2, …, i – 1 and k = 1, 2, …, m) are known functions of x, t, u(x, t), $\frac{\partial}{\partial x} u(x, t)$, $\frac{\partial}{\partial t} u(x, t)$, $\frac{\partial^2}{\partial x^2} u(x, t)$, $\frac{\partial^2}{\partial t^2} u(x, t)$, $\frac{\partial^2}{\partial x \partial t} u(x, t)$, …, $\frac{\partial^n}{\partial x^n} u(x, t)$, $\frac{\partial^n}{\partial t^n} u(x, t)$, $\frac{\partial^n}{\partial x^{n-1} \partial t} u(x, t)$, $\frac{\partial^n}{\partial x^{n-2} \partial t^2} u(x, t)$, …, $\frac{\partial^n}{\partial x \partial t^{n-1}} u(x, t)$.

Another form of partial functional-differential equations is:

$$F(x, t; u(x,t), u(k_{01}x,p_{01}t), u(k_{02}x,p_{02}t), \ldots, u(k_{0m}x,p_{0m}t), \frac{\partial}{\partial x} u(x,t),$$

$$\frac{\partial}{\partial x} u(k^x_{11} x,\ p^x_{11} t), \frac{\partial}{\partial x} u(k^x_{12} x,\ p^x_{12} t), \ldots, \frac{\partial}{\partial x} u(k^x_{1m} x,\ p^x_{1m} t),$$

$$\frac{\partial}{\partial t} u(x, t), \frac{\partial}{\partial t} u(k^t_{11} x,\ p^t_{11} t), \frac{\partial}{\partial t} u(k^t_{12} x,\ p^t_{12} t), \ldots, \frac{\partial}{\partial t} u(k^t_{1m} x,$$

$$p^t_{1m} t), \frac{\partial^2}{\partial x^2} u(x, t), \frac{\partial^2}{\partial x^2} u(k^x_{21} x,\ p^x_{21} t), \frac{\partial^2}{\partial x^2} u(k^x_{22} x,\ p^x_{22} t), \ldots,$$

$$\frac{\partial^2}{\partial x^2} u(k^x_{2m} x,\ p^x_{2m} t), \frac{\partial^2}{\partial t^2} u(x, t), \frac{\partial^2}{\partial t^2} u(k^t_{21} x,\ p^t_{21} t), \frac{\partial^2}{\partial t^2} u(k^t_{22} x,$$

$$p^t_{22} t), \ldots, \frac{\partial^2}{\partial t^2} u(k^t_{2m} x,\ p^t_{2m} t), \frac{\partial^2}{\partial x \partial t} u(x, t), \frac{\partial^2}{\partial x \partial t} u(k^{xt}_{2(1)(1)} x,$$

$$p^{xt}_{2(1)(1)} t), \frac{\partial^2}{\partial x \partial t} u(k^{xt}_{2(1)(2)} x,\ p^{xt}_{2(1)(2)} t), \ldots, \frac{\partial^2}{\partial x \partial t} u(k^{xt}_{2(1)(m)} x,$$

$$p^{xt}_{2(1)(m)}t), \ldots, \frac{\partial^n}{\partial x^n}u(x, t), \frac{\partial^n}{\partial x^n}u(k^x_{n1}x, p^x_{n1}t), \frac{\partial^n}{\partial x^n}u(k^x_{n2}x, p^x_{n2}t),$$

$$\ldots, \frac{\partial^n}{\partial x^n}u(k^x_{nm}x, p^x_{nm}t), \frac{\partial^n}{\partial t^n}u(k^x_{n1}x, p^x_{n1}t), \frac{\partial^n}{\partial t^n}u(k^x_{n2}x, p^x_{n2}t),$$

$$\ldots, \frac{\partial^n}{\partial t^n}u(k^x_{nm}x, p^x_{nm}t), \frac{\partial^n}{\partial x^{n-1}\partial t}u(x, t), \frac{\partial^n}{\partial x^{n-1}\partial t}u(k^{xt}_{n(1)(1)}x,$$

$$p^{xt}_{n(1)(1)}t), \frac{\partial^n}{\partial x^{n-1}\partial t}u(k^{xt}_{n(1)(2)}x, p^{xt}_{n(1)(2)}t), \ldots, \frac{\partial^n}{\partial x^{n-1}\partial t}u(k^{xt}_{n(1)(m)}x,$$

$$p^{xt}_{n(1)(m)}t), \frac{\partial^n}{\partial x^{n-2}\partial t^2}u(k^{xt}_{n(2)(1)}x, p^{xt}_{n(2)(1)}t), \frac{\partial^n}{\partial x^{n-2}\partial t^2}u(k^{xt}_{n(2)(2)}x,$$

$$p^{xt}_{n(2)(2)}t), \ldots, \frac{\partial^n}{\partial x^{n-2}\partial t^2}u(k^{xt}_{n(2)(m)}x, p^{xt}_{n(2)(m)}t)\ldots, \frac{\partial^n}{\partial x\partial t^{n-1}}u(x, t),$$

$$\frac{\partial^n}{\partial x\partial t^{n-1}}u(k^{xt}_{n(n-1)(1)}x, p^{xt}_{n(n-1)(1)}t), \frac{\partial^n}{\partial x\partial t^{n-1}}u(k^{xt}_{n(n-1)(2)}x,$$

$$p^{xt}_{n(n-1)(2)}t), \ldots, \frac{\partial^n}{\partial x\partial t^{n-1}}u(k^{xt}_{n(n-1)(m)}x, p^{xt}_{n(n-1)(m)}t)) = g(x, t),$$

where $k^{x,t}_{(i)(j)k}$ and $p^{x,t}_{(i)(j)k}$ for all (i = 0, 1, …, n; j = 1, 2, …, i – 1 and k = 1, 2, …, m), are known functions of x, t; u(x, t), $\frac{\partial}{\partial x}u(x, t)$, $\frac{\partial}{\partial t}u(x, t)$, …, $\frac{\partial^2}{\partial x^2}u(x, t)$, $\frac{\partial^2}{\partial t^2}u(x, t)$, $\frac{\partial^2}{\partial x\partial t}u(x, t)$, …, $\frac{\partial^n}{\partial x^n}u(x, t)$, $\frac{\partial^n}{\partial t^n}u(x, t)$, $\frac{\partial^n}{\partial x^{n-1}\partial t}u(x, t)$, $\frac{\partial^n}{\partial x^{n-2}\partial t^2}u(x, t)$,… and $\frac{\partial^n}{\partial x\partial t^{n-1}}u(x, t)$.

As in ODDEv's, we can classify PDDEv's to be homogeneous or nonhomogeneous, linear or nonlinear, etc., such that the n-th order linear PDDE with variable multiple delays and variable coefficients, takes the form:

$$\sum_{i=0}^{m} a_i(x,t)u(x-\varphi_{i0}, t-\psi_{i0}) + \sum_{i=1}^{n}\sum_{j=1}^{m}\left\{ b_{ij}(x,t)\frac{\partial^i}{\partial x^i}u(x-\varphi_{ij}^{x}, t-\psi_{ij}^{x}) + c_{ij}(x,t)\frac{\partial^i}{\partial t^i}u(x-\varphi_{ij}^{t}, t-\psi_{ij}^{t})\right\} + \sum_{i=2}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{m} d_{ijk}(x,t)$$

$$\frac{\partial^i}{\partial x^{i-j}\partial t^j}u(x-\varphi_{ijk}^{xt}, t-\psi_{ijk}^{xt}) = g(x, t), \text{ for } \varphi_{00} = \psi_{00} = 0,$$

where a_i, b_{ij}, c_{ij} and d_{ijk} are assumed to be known functions of x and t.

So, the first order linear PDDE of single delays is:

$$a_0(x, t)u(x, t) + a_1(x, t)u(x-\varphi_1, t-\psi_1) + b_0(x, t)\frac{\partial}{\partial x}u(x, t) + b_1(x, t)\frac{\partial}{\partial x}u(x-\varphi_1^{x}, t-\psi_1^{x}) + c_0(x, t)\frac{\partial}{\partial t}u(x, t) + c_1(x, t)\frac{\partial}{\partial t}u(x-\varphi_1^{t}, t-\psi_1^{t}) = g(x, t) \quad \text{.........(1.5)}$$

if $a_1 = b_1 = c_1 = 0$, then eq.(1.5) is reduced to the general form of the first order linear PDE. Also, eq.(1.5) may be classified into retarded, neutral and advanced PDDEv. A RPDDEv may be characterized by the fact that F is dependent on delayed terms without partial derivatives, such as $u(x-\varphi_0, t-\psi_0)$. While NPDDEv may be characterized by the fact that F is dependent on delayed partial derivative terms, such as $\frac{\partial}{\partial x}u(x-\varphi_1^{x}, t-\psi_1^{x})$, $\frac{\partial}{\partial t}u(x-\varphi_1^{t}, t-\psi_1^{t})$ and finally APDDEv is the combination of the two types (RPDDEv and NPDDEv).

The second order linear PDDE with variable delays and variable coefficients takes the form:

$$a_0(x,t)u(x, t) + a_1(x,t)u(x - \varphi_1, t - \psi_1) + b_0(x,t)\frac{\partial}{\partial x}u(x, t) + b_1(x,t)$$

$$\frac{\partial}{\partial x}u(x - \varphi_1^x, t - \psi_1^x) + b_2(x, t)\frac{\partial^2}{\partial x^2}u(x - \varphi_2^x, t - \psi_2^x) + c_0(x, t)$$

$$\frac{\partial}{\partial t}u(x,t) + c_1(x,t)\frac{\partial}{\partial t}u(x-\varphi_1^t,t-\psi_1^t) + c_2(x,t)\frac{\partial^2}{\partial t^2}u(x-\varphi_2^t,t-\psi_2^t) +$$

$$d_0(x, t)\frac{\partial^2}{\partial x\partial t}u(x, t) + d_1(x, t)\frac{\partial^2}{\partial x\partial t}u(x-\varphi_{11}^{xt}, t-\psi_{11}^{xt}) = g(x, t) \quad ...(1.6)$$

A special case of eq.(1.6) is the following PDDE of single delays:

$$a_0u(x, t) + a_1u(x - \tau, t - \xi) + b_1\frac{\partial}{\partial x}u(x, t) + b_2\frac{\partial^2}{\partial x^2}u(x, t) +$$

$$c_1\frac{\partial}{\partial t}u(x, t) + c_2\frac{\partial^2}{\partial t^2}u(x, t) + d_2\frac{\dot{\partial}^2}{\partial x\partial t}u(x, t) = g(x, t) \quad ....(1.7)$$

and the following cases may be considered, [Al-Esawi, 2006]:

(i) If $(d_2^2 - 4b_2c_2) > 0$, then eq.(1.7) is said to be of hyperbolic type.

(ii) If $(d_2^2 - 4b_2c_2) = 0$, then eq.(1.7) is said to be of parabolic type.

(iii) If $(d_2^2 - 4b_2c_2) < 0$, then eq.(1.7) is said to be of elliptic type.

Variants of nonlinear PDDEv's are used in some models, for example, the following NPDDEv:

$$\frac{\partial}{\partial t}D_{u_t} = A D_{u_t} + F(t, u_t), \ t \geq 0$$

with the initial condition:

$$u_0 = \Phi \in B,$$

where $A : D(A) \subseteq E \longrightarrow E$ is a linear operator on a Banach space $(E, |.|)$, B is the phase space of functions mapping $(-\infty, 0)$ into E, D is a bounded linear operator from B to E defined by:

$$D_\varphi = \varphi(0) = D_0\varphi, \text{ for } \varphi \in B;$$

the operator D_0 is bounded and linear from B to E and for each $u : (-\infty, b] \longrightarrow E$, $b > 0$ and $t \in [0, b]$, u_t represent, as usual the mapping defined from $(-\infty, 0)$ to E by:

$$u_t(\varphi) = u(t - \varphi), \text{ for } \varphi \in (-\infty, 0]$$

and F a nonlinear continuous mapping onto $\mathbb{R}^+ \times B$. This model has been suggested in the description of the heat flow models, [Bouzahir, 2006], and of the viscoelastic and thermoviscoelsatic materials dynamics, [Desch, 1988].

One of the most important methods that can be used to solve PDDE's is the method of separation of variables, which is used to solve that following types or problems by letting $u(x, t) = X(x)T(t - \tau)$:

(i) The generalized diffusion equation:

$$\frac{\partial}{\partial x} u(x, t) = a_0^2 \frac{\partial^2}{\partial x^2} u(x, t) + a_1^2 \frac{\partial^2}{\partial x^2} u(x, t - \tau),$$

where a_0, a_1 and τ are constants and $\tau > 0$ together with the initial and boundary conditions.

$$u(x, t) = \varphi(x, t), \text{ for } 0 \le x \le \ell, 0 \le t \le \tau$$

and

$$u(0, t) = 0, u(\ell, t) = 0.$$

(ii) The generalized wave equation:

$$\frac{\partial^2}{\partial t^2}u(x, t) = a_0^2 \frac{\partial^2}{\partial x^2}u(x, t) + a_1^2 \frac{\partial^2}{\partial x^2}u(x, t - \tau),$$

where a_0, a_1 and τ are constants and $\tau > 0$ together with the initial and boundary conditions.

$u(x, t) = \varphi(x, t)$, for $0 \leq x \leq \ell$, $0 \leq t \leq \tau$

and

$u(0, t) = 0$, $u(\ell, t) = 0$.

Also, Vandewalle S. and Gander M. in 2003 used this method to solve the parabolic PDDE's, [Vandewall, 2003].

2 Variational Formulation of Ordinary Delay-Differential Equations with Variable Delays

The decision on which the type of the method is best for solving DDE's depends on whether one is concerned with the general DDE's or with a special class of DDE's, [Shampine, 2000]. In addition to that, numerical methods for solving both ODDE's and PDDE's are intended for problems with solution that have several continuous derivatives, and the discontinuities in low-order derivatives affect the analytical methods which are employed to solve them. In treating such issues, one can really use optimization methods, [Baker, 1994]. Optimization is a universal human goal, typically, our try to maximize profit, minimize cost, travel a destination in the quickest time, all are the optimization problems which appear in real life problems.

So, our natural propensity in optimization, has led to a long standing effort to systematically determine the optimal realization of variety of activities in science and engineering. This continuity effort has created a body of mathematical method called (the mathematics of optimization), [Wan, 1995]. In mathematics, optimization makes sense when formulated in terms of the functional v[x]. The mathematical theory for this type of optimization problems is called variational problems.

There are countless problems of this type in science and engineering and many others which can recast into the same form, [Lebedev, 2003]. So, in this chapter, we will focus our attention mainly on the variational formulation of ODDEv's by Magrie's approach, which can be minimized by different solution techniques of theory, such as; by Euler-Lagrange equation or by using the direct Ritz method.

This chapter consists of five sections. In section one, some basic concepts related to the subject of calculus of variation are surveyed. In section two, the Magrie's approach is used to find the variational formulation of ODDEv's. In section three, some related results for evaluating the necessary condition for an extremum of different kinds of the variational problems of ODDEv's are derived. Section four concerned with the sufficiency condition for functionals to have a minimum points. Finally, section five presents the direct-Ritz method to find the approximate solution of the variational problem related to ODDEv's.

2.1 Basic Concepts in Calculus of Variation

The calculus of variation is the most important branch of functional analysis since it can be applied with great power to the range of problems in pure mathematics, and can be used to express the fundamental principles of applied mathematics and mathematical physics in unusually simple and elegant forms [Fox, 1987]. It deals with problems of maxima and minima. But while in the ordinary theory of maxima and minima, the problem, is to determine those values of the independent variables for which a given function of these variables takes a maximum or minimum value, in the calculus of variations integrals involving one or more unknown functions are considered, and it is required so to determine these unknown functions that the definite integrals shall take maximum or minimum values, [Boiza, 2001]. In language of geometry, we may say that this calculus deals with the problem of finding paths of integration for which integrals admits maximum or minimum values, [Berechtken, 1991].

In addition, the calculus of variation may be considered as one of the classical branches of mathematics. It was Euler who, looking at the work of Lagrange and gave the present, not really self explanatory, to this field of mathematics.

In fact, the subject is much older than it appears; it starts when several more or less rigorous proofs were known since the times of Zenodorus around 200 B.C., who proved the inequality for polygons. There are also significant contributions by Archimedes and Poppus. Important attempts for proving the inequality are due to Euler, Galileo, Legender and L'Huilier, etc. The first proof that agrees with modern standards is due to Weierstrass and it has been extended or proved with different tools by Blaschke, Bonnesen and Euler, etc.

Other important problems in calculus of variation were considered in the seventeenth century in Europe, such as the work of Fermat on geometrical optics (1662), the problem of Newton (1685), for the study of body moving in fluids or the problem of the brachistochrone formulated by Galileo in (1638). This last problem had a very strong influence on the development of the calculus of variation. It was resolved by John Bernoulli in (1696), and almost immediately after, by James (his brother), Leibniz and Newton A.

In the nineteenth century and in parallel to the work that was mentioned above, many important contributions were made by Drichlet, Gauss, Thompson and Riemann among others. It was Hilbert who, at the turn of the twentieth century, solved the problem of multiple integrals and it was immediately after initiated by Lebsegue and then Tonelli. Their methods for solving the problem were, essentially, what are now known as the direct methods of the calculus of variation, should also emphasize that the problem was very important in the development of analysis in general, and more notably in functional analysis, measure theory and distribution theory.

In 1900, at the International Congress of Mathematics in Paris, Hilbert formulated 23 problems that he considered to be important for the development of mathematics in the twentieth century. Three of them were devoted to the subject of calculus of variation and minimizing the functionals by using the conventions of calculus of variations. These predictions of Hilbert have been amply justified all along the twentieth century, and the field is at the turn of the twenty first one, as active as in the previous century, [Dacorogna, 2004].

2.1.1 The Fundamental Problem of Calculus of Variation:

The fundamental problem in solving ODE's using the subject of calculus of variation is to find the extremal function (curve) x = x(t), amongst a set (space) of admissible functions, which maximizes or minimizes a given functional v[x(t)].

The simplest variational problem is to consider the functional:

$$v[x(t)] = \int_{t_0}^{t_1} F(t; x(t), x'(t))\, dt \quad \text{...(2.1)}$$

where $t \in [t_0, t_1]$, $t_0 < t_1$ be a given bounded subinterval of the real line $\mathbb{R}$, while the set of admissible functions x: $[t_0,t_1] \longrightarrow \mathbb{R}$, satisfy the specified constraints. For the functional (2.1) to have a minimum corresponding to some particular curve x= x(t), the inequality:

$$v[x(t)] = \int_{t_0}^{t_1} F(t; x, x')\, dt$$

$$\leq \int_{t_0}^{t_1} F(t; x + \delta x, x' + \delta x')\, dt$$

$$= \int_{t_0}^{t_1} F(t; x^*, x^{*\prime})\, dt = v[x^*(t)];$$

must be hold for every admissible functions $x^* = x(t) + \varepsilon\delta x(t)$, so that $x^{*\prime} = x'(t) + \varepsilon\delta x'(t)$, where $|\varepsilon|$ is sufficiently small and $\delta x(t)$ is an arbitrary admissible function satisfying $\delta x(t_0) = \delta x(t_1) = 0$, for the case of fixed end points, [Gelfand, 1963].

Let:

$$\delta v[x] = v[x + \delta x] - v[x]$$

$$= \int_{t_0}^{t_1} \left\{ F(t; x + \delta x, x' + \delta x') - F(t; x, x') \right\} dt$$

$$= \int_{t_0}^{t_1} \Delta F \, dt,$$

where ΔF is the linear part of the increment, defined as:

$$\Delta F = F(t;\, x + \delta x,\, x' + \delta x') - F(t;\, x,\, x')$$

$$= \delta F + \delta^2 F + \ldots + \delta^r F + \ldots$$

where, from the Taylor series expansion:

$$\delta F = F_x \delta x + F_{x'} \delta x'$$

and

$$\delta^2 F = \frac{1}{2!} \{F_{xx} \delta x^2 + 2F_{xx'} \delta x \delta x' + F_{x'x'} \delta x'^2\}.$$

Then:

$$\delta v[x] = \int_{t_0}^{t_1} \left\{ \delta F + \delta^2 F + \ldots + \delta^r F + \ldots \right\} dt$$

$$= \delta v + \delta^2 v + \ldots + \delta^r v + \ldots$$

where

$$\delta v = \int_{t_0}^{t_1} \delta F \, dt \quad \text{(the first variation of } v[x]) \text{ ....(2.2)}$$

$$\delta^2 v = \int_{t_0}^{t_1} \delta^2 F \, dt \quad \text{(the second variation of } v[x]) \text{ ....(2.3)}$$

According to the last equations, the first variation for the functional (2.1) may be written as:

$$\delta v = \int_{t_0}^{t_1} \delta F \, dt$$

$$= \int_{t_0}^{t_1} \left\{ F_x \delta x + F_{x'} \delta x' \right\} dt .$$

Integrating the second term by parts and recalling that the admissible curve x* passes through the fixed boundary points t_0 and t_1, then:

$$\int_{t_0}^{t_1} F_{x'} \delta x' \, dt = -\int_{t_0}^{t_1} \left\{ \frac{d}{dt} F_{x'} \delta x \right\} dt .$$

Hence:

$$\delta v = \int_{t_0}^{t_1} \left\{ F_x - \frac{d}{dt} F_{x'} \right\} \delta x \, dt .$$

A necessary condition for v[x*] = v[x + δx] to have an extremum at x = x*, is that δv[x] = 0, for an admissible δx, [Elsgolc, 1962]. Hence:

$$\delta v[x] = \int_{t_0}^{t_1} \left\{ F_x - \frac{d}{dt} F_{x'} \right\} \delta x \, dt ,$$

implies that:

$$\int_{t_0}^{t_1} \left\{ F_x - \frac{d}{dt} F_{x'} \right\} \delta x \, dt = 0 .$$

Since δx is arbitrary, then by the fundamental Lemma of calculus of variation, the fundamental Euler-Lagrange equation is;

$$\left(F_x - \frac{d}{dt}F_{x'}\right) = 0.$$

Also, other types of the fundamental ODE problems with different types of functionals, such as functional with higher derivatives, or with several dependent variables, etc., may occur depends on the essential of the boundary conditions, and one may consider four cases:

i) $\delta x(t_0) = \delta x(t_1) = 0$, when the two end points are fixed, is often referred to as point-point problems.

ii) $\delta x(t_0) \neq 0$, $\delta x(t_1) = 0$, when t_0 is movable and t_1 is fixed, is often referred to as curve-point problems.

iii) $\delta x(t_0) = 0$, $\delta x(t_1) \neq 0$, when t_0 is fixed and t_1 is movable, is often referred to as point-curve problems.

iv) $\delta x(t_0) \neq 0$, $\delta x(t_1) \neq 0$, when the two end points are free, is often referred to as curve- curve problems, [Memarbashi, 2006].

Hereby some theorems, which will be needed later in this work.

Lemma (2.1), [El'sgol'c, 1964]:

Let the function $\delta x(t)$ satisfy the following conditions:

1. $\delta x(t_0 - \varphi_i(t_0; x(t_0), x'(t_0))) = 0$ and $\delta x(t_1 - \varphi_i(t_1; x(t_1), x'(t_1))) = 0$, for each $(i = 1, 2, \ldots, n)$.

2. $\delta x(t)$ has continuous derivatives up to order p in the interval $t_0 - \eta_0 \le t \le t_1$, where

   $\eta_0 = \text{Max}\ \{\varphi_i(t_0; x(t_0), x'(t_0))\}$, $(i = 1, 2, \ldots, n)$

3. $|\delta x^{(k)}(t)| < \varepsilon$, $(k = 0, 1, \ldots, s)$, $s \le p$, then if for arbitrary functions δx, the integral:

   $$\int_{t_0}^{t_1} \sum_{i=1}^{n} x_i(t)\delta x(t - \varphi_i(t, x(t), x'(t)))\, dt = 0,$$

   where $x_i(t)$ is continuous in the interval $[t_0, t_1]$, and $\varphi_i(t, x(t), x'(t))$ are non-negative continuous differentiable functions, with first derivatives that satisfy the inequalities $\varphi_i' < d < 1$, $(i = 1,2,\ldots,n)$, then:

   $$\sum_{i=1}^{n} \frac{x_i(f_i(t, x(t), x'(t)))}{1 - \varphi_i(f_i(t, x(t), x'(t)))} = 0, \text{ where } 1 - \varphi_i(f_i(t, x(t), x'(t))) \neq 0$$

for $t_0 - \eta_0 \le t \le t_1$, where $f_i(t)$, is inverse to the function of $z = t - \varphi_i(t, x(t), x'(t))$. Outside the interval $t_0 \le t \le t_1$, all the functions $x_i(t)$ are assumed to vanish.

Theorem (2.2) (Legendre Theorem), [Gelfand, 1963]:

A necessary condition for the functional:

$$v[x] = \int_{t_0}^{t_1} F(t; x, x')\, dt,\ x(t_0) = x_0,\ x(t_1) = x_1\ ;$$

to have a minimum for the curve $x = x(t)$ is that the inequality

$F_{x'x'} \ge 0$, (Legendre's condition),

be satisfied at every point of the curve.

2.2 The Magrie's Approach, [Magri, 1974]

This Magrie's approach deals with the problem of finding the variational formulation for the linear equations associated with initial or boundary conditions denote such equations as:

$Lx = f$..(2.4)

where x denotes a scalar or vector valued function and L denotes a linear operator, with domain D(L) in a linear space X and range R(L) in a second linear space Y and f a given real valued function.

The aim of this approach is to search for a functional v[x] defined on the domain of the linear operator L, whose critical points are solutions of the given equation (2.4).

This problem is called the inverse problem of calculus of variation, while the usual problem of finding the critical points of a pre-assigned functional may be called the direct problem.

In order to solve eq.(2.4), given two linear spaces X and Y and a bilinear form, defined on them; a functional G[x, y] is linear in both x and y, where x and y are elements of X and Y, respectively. The functional is usually denoted by the symbol $<x, y>$, which is called non-degenerate on X and Y if the following two conditions are satisfied:

i) If $<x, \bar{y}> = 0$, then $\bar{y} = 0$, for every $x \in X$.

ii) If $<\bar{x}, y> = 0$, then $\bar{x} = 0$, for every $y \in Y$.

As it is pointed out by Magri in 1974, an operator L is said to be symmetric with respect to the chosen bilinear form if the condition $<Lx, y> = <Ly, x>$ is satisfied for every pair of elements x and y of

D(L), and if the operator is not symmetric with respect to the chosen bilinear form, one can make the transformation <x, y> = (x, Ly), to get <., .> is symmetric. Therefore, it is possible to prove that the bilinear form with the linear operator L is symmetric whatever the chosen bilinear form <x, y>, since:

$$\langle Lx, y\rangle = (Lx, Ly) = (Ly, Lx) = \langle Ly, x\rangle,$$

so, there is a variational formulation corresponding to the linear equation (2.4) if and only if the operator L is symmetric relative to the chosen bilinear form which is non-degenerate and the functional v[x] is given by

$$v[x] = \frac{1}{2}\langle Lx, x\rangle - \langle f, x\rangle.$$

2.3 The Fundamental Necessary Condition for an Extremum of the Variational Problems of ODDEv's

The fundamental necessary conditions for an extremum are the true work of variational problem's theory, in which variational formulation yields to only necessary conditions of optimality, because it is assumed that the compared trajectories are close to each other in some sense, while exact sufficient conditions are rare remarkable exception.

The necessary condition for x(t) established to provide a local minimum tells us that there is no other curve $x + \delta x$, which

i) Is sufficiently close to the chosen curve x.

ii) Satisfies the same boundary conditions, and

iii) Corresponds to a smaller value $v[x] < v[x+ \delta x]$ of the objective functional, [Cherkaev, 2003].

In other wards, a necessary condition for an extremum $x = x^*$ of the functional v[x] is that; its first variation vanishes for $x = x^*$, i.e., is that $\delta v = 0$ for $x = x^*$ and all admissible x, [Gelfand, 1963].

In this section, several results will be introduced for extremizing the variational formulation of ODDEv's, which differs from popular problems in terms of variable delay in x which is constructed from the natural of the boundary conditions.

Remark (2.3):

The maximization problem for the functional v[x] is equivalent to the minimization problem for $-v[x]$ and therefore, the variational problem always formulated in terms of minimizing the functional, [Elsgolc, 1962].

Remark (2.4):

For simplicity, the notation F is written as the shorthand for the general form of $F(t; x, x', x(t - \varphi_0), x'(t - \varphi_1))$, where $\varphi_i = \varphi_i(t; x(t), x'(t))$, $(i = 0, 1)$, unless it is stated.

2.3.1 The Point-Point Problems of ODDEv's:

In this subsection, the necessary condition for an extremum of different types of functionals will be established. In case (t_0, x_0) and (t_1, x_1) are the fixed end points of the curve. This kind of problems classified into two type problems with known boundary conditions and for undetermined boundary conditions.

2.3.1.1 Problems with Known Boundary Conditions:

The first kind of the point-point problems starts by considering what might be called the "simplest" variational problem with known boundary conditions, which can be formulated as the minimizing of the functional:

$$v[x] = \int_{t_0}^{t_1} F(t; x, x', x(t-\varphi_{01}), x(t-\varphi_{02}), \ldots, x(t-\varphi_{0m}), x'(t-\varphi_{11}), x'(t-\varphi_{12}), \ldots, x'(t-\varphi_{1m}))\, dt \quad \ldots\ldots(2.5)$$

satisfying the known boundary conditions:

$$\left.\begin{array}{l} x(t_0) = x_0,\ x(t_1) = x_1, \\ x(t_0-\eta_0) = x_{\eta_0} \text{ and } x(t_1-\eta_1) = x_{\eta_1} \end{array}\right\} \quad \ldots\ldots(2.6)$$

where $\eta_0 = \text{Max}\{\varphi_{ij}(t_0; x(t_0), x'(t_0)\}$, $\eta_1 = \text{Max}\{\varphi_{ij}(t_1; x(t_1), x'(t_1)\}$, x_0, x_1, x_{η_0}, x_{η_1} are given and called the boundary values of the problem and φ_{ij} are the delay functions when i refers to the order of the derivative with respect to the dependent variables for each (j = 1, 2, …, m), [Russak, 2002]. In the next theorem the extremum of the functional (2.5), where the class of admissible curves consist of all smooth functions joining its two fixed end points, will be found.

Theorem (2.5):

If F is a function with continuous first and second derivatives with respect to the argument t, and of the variables $x(t)$, $x(t-\varphi_{ij})$, $x'(t)$, $x'(t-\varphi_{ij})$, for (i = 0, 1 and j = 1, 2, …, m) satisfies the boundary conditions defined by eq.(2.6). Then the necessary condition for v[x] defined by eq.(2.5) to have an extremum is that x(t) must satisfies the following Euler's equation:

$$\left(F_x - \frac{d}{dt}F_{x'}\right) + \sum_{j=1}^{m}\left(\overline{F}_{x(t-\varphi_{0j})}(1-\varphi'_{0j}) - \frac{d}{dt}\left\{\overline{F}_{x'(t-\varphi_{1j})}(1-\varphi'_{1j})\right\}\right) = 0,$$

.................................(2.7)

where $\overline{F}$ is a symbol used for the function F with time lag functions.

Proof:

The first variation of v is:

$$\delta v = \int_{t_0}^{t_1} \delta F \, dt$$

$$= \int_{t_0}^{t_1} \{F_x\delta x + F_{x'}\delta x' + F_{x(t-\varphi_{01})}\delta x(t-\varphi_{01}) + F_{x(t-\varphi_{02})}\delta x(t-\varphi_{02})$$

$$+ \ldots + F_{x(t-\varphi_{0m})}\delta x(t-\varphi_{0m}) + F_{x'(t-\varphi_{11})}\delta x'(t-\varphi_{11}) +$$

$$F_{x'(t-\varphi_{12})}\delta x'(t-\varphi_{12}) + \ldots + F_{x'(t-\varphi_{1m})}\delta x'(t-\varphi_{1m})\} \, dt \quad .(2.8)$$

In this case, where the necessary condition for the functional to have an extremum; is that its first variation should be vanished, thus eq.(2.8) becomes:

$$\int_{t_0}^{t_1} \{F_x\delta x + F_{x'}\delta x'\}dt = -\int_{t_0}^{t_1} \{F_{x(t-\varphi_{01})}\delta x(t-\varphi_{01}) + F_{x(t-\varphi_{02})}\delta x(t-\varphi_{02}) +$$

$$\ldots + F_{x(t-\varphi_{0m})}\delta x(t-\varphi_{0m}) + F_{x'(t-\varphi_{11})}\delta x'(t-\varphi_{11}) +$$

$$F_{x'(t-\varphi_{12})}\delta x'(t-\varphi_{12}) + \ldots + F_{x'(t-\varphi_{1m})}\delta x'(t-\varphi_{1m})\} \, dt \quad .....(2.9)$$

Let $z_{ij} = (t - \varphi_{ij})$, i.e., $z_{ij} = z_{ij}(t; x, x')$, for ($i = 0, 1$ and $j = 1, 2, \ldots, m$) or equivalently $t = (z_{ij} + \varphi_{ij})$ with the same index of z and φ in each equation.

Let f_{ij} be the inverse function of z_{ij}, such that $t = f_{ij}(z_{ij})$, and hence:

$$dt = f'_{ij}(z_{ij})z'_{ij}\, dz = (1 - \varphi'_{ij})\, dz, \text{ for } (i = 0, 1 \text{ and } j = 1, 2, \ldots, m).$$

with the shifted boundary conditions (and hence shifted limits of integration) to $z_0 = t_0 - \eta_0$ and $z_1 = t_1 - \eta_1$.

Producing a function $\overline{F}$ that satisfies all values of time, and can be written as:

$$\overline{F} = F(t; x, x', x(z_{01}), x(z_{02}), \ldots, x(z_{0m}), x'(z_{11}), x'(z_{12}), \ldots, z'(z_{1m})) \qquad (2.9.a)$$

Consequently on using (2.9.a) and by renaming the variable of integration as t, the right hand side of eq.(2.9) can be represented as:

$$\int_{t_0}^{t_1} (F_{x(t-\varphi_{01})}\delta x(t-\varphi_{01}) + F_{x(t-\varphi_{02})}\delta x(t-\varphi_{02}) + \ldots + F_{x(t-\varphi_{0m})}\delta x(t-\varphi_{0m}) + F_{x'(t-\varphi_{11})}\delta x'(t-\varphi_{11}) + F_{x'(t-\varphi_{12})}\delta x'(t-\varphi_{12}) + \ldots + F_{x'(t-\varphi_{1m})}\delta x'(t-\varphi_{1m}))\, dt = \int_{t_0-\eta_0}^{t_1-\eta_1} (\overline{F}_{x(z_{01})}(1-\varphi'_{01})\delta x(z_{01}) + \overline{F}_{x(z_{02})}(1-\varphi'_{02})\delta x(z_{02}) + \ldots + \overline{F}_{x(z_{0m})}(1-\varphi'_{0m})\delta x(z_{0m}) + \overline{F}_{x'(z_{11})}(1-\varphi'_{11})\delta x'(z_{11}) + \overline{F}_{x'(z_{12})}(1-\varphi'_{12})\delta x'(z_{12}) + \ldots + \overline{F}_{x'(z_{1m})}(1-\varphi'_{1m})\delta x'(z_{1m}))\, dt \qquad (2.9.b)$$

Substituting eq.(2.9.b) in eq.(2.9), yields to:

$$\int_{t_0}^{t_1} (F_x\delta x + F_{x'}\delta x')\, dt = -\int_{t_0-\eta_0}^{t_1-\eta_1} (\overline{F}_{x(z_{01})}(1-\varphi'_{01})\delta x(z_{01}) +$$

$$\overline{F}_{x(z_{02})}(1-\varphi'_{02})\delta x(z_{02}) + \ldots + \overline{F}_{x(z_{0m})}(1-\varphi'_{0m})\delta x(z_{0m}) +$$

$$\overline{F}_{x'(z_{11})}(1-\varphi'_{11})\delta x'(z_{11}) + \overline{F}_{x'(z_{12})}(1-\varphi'_{12})\delta x'(z_{12}) + \ldots +$$

$$\overline{F}_{x'(z_{1m})}(1-\varphi'_{1m})\delta x'(z_{1m}))\, dt.$$

Integrating by parts the terms of the arbitrary functions $\delta x'$, and taking the boundary conditions into account, one can get:

$$\int_{t_0}^{t_1} F_{x'}\delta x'\, dt = F_{x'}\delta x\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left\{\frac{d}{dt}F_{x'}\delta x\right\} dt$$

$$= -\int_{t_0}^{t_1} \frac{d}{dt}F_{x'}\delta x$$

and

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\delta x'(z_{1j})\, dt = \overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\delta x(z_{1j})\Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d}{dt}\left\{\overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\right\} \delta x(z_{1j})\, dt$$

$$= \int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d}{dt}\left\{\overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\right\} \delta x(z_{1j})\, dt\,;$$

for all ($j = 1, 2, \ldots, m$).

Therefore, the increment of the first variation δv, becomes:

$$\delta v = \int_{t_0}^{t_1} \left\{ F_x - \frac{d}{dt} F_{x'} \right\} \delta x(t)\, dt + \int_{t_0-\eta_0}^{t_1-\eta_1} \sum_{j=1}^{m} \left\{ \overline{F}_{x(z_{0j})} (1-\varphi'_{0j}) \delta x(z_{0j}) - \right.$$

$$\left. \frac{d}{dt} \left\{ \overline{F}_{x(z_{1j})} (1-\varphi'_{1j}) \right\} \delta x(z_{1j}) \right\} dt\,.$$

By lemma (2.1), the fundamental necessary condition for v[x*] = v[x + δx] to have an extremum at x = x*, is:

$$\left(F_x - \frac{d}{dt} F_{x'} \right) + \sum_{j=1}^{m} \left(\overline{F}_{x(z_{0j})} (1-\varphi'_{0j}) - \frac{d}{dt} \left\{ \overline{F}_{x'(z_{1j})} (1-\varphi'_{1j}) \right\} \right) = 0\,,$$

where $z_{ij} = (t - \varphi_{ij})$, for (i = 0, 1 and j = 1, 2, ..., m). ■

2.3.1.2 Problems with Undetermined Boundary Conditions:

The second kind of the point-point problems are called problems with undetermined boundary conditions, where one or more values of the boundary conditions at the points t_0 and/or t_1 and/or $t_0 - \eta_0$ and/or $t_1 - \eta_1$ which are called the boundary values of the extremals, are not specified, i.e., one or more of the ends of extremals:

$$x(t_0) = x_0,\ x(t_1) = x_1,\ x(t_0 - \eta_0) = x_{\eta_0} \ \text{ and } \ x(t_1 - \eta_1) = x_{\eta_1}\,,$$

are not specified, [Russak, 2002]. In this case the Euler's equation for this problem remains the same, but additional conditions must be supplemented by the terms of the function F at the undetermined end points. For example, the problem with undetermined boundary values x_1 and x_{η_1}, which can be formed as $\min\limits_{\substack{x(t):x(t_0)=x_0,\\ x(t_0-\eta_0)=x_{\eta_0}\\ x(t_1)=x|_{t=t_1}\\ x(t_1-\eta_1)=x|_{t=t_1-\eta_1}}} v[x]$, where:

$$v[x] = \int_{t_0}^{t_1} F(t; x, x', x(t-\varphi_{01}), x(t-\varphi_{02}), \ldots, x(t-\varphi_{0m}), x'(t-\varphi_{11}), x'(t-\varphi_{12}), \ldots, x'(t-\varphi_{1m}))\, dt.$$

The necessary condition of the above functional is given in the next theorem:

Theorem (2.6):

If F is a function with continuous first and second derivatives with respect to the argument t, and of the variables x(t), $x(t-\varphi_{ij})$, $x'(t)$, $x'(t-\varphi_{ij})$, for (i = 0, 1 and j = 1, 2, …, m) satisfies the boundary conditions:

$$x(t_0) = x_0 \text{ and } x(t_0-\eta_0) = x_{\eta_0}.$$

Then the necessary condition for v[x] defined by eq.(2.5) to have an extremum is that x(t) must satisfies the following Euler's equation:

$$\left(F_x - \frac{d}{dt}F_{x'}\right) + \sum_{j=1}^{m}\left(\overline{F}_{x(t-\varphi_{0j})}(1-\varphi'_{0j}) - \frac{d}{dt}\left\{\overline{F}_{x'(t-\varphi_{1j})}(1-\varphi'_{1j})\right\}\right) = 0,$$

as well as:

$$F_{x'}\Big|_{t=t_1} = 0 \text{ and } \overline{F}_{x'(t-\varphi_{1j})}(1-\varphi'_{1j})\Big|_{t=t_1-\eta_1} = 0 \quad \text{..........................(2.10)}$$

for all (j = 1, 2,…, m).

Proof:

The first variation of the functional (2.5) is given by:

$$\delta v = \int_{t_0}^{t_1} \delta F \, dt$$

$$= \int_{t_0}^{t_1} \{F_x \delta x + F_{x'} \delta x' + F_{x(t-\varphi_{01})} \delta x(t-\varphi_{01}) + F_{x(t-\varphi_{02})} \delta x(t-\varphi_{02}) + \ldots + F_{x(t-\varphi_{0m})} \delta x(t-\varphi_{0m}) + F_{x'(t-\varphi_{11})} \delta x'(t-\varphi_{11}) + F_{x'(t-\varphi_{12})} \delta x'(t-\varphi_{12}) + \ldots + F_{x'(t-\varphi_{1m})} \delta x'(t-\varphi_{1m})\} \, dt \quad \ldots(2.11)$$

Since for the extremal solution, one must have $\delta v[x] = 0$, then eq.(2.11) may be rewritten as:

$$\int_{t_0}^{t_1} \{F_x \delta x + F_{x'} \delta x'\} \, dt = -\int_{t_0}^{t_1} \{F_{x(t-\varphi_{01})} \delta x(t-\varphi_{01}) + F_{x(t-\varphi_{02})} \delta x(t-\varphi_{02}) + \ldots + F_{x(t-\varphi_{0m})} \delta x(t-\varphi_{0m}) + F_{x'(t-\varphi_{11})} \delta x'(t-\varphi_{11}) + F_{x'(t-\varphi_{12})} \delta x'(t-\varphi_{12}) + \ldots + F_{x'(t-\varphi_{1m})} \delta x'(t-\varphi_{1m})\} \, dt \quad \ldots(2.12)$$

Using the same functions defined in eq.(2.9.a) and substitute in eq.(2.12), one can get:

$$\int_{t_0}^{t_1} \{F_x \delta x + F_{x'} \delta x'\} \, dt = -\int_{t_0-\eta_0}^{t_1-\eta_1} \{\overline{F}_{x(z_{01})}(1-\varphi'_{01})\delta x(z_{01}) + \overline{F}_{x(z_{02})}(1-\varphi'_{02})\delta x(z_{02}) + \ldots + \overline{F}_{x(z_{0m})}(1-\varphi'_{0m})\delta x(z_{0m}) + \overline{F}_{x'(z_{11})}(1-\varphi'_{11})\delta x'(z_{11}) + \overline{F}_{x'(z_{12})}(1-\varphi'_{12})\delta x'(z_{12}) + \ldots + \overline{F}_{x'(z_{1m})}(1-\varphi'_{1m})\delta x'(z_{1m})\} \, dt.$$

Similarly, integrating by parts and taking the boundary conditions into account, gives:

$$\int_{t_0}^{t_1} F_{x'}\delta x'\, dt = F_{x'}\delta x\Big|_{t=t_1} - \int_{t_0}^{t_1}\left\{\frac{d}{dt}F_{x'}\right\}\delta x\, dt$$

and

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\delta x'(z_{1j})\, dt = \overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\delta x(z_{1j})\Big|_{t=t_1-\eta_1} - \int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d}{dt}\left\{\overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\right\}\delta x(z_{1j})\, dt\,;$$

for all (j = 1, 2, …, m).

Therefore, the increment of the functional δv, becomes:

$$\delta v[x] = F_{x'}\delta x\Big|_{t=t_1} + \int_{t_0}^{t_1}\left\{F_x - \frac{d}{dt}F_{x'}\right\}\delta x\, dt + \sum_{j=1}^{m}\left\{\overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\delta x(z_{1j})\Big|_{t=t_1-\eta_1} + \int_{t_0-\eta_0}^{t_1-\eta_1}\left\{\overline{F}_{x(z_{0j})}(1-\varphi'_{0j})\delta x(z_{0j}) - \frac{d}{dt}\left\{\overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\right\}\delta x(z_{1j})\right\}\right\} dt\,.$$

Thus, using lemma (2.1), the stationary condition associated with Euler's equation (2.7) is:

$$F_{x'}\Big|_{t=t_1} = 0 \text{ and } \overline{F}_{x'(z_{1j})}(1-\varphi'_{1j})\Big|_{t=t_1-\eta_1} = 0; \text{ for all } (j = 1, 2, \ldots, m),$$

where $z_{1j} = (t - \varphi_{1j})$, for all (j = 1, 2, …, m). ■

Remark (2.7):

Similar conditions to (2.10) may be derived for the point $t = t_0$ and $t = t_0 - \eta_0$, if the value of the dependent variables at these points are not prescribed.

2.3.1.3 Generalization to Functionals of More Complicated Forms:

In this subsection, additional fundamental necessary conditions for functionals with more complicated integrands containing time-lag functions are derived.

In order to deal with such type of functionals, for simplicity the functional v[x] will be reduced in problems of ODDEv's with single delay variables.

Theorem (2.8):

Consider the functional:

$$v[x] = \int_{t_0}^{t_1} F(t; x, x', \ldots, x^{(n)}, x(t - \varphi_0), x'(t - \varphi_1), \ldots, x^{(n)}(t - \varphi_n))\, dt$$

................................(2.13)

where $x \in C^1[t_0, t_1]$, and satisfy the boundary conditions:

$x(t_0) = x_0,\ x'(t_0) = x'_0,\ \ldots,\ x^{(n-1)}(t_0) = x_0^{(n-1)}$,

$$x(t_1) = x_1,\ x'(t_1) = x'_1,\ \ldots,\ x^{(n-1)}(t_1) = x_1^{(n-1)},$$

$$x(t_0 - \eta_0) = x_{\eta_0},\ x'(t_0 - \eta_0) = x'_{\eta_0},\ \ldots,\ x^{(n-1)}(t_0 - \eta_0) = x_{\eta_0}^{(n-1)}$$

and

$$x(t_1 - \eta_1) = x_{\eta_1},\ x'(t_1 - \eta_1) = x'_{\eta_1},\ \ldots,\ x^{(n-1)}(t_1 - \eta_1) = x_{\eta_1}^{(n-1)} \quad \text{.....(2.14)}$$

Then the necessary condition for v[x] to have an extremum is that x(t) satisfies the following Euler's equation:

$$\left(F_x + \sum_{i=1}^{n}(-1)^i \frac{d^i}{dt^i} F_{x^{(i)}}\right) + \left(\overline{F}_{x(t-\varphi_0)}(1-\varphi'_0) + \sum_{i=1}^{n}(-1)^i \frac{d^i}{dt^i}\left\{\overline{F}_{x^{(i)}(t-\varphi_i)}(1-\varphi'_i)\right\}\right) = 0 \quad \text{..................(2.15)}$$

Proof:

The first variation of the functional v[x] is:

$$\delta v = \int_{t_0}^{t_1} (F_x\delta x + F_{x'}\delta x' + \ldots + F_{x^{(n)}}\delta x^{(n)} + F_{x(t-\varphi_0)}\delta x(t-\varphi_0) + F_{x'(t-\varphi_1)}\delta x'(t-\varphi_1) + \ldots + F_{x^{(n)}(t-\varphi_n)}\delta x^{(n)}(t-\varphi_n))\, dt,$$

and since $\delta v = 0$, then:

$$\int_{t_0}^{t_1} (F_x\delta x + F_{x'}\delta x' + \ldots + F_{x^{(n)}}\delta x^{(n)})\, dt = -\int_{t_0}^{t_1} (F_{x(t-\varphi_0)}\delta x(t-\varphi_0) + F_{x'(t-\varphi_1)}\delta x'(t-\varphi_1) + \ldots + F_{x^{(n)}(t-\varphi_n)}\delta x^{(n)}(t-\varphi_n))\, dt.$$

Letting $z_i = (t - \varphi_i)$, for $(i = 0, 1, \ldots, n)$ or equivalently $t = (z_i + \varphi_i) = f_i(z_i)$, where $f_i(z_i)$ is the inverse function of z_i with the same index of z and φ in each equation. Hence $dt = f_i'(z_i) z_i' dz = (1 - \varphi'_i)dz$, for $(i = 0, 1, \ldots, n)$.

Letting

$$\overline{F} = F\{t; x, x', \ldots, x^{(n)}, x(z_0), x'(z_1), \ldots, x^{(n)}(z_n)\} \quad \ldots\ldots(2.16)$$

Therefore:

$$\int_{t_0}^{t_1} (F_x\delta x + F_{x'}\delta x' + \ldots + F_{x^{(n)}}\delta x^{(n)})\, dt = -\int_{t_0-\eta_0}^{t_1-\eta_1} (\overline{F}_{x(z_0)}(1 - \varphi'_0)\delta x(z_0) + \overline{F}_{x'(z_1)}(1 - \varphi'_1)\delta x'(z_1) + \ldots + \overline{F}_{x^{(n)}(z_n)}(1 - \varphi'_n)\delta x^{(n)}(z_n))\, dt \quad \ldots\ldots(2.17)$$

Integrating each term in eq.(2.17), containing $\delta x', \delta x'', \ldots, \delta x^{(n)}$, by parts n times where n refers to the n number of derivation in that term , give:

$$\int_{t_0}^{t_1} F_{x'}\delta x'\, dt = F_{x'}\delta x\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\left\{\frac{d}{dt}F_{x'}\right\}\delta x\, dt,$$

$$\int_{t_0}^{t_1} F_{x''}\delta x''\, dt = F_{x''}\delta x'\Big|_{t_0}^{t_1} - \frac{d}{dt}F_{x''}\delta x\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\left\{\frac{d^2}{dt^2}F_{x''}\right\}\delta x\, dt,$$

$$\vdots$$

$$\int_{t_0}^{t_1} F_{x^{(n)}}\delta x^{(n)}\, dt = F_{x^{(n)}}\delta x^{(n-1)}\Big|_{t_0}^{t_1} - \frac{d}{dt}F_{x^{(n)}}\delta x^{(n-2)}\Big|_{t_0}^{t_1} + \ldots + (-1)^n\int_{t_0}^{t_1}\left\{\frac{d^n}{dt^n}F_{x^{(n)}}\right\}\delta x\, dt.$$

Doing similarly, for the right hand side of eq.(2.17), yields to:

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \{\overline{F}_{x'(z_1)}(1-\varphi'_1)\,\delta x'(z_1) = \overline{F}_{x'}(1-\varphi'_1)\delta x(z_1)\Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \left\{\frac{d}{dt}\overline{F}_{x'(z_1)}(1-\varphi'_1)\right\}\delta x(z_1)\,dt\,,$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \{\overline{F}_{x''(z_2)}(1-\varphi'_2)\delta x''(z_2) = \overline{F}_{x''(z_2)}(1-\varphi'_2)\delta x'(z_2)\Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\frac{d}{dt}\left\{\overline{F}_{x''(z_2)}(1-\varphi'_2)\right\}\delta x(z_2)\Big|_{t_0-\eta_0}^{t_1-\eta_1} +$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d^2}{dt^2}\left\{\overline{F}_{x''(z_2)}(1-\varphi'_2)\right\}\delta x(z_2)\,dt\,,$$

$$\vdots$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \{\overline{F}_{x^{(n)}(z_n)}(1-\varphi'_n)\delta x^{(n)}(z_n) = \overline{F}_{x^{(n)}(z_n)}(1-\varphi'_n)\delta x^{(n-1)}(z_n)\Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\frac{d}{dt}\left\{\overline{F}_{x^{(n)}(z_n)}(1-\varphi'_n)\right\}\delta x^{(n-2)}(z_n)\Big|_{t_0-\eta_0}^{t_1-\eta_1} + \ldots +$$

$$(-1)^n \int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d^n}{dt^n}\left\{\overline{F}_{x^{(n)}(z_n)}(1-\varphi'_n)\right\}\delta x(z_n)\,dt\,.$$

Taking the boundary conditions into account, then the first variation δv may be rewritten as:

$$\delta v = \int_{t_0}^{t_1} \left\{ F_x - \frac{d}{dt} F_{x'} + \frac{d^2}{dt^2} F_{x''} + \ldots + (-1)^n \frac{d^n}{dt^n} F_{x^{(n)}} \right\} \delta x \, dt \; +$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \left\{ \overline{F}_{x(z_0)} (1-\varphi_0') \delta x(z_0) - \frac{d}{dt} \left\{ \overline{F}_{x'(z_1)} (1-\varphi_1') \right\} \delta x(z_1) + \right.$$

$$\frac{d^2}{dt^2} \left\{ \overline{F}_{x''(z_2)} (1-\varphi_2') \right\} \delta x(z_2) - \ldots +$$

$$\left. (-1)^n \frac{d^n}{dt^n} \left\{ \overline{F}_{x^{(n)}(z_n)} (1-\varphi_n') \right\} \delta x(z_n) \right\} dt = 0.$$

Since δx is arbitrary; therefore, according to lemma (2.1), $\delta v = 0$ which give the fundamental necessary condition:

$$\left(F_x + \sum_{i=1}^{n} (-1)^i \frac{d^i}{dt^i} F_{x^{(i)}} \right) + \left(\overline{F}_{x(z_0)} (1-\varphi_0') + \right.$$

$$\left. \sum_{i=1}^{n} (-1)^i \frac{d^i}{dt^i} \left\{ \overline{F}_{x^{(i)}(z_i)} (1-\varphi_i') \right\} \right) = 0,$$

where $z_i = (t - \varphi_i)$, for each $(i = 0, 1, \ldots, n)$. ■

Theorem (2.9):

Consider the functional:

$$v[x_1, x_2, \ldots, x_m] = \int_{t_0}^{t_1} F(t; x_1, x_2, \ldots, x_m, x'_1, x'_2, \ldots, x'_m, \ldots, x_1^{(n)},$$

$$x_2^{(n)}, \ldots, x_m^{(n)}, \ldots, x_1(t-\varphi_{10}), x_2(t-\varphi_{20}), \ldots, x_m(t-\varphi_{m0}),$$

$$x'_1(t-\varphi_{11}), x'_2(t-\varphi_{21}), \ldots, x'_m(t-\varphi_{m1}), \ldots, x_1^{(n)}(t-\varphi_{1n}),$$

$$x_2^{(n)}(t-\varphi_{2n}), \ldots, x_m^{(n)}(t-\varphi_{mn}))\, dt \; \ldots\ldots\ldots\ldots(2.18)$$

where φ_{ij} (i = 1, 2, …, m and j = 0, 1, …, n) are delay functions where j refers to derivative of the i-th dependent variable, defined on the set of functions $x_i(t)$, where $x_i \in C^n[t_0, t_1]$ for (i = 1, 2, …, m), and satisfy the boundary conditions:

$$x_i(t_0) = x_{i_0}, \; x'_i(t_0) = x'_{i_0}, \; \ldots, \; x_i^{(n-1)}(t_0) = x_{i_0}^{(n-1)},$$

$$x_i(t_1) = x_{i_1}, \; x'_i(t_1) = x'_{i_1}, \; \ldots, \; x_i^{(n-1)}(t_1) = x_{i_1}^{(n-1)},$$

$$x_i(t_0 - \eta_{i_0}) = x_{\eta_{i_0}}, \; x'_i(t_0 - \eta_{i_0}) = x'_{\eta_{i_0}}, \; \ldots, \; x_i^{(n-1)}(t_0 - \eta_{i_0}) = x_{\eta_{i_0}}^{(n-1)}$$

and

$$x_i(t_1 - \eta_{i_1}) = x_{\eta_{i_1}}, \; x'_i(t_1 - \eta_{i_1}) = x'_{\eta_{i_1}}, \; \ldots, \; x_i^{(n-1)}(t_1 - \eta_{i_1}) = x_{\eta_{i_1}}^{(n-1)},$$

for all (i = 1, 2, …, m). Then the necessary condition for $v[x_1, x_2, \ldots, x_m]$ to have an extremum is that $x_i(t)$ must satisfies the following system:

$$\left(F_{x_1} + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} F_{x_1^{(j)}} \right) + \left(\overline{F}_{x_1(t-\varphi_{10})}(1-\varphi'_{10}) + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} \left\{ \overline{F}_{x_1^{(j)}(t-\varphi_{1j})}(1-\varphi'_{1j}) \right\} \right) = 0$$

$$\left(F_{x_2} + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} F_{x_2^{(j)}} \right) + \left(\overline{F}_{x_2(t-\varphi_{20})}(1-\varphi'_{20}) + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} \left\{ \overline{F}_{x_2^{(j)}(t-\varphi_{2j})}(1-\varphi'_{2j}) \right\} \right) = 0$$

$$\vdots$$

$$\left(F_{x_m} + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} F_{x_m^{(j)}} \right) + \left(\overline{F}_{x_m(t-\varphi_{m0})}(1-\varphi'_{m0}) + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} \left\{ \overline{F}_{x_m^{(j)}(t-\varphi_{mj})}(1-\varphi'_{mj}) \right\} \right) = 0$$

...............................(2.19)

Proof:

In order to find the necessary condition for a functional depending on several functions of one independent variable and involving derivatives of higher order, only one of the functions $x_i(t)$ for some (i = 1, 2, …, m) will be varied, at a time, keeping all the other functions fixed.

By so doing, the functional $v[x_1, x_2, \ldots, x_m]$ turns into a functional depending only on one function, that is being varied, for instance on $x_k(t)$, for some k, then:

$$v[x_1, x_2, \ldots, x_m] = \overline{v}[x_k] \quad \text{...(2.20)}$$

for some k.

Consequently, from theorem (2.8), the function $x_k = x_k^*$ that makes the functional (2.20) has an extremum must satisfy the necessary condition:

$$\left(F_{x_k} + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} F_{x_k^{(j)}} \right) + \left(\overline{F}_{x_k(t-\varphi_{k0})}(1-\varphi'_{k0}) + \sum_{j=1}^{n} (-1)^j \frac{d^j}{dt^j} \left\{ \overline{F}_{x_k^{(j)}(t-\varphi_{kj})}(1-\varphi'_{kj}) \right\} \right) = 0$$

...............................(2.21)

for some k.

Since, the argument applies to each function x_i, for (i = 1, 2, …, m), then a system of m delay-differential equations (2.21) is obtained, which is the fundamental necessary condition of the functional (2.18). ■

2.3.2 The Point-Curve Problems of ODDEv's:

The second kind of variational problems with time-lag functions are called the point-curve problems with moving boundaries, which are variational problems in which the upper point of the admissible functions is moving along certain boundary curve. Which means that the end point $(t_1; x_1)$ is variable and can move turning to $(t_1 + \delta t_1; x_1 + \delta x_1)$ and the other point $(t_0; x_0)$ is fixed, [Elsgolc, 1962]. For this type of problems two cases may occur:

Case (i):

In this case; problems for which the movable end point t_1 can move freely along certain line parallel to the y-axis, in fact at this point the admissible function x is not specified. In this case admissible functions must fulfill the natural boundary conditions which will be discussed later.

Case (ii):

In this case; problems with movable end point move freely along a given curve $x = G(t)$. In this case, the admissible function $x(t)$ and the end point must satisfy the necessary conditions, called the transversility conditions, [Memarbashi, 2006].

Since, in the above two cases the class of admissible functions is more rich than the case of point-point problems, the function x that realizes the extremum in the above cases must satisfy the fundamental necessary conditions for the case of the point-point problems

(Euler's equations), in addition to the several conditions for movable boundary points which will be arise.

Hereinafter some theorems will be introduced, which establish the natural and transversility conditions for some type of functionals.

Theorem (2.10):

If F is a function with continuous first and second derivatives with respect to the argument t, and of the variables x(t), $x(t - \varphi_0)$, $x'(t)$, $x'(t - \varphi_1)$ satisfies the boundary conditions $x(t_0) = x_0$, $x(t_0 - \eta_0) = x_{\eta_0}$. Then the fundamental necessary condition for an extremum of the functional:

$$v[x] = \int_{t_0}^{t_1} F(t; x, x', x(t-\varphi_0), x'(t-\varphi_1))\, dt \qquad (2.22)$$

is given by:

$$\left(F - x'F_{x'}\big|_{t=t_1} - x'(t_1 - \eta_1)\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} \right) = 0,$$

$$F_{x'}\big|_{t=t_1} = 0 \text{ and } \left(\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} \right) = 0;$$

or by the transversility condition:

$$\left(F\big|_{t=t_1} + (G' - x')F_{x'}\big|_{t=t_1} + (H' - x')\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} \right) = 0;$$

both with Euler's equation:

$$\left(F_x - \frac{d}{dt}F_{x'} \right) + \left(\overline{F}_{x(t-\varphi_0)}(1-\varphi_0') - \frac{d}{dt}\left\{ \overline{F}_{x'(t-\varphi_1)}(1-\varphi_1') \right\} \right) = 0.$$

Proof:

The linear part of the increment δv of the functional v[x] may be expressed as:

$$\delta v = \int_{t_0}^{t_1+\delta t_1} F(t; x + \delta x, x' + \delta x', x(t-\varphi_0) + \delta x(t-\varphi_0), x'(t-\varphi_1) + \delta x'(t-\varphi_1))\, dt - \int_{t_0}^{t_1} F\, dt$$

$$= \int_{t_1}^{t_1+\delta t_1} F(t; x + \delta x, x' + \delta x', x(t-\varphi_0) + \delta x(t-\varphi_0), x'(t-\varphi_1) + \delta x'(t-\varphi_1))\, dt,$$

or equivalently:

$$\delta v = \int_{t_1}^{t_1+\delta t_1} F(t; x + \delta x, x' + \delta x', x(t-\varphi_0) + \delta x(t-\varphi_0), x'(t-\varphi_1) + \delta x'(t-\varphi_1))\, dt + \int_{t_0}^{t_1} \Delta F\, dt \qquad \text{....................................(2.23)}$$

where ΔF refers to the linear part of F. Since t_0 and t_1 are fixed, then this implies that:

$$\delta v = \int_{t_0}^{t_1} \Delta F\, dt \longrightarrow 0$$

and with the aid of the Mean Value theorem, the first integral of eq.(2.23) will be transformed to:

$$\int_{t_1}^{t_1+\delta t_1} F(t;\, x + \delta x,\, x' + \delta x',\, x(t - \varphi_0) + \delta x(t - \varphi_0),\, x'(t - \varphi_1) +$$

$$\delta x'(t - \varphi_1))\, dt = F(t; x, x', x(t-\varphi_0), x'(t-\varphi_1))\Big|_{t=t_1} \delta t_1$$

$$= F\Big|_{t=t_1} \delta t_1 \quad \text{...} (2.23.a)$$

The second integral of eq.(2.23) can be transformed by using the Taylor series as:

$$\int_{t_0}^{t_1} \Delta F\, dt = \int_{t_0}^{t_1} \delta F\, dt + R_1 \quad \text{..} (2.23.b)$$

where R_1 represents the higher variation in F.

From (2.23.a) and (2.23.b), the following result is obtained:

$$\delta v = F\Big|_{t=t_1} \delta t_1 + \int_{t_0}^{t_1} \delta F\, dt.$$

Justification for $\delta v = 0$, as the solution of the necessary condition, then:

$$F\Big|_{t=t_1} \delta t_1 + \int_{t_0}^{t_1} \{F_x \delta x + F_{x'} \delta x'\}\, dt = -\int_{t_0-\eta_0}^{t_1-\eta_1} \Big\{F_{x(t-\varphi_0)} \delta x(t-\varphi_0) +$$

$$F_{x'(t-\varphi_1)} \delta x'(t-\varphi_1)\Big\}\, dt\,.$$

Using the same functions defined in eq.(2.16), for (i = 0, 1), gives:

$$F\Big|_{t=t_1} \delta t_1 + \int_{t_0}^{t_1} \{F_x \delta x + F_{x'} \delta x'\}\, dt = -\int_{t_0-\eta_0}^{t_1-\eta_1} \Big\{\overline{F}_{x(z_0)} (1-\varphi_0') \delta x(z_0) +$$

$$\overline{F}_{x'(z_1)} (1-\varphi_1') \delta x'(z_1)\Big\}\, dt\,.$$

Then after integrating by parts, one may have:

$$F\Big|_{t=t_1}\delta t_1 + F_{x'}\delta x\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\left\{F_x - \frac{d}{dt}F_{x'}\right\}\delta x\,dt = -\overline{F}_{x'(z_1)}(1-\varphi_1')\delta x(z_1)\Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1}\left\{\overline{F}_{x(z_0)}(1-\varphi_0')\delta x(z_0) - \frac{d}{dt}\left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\right\}\delta x'(z_1)\right\}dt\ ,$$

where,

$$\int_{t_0}^{t_1}\left\{F_x - \frac{d}{dt}F_{x'}\right\}\delta x\,dt\ +$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1}\left\{\overline{F}_{x(z_0)}(1-\varphi_0')\delta x(z_0) - \frac{d}{dt}\left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\right\}\delta x'(z_1)\right\}dt\ ,$$

will leads to the fundamental necessary condition with nonmovable boundary points for the general Euler's equation:

$$\left(F_x - \frac{d}{dt}F_{x'}\right) + \left(\overline{F}_{x(z_0)}(1-\varphi_0') - \frac{d}{dt}\left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\right\}\right) = 0\,.$$

Since the lower end points are fixed, it follows that $\delta x\Big|_{\substack{t=t_0\\ t=t_0-\eta_0}} = 0$, and the values of the functional are taken only along extremals. Therefore:

$$\delta v = F\Big|_{t=t_1}\delta t_1 + F_{x'}\delta x\Big|_{t=t_1} + \overline{F}_{x'(z_1)}(1-\varphi_1')\delta x(z_1)\Big|_{t=t_1-\eta_1} \quad ....(2.24)$$

Since, [El'sgol'c, 1964]:

$$\delta x\Big|_{t=t_1} = \delta x_1 - x'(t_1)\delta t_1$$

and

$$\delta x(z_1)\Big|_{t=t_1-\eta_1} = \delta x_{\eta_1} - x'(t_1-\eta_1)\delta t_1 \quad ....(2.24.a)$$

Substituting the result (2.24.a) in eq.(2.24), one can get:

$$\delta v = F\big|_{t=t_1}\delta t_1 + F_{x'}\big|_{t=t_1}(\delta x_1 - x'(t_1)\delta t_1) + \overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1}$$

$$(\delta x_{\eta_1} - x'(t_1-\eta_1)\delta t_1)$$

$$= \left\{ F\big|_{t=t_1} - x' F_{x'}\big|_{t=t_1} - x'(t_1-\eta_1)\overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} \right\}\delta t_1 +$$

$$F_{x'}\big|_{t=t_1}\delta x_1 + \overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1}\delta x_{\eta_1}.$$

Then the fundamental necessary condition for an extremum will takes the form:

$$\{ F\big|_{t=t_1} - x' F_{x'}\big|_{t=t_1} - x'(t_1-\eta_1)\overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} \}\delta t_1 +$$

$$F_{x'}\big|_{t=t_1}\delta x_1 + \overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1}\delta x_{\eta_1} = 0 \qquad \text{.............................(2.25)}$$

If the arbitrary functions δt_1, δx_1 and δx_{η_1} are independent, then the fundamental necessary condition which is called the natural condition is:

$$F\big|_{t=t_1} - x' F_{x'}\big|_{t=t_1} - x'(t_1-\eta_1)\overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} = 0,\ F_{x'}\big|_{t=t_1} = 0$$

and $\overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} = 0.$

While, if δx_1 is dependent on δt_1, and the end point $(t_1; x_1)$ can move along a certain curve $x_1 = G(t_1)$, implies that; $\delta x_1 = G'(t_1)\delta t_1$, and hence eq.(2.25) becomes:

$$\{ F\big|_{t=t_1} - x' F_{x'}\big|_{t=t_1} - x'(t_1-\eta_1)\overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1} \}\delta t_1 +$$

$$F_{x'}\big|_{t=t_1}(G'(t_1)\delta t_1) + \overline{F}_{x'(z_1)}(1-\varphi_1')\big|_{t=t_1-\eta_1}\delta x_{\eta_1} = 0.$$

Then the necessary condition in this case is:

$$F\Big|_{t=t_1} + (G' - x')\,F_{x'}\Big|_{t=t_1} - x'(t_1 - \eta_1)\,\overline{F}_{x'(z_1)}(1-\varphi_1')\Big|_{t=t_1-\eta_1} = 0$$

and

$$\overline{F}_{x'(z_1)}(1-\varphi_1')\Big|_{t=t_1-\eta_1} = 0.$$

Alternatively, if δx_{η_1} is dependent on δt_1, such that $x_{\eta_1} = H(t_1 - \eta_1)$, implies that $\delta x_{\eta_1} = H'(t_1 - \eta_1)\delta t_1$, then the necessary condition will takes the form:

$$\left(F\Big|_{t=t_1} - x'\,F_{x'}\Big|_{t=t_1} + (H' - x')\,\overline{F}_{x'(z_1)}(1-\varphi_1')\Big|_{t=t_1-\eta_1} \right)=0$$

and

$$F_{x'}\Big|_{t=t_1}=0.$$

If both (δx_1 and δx_{η_1}) are dependent on δt_1, then the necessary condition becomes:

$$\left(F\Big|_{t=t_1} + (G' - x')\,F_{x'}\Big|_{t=t_1} + (H' - x')\,\overline{F}_{x'(z_1)}(1-\varphi_1')\Big|_{t=t_1-\eta_1} \right) = 0,$$

where $z_1 = (t - \varphi_1)$, and the last condition is called the transversility condition.

All the above cases of necessary conditions are associated with Euler's equation (2.7). ■

Theorem (2.11):

Let F be a function with continuous first and second partial derivatives with respect to the argument t and to the variables $x_i(t)$,

$x_i(t - \varphi_{ij})$, (i =1, 2, …, m and j =0, 1, …, n); and their derivatives with respect to the movable boundary at the point $(t_1; x_{11}, x_{21}, \ldots, x_{m1})$, and satisfies the boundary conditions:

$$x_i(t_0) = x_{i_0},\ x'_i(t_0) = x'_{i_0},\ \ldots,\ x_i^{(n-1)}(t_0) = x_{i_0}^{(n-1)}$$

and

$$x_i(t_0 - \eta_{i_0}) = x_{\eta_{i_0}},\ x'_i(t_0 - \eta_0) = x'_{\eta_{i_0}},\ldots,\ x_i^{(n-1)}(t_0 - \eta_{i_0}) = x_{\eta_{i_0}}^{(n-1)}.$$

Then the fundamental necessary condition for an extremum for the functional (2.18), is given by:

$$\left(F\Big|_{t=t_1} - \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{(j)} \sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}F_{x_i^{(k)}}\right\}\Bigg|_{t=t_1} - \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{(j)}(t_1-\eta_{i_1}) \right.$$
$$\left. \sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}\left\{\overline{F}_{x_i^{(k)}(t-\varphi_{i_k})}(1-\varphi'_{i_k})\right\}\right\}\Bigg|_{t=t_1-\eta_1} \right) = 0,$$

$$\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}F_{x_i^{(k)}}\right\}\Bigg|_{t=t_1} = 0$$

and

$$\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}\left\{\overline{F}_{x_i^{(k)}(t-\varphi_{i_k})}(1-\varphi'_{i_k})\right\}\right\}\Bigg|_{t=t_1-\eta_1} = 0,$$

for each (i = 1, 2, …, m and j = 1, 2, …, n); and the transversility condition is:

$$\left(F\Big|_{t=t_1} + \sum_{i=1}^{m}\sum_{j=1}^{n}(G'_{ij} - x_i^{(j)})\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}F_{x_i^{(k)}}\right\}\Bigg|_{t=t_1} + \right.$$

$$\left.\sum_{i=1}^{m}\sum_{j=1}^{n}(H'_{ij} - x_i^{(j)})\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}\left\{\overline{F}_{x_i^{(k)}(t-\varphi_{i_k})}(1-\varphi'_{i_k})\right\}\right\}\Bigg|_{t=t_1-\eta_1}\right)=0;$$

all with general Euler's equation (2.19).

Proof:

The first variation of the functional v is:

$$\delta v = \int_{t_1}^{t_1+\delta t_1} F(t; x_1 + \delta x_1, x_2 + \delta x_2, \ldots, x_m + \delta x_m, x'_1 + \delta x'_1, x'_2 +$$

$$\delta x'_2, \ldots, x'_m + \delta x'_m, \ldots, x_1^{(n)} + \delta x_1^{(n)}, x_2^{(n)} + \delta x_2^{(n)}, \ldots,$$

$$x_m^{(n)} + \delta x_m^{(n)}, x_1(t - \varphi_{10}) + \delta x_1(t - \varphi_{10}), x_2(t - \varphi_{20}) + \delta x_2(t -$$

$$\varphi_{20}), \ldots, x_m(t - \varphi_{m0}) + \delta x_m(t - \varphi_{m0}), x'_1(t - \varphi_{11}) + \delta x'_1(t -$$

$$\varphi_{11}), x'_2(t - \varphi_{21}) + \delta x'_2(t - \varphi_{21}), \ldots, x'_m(t - \varphi_{m1}) + \delta x'_m(t -$$

$$\varphi_{m1}),\ldots, x_1^{(n)}(t - \varphi_{1n}) + \delta x_1^{(n)}(t - \varphi_{1n}), x_2^{(n)}(t - \varphi_{2n}) + \delta x_2^{(n)}(t$$

$$- \varphi_{2n}), \ldots, x_m^{(n)}(t - \varphi_{mn}) + \delta x_m^{(n)}(t - \varphi_{mn}))\, dt + \int_{t_0}^{t_1}\Delta F\, dt \quad \ldots(2.26)$$

Applying multivariable Mean Value theorem on the 1st integral of eq.(2.26), and Taylor series on the remaining terms, gives:

$$\delta v = F\Big|_{t=t_1}\delta t_1 + \int_{t_0}^{t_1}\delta F\, dt,$$

where $\delta F = \sum_{i=1}^{m}\sum_{j=0}^{n}\left\{F_{x_i^{(j)}}\delta x_i^{(j)} + \overline{F}_{x_i^{(j)}(z_{ij})}(1-\varphi'_{ij})\delta x^{(j)}(z_{ij})\right\}$, $z_{ij} = (t - \varphi_{ij})$ and

$\overline{F}$ is a symbol used for F with time lags.

Taking the increment $\delta v = 0$, it follows that:

$$F\Big|_{t=t_1}\delta t_1 + \int_{t_0}^{t_1}\sum_{i=1}^{m}\sum_{j=0}^{n}\left\{F_{x_i^{(j)}}\delta x_i^{(j)}\right\}dt = -\int_{t_0-\eta_0}^{t_1-\eta_1}\left\{\sum_{i=1}^{m}\sum_{j=0}^{n}\overline{F}_{x_i^{(j)}(z_{ij})}(1-\varphi'_{ij})\delta x^{(j)}(z_{ij})\right\}dt\,.$$

Integrating by parts terms of the arbitrary functions with the higher derivatives and taking into consideration that $\delta x_i = \delta x'_i = \ldots = \delta x_i^{(n-1)}\Big|_{t=t_0} = 0$, for all (i = 1, 2, …, m), one obtains:

$$\delta v = F\Big|_{t=t_1}\delta t_1 + \int_{t_0}^{t_1}\sum_{i=1}^{m}\sum_{j=0}^{n}\left\{(-1)^j\frac{d^j}{dt^j}F_{x_i^{(j)}}\right\}\delta x_i\,dt + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}F_{x_i^{(k)}}\right\}\delta x_i^{(j-1)} + \int_{t_0-\eta_0}^{t_1-\eta_1}\sum_{i=1}^{m}\sum_{j=0}^{n}\left\{(-1)^j\frac{d^j}{dt^j}\left\{\overline{F}_{x_i^{(j)}(z_{ij})}(1-\varphi'_{ij})\right\}\right\}\delta x_i(z_{ij})dt + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}\left\{\overline{F}_{x_i^{(k)}(z_{ik})}(1-\varphi'_{ij})\right\}\right\}\delta x_i^{(j-1)}(z_{ik}) \quad ..(2.27)$$

where:

$$\int_{t_0}^{t_1}\sum_{i=1}^{m}\sum_{j=0}^{n}\left\{\left\{(-1)^j\frac{d^j}{dt^j}F_{x_i^{(j)}}\right\}\delta x_i+(-1)^j\frac{d^j}{dt^j}\left\{\overline{F}_{x_i^{(j)}(z_{ij})}(1-\varphi'_{ij})\right\}\delta x_i(z_{ij})\right\}dt$$

............................(2.27.a)

is the fundamental necessary condition for nonmovable boundary points, and making use of:

$$\delta x_i\big|_{t=t_1}=\delta x_{i1}-x'_i(t_1)\delta t_1$$

and

$$\delta x_i(z_{i0})\big|_{t=t_1-\eta_{i1}}=\delta x_{\eta_{i1}}-x'_i(t_1-\eta_1)\delta t_1;$$

to obtain the same result for $\delta x'_i$, …, $\delta x_i^{(n-1)}$, for all (i = 1, 2, …, m), as:

$$\delta x_i^{(j)}\Big|_{t=t_1}=\delta x_{i1}^{(j)}-x_i^{(j+1)}(t_1)\delta t_1$$

and

$$\delta x_i^{(j)}(z_{ij})\Big|_{t=t_1-\eta_1}=\delta x_{\eta_{i1}}^{(j)}-x_i^{(j+1)}(t_1-\eta_{i1})\delta t_1;$$

for all (j = 1, 2, …, n – 1).

Substituting the values of the arbitrary functions at the end points t_1 and $t_1-\eta_1$ back in eq.(2.27), yields to:

$$\delta v=F\big|_{t=t_1}\delta t_1+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}F_{x_i^{(k)}}\right|_{t=t_1}\left(\delta x_{i1}^{(j-1)}-x_i^{(j)}(t_1)\delta t_1\right)+$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}\left\{\overline{F}_{x_i^{(k)}(z_{ik})}(1-\varphi'_{ik})\right\}\right|_{t=t_1}\left(\delta x_{\eta_{i1}}^{(j-1)}-\right.$$

$$\left.x_i^{(j)}(t_1-\eta_{i1})\delta t_1\right).$$

Consequently:

$$\delta v = \left(F\Big|_{t=t_1} - \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{(j)} \sum_{k=j}^{n} \left\{ (-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} F_{x_i^{(k)}} \right\}\Bigg|_{t=t_1} - \right.$$

$$\left. \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{(j)}(t_1-\eta_{i1}) \sum_{k=j}^{n} \left\{ (-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} \left\{ \overline{F}_{x_i^{(k)}(z_{ik})} (1-\varphi'_{ik}) \right\} \right\}\Bigg|_{t=t_1-\eta_1} \right) \delta t_1 +$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=j}^{n} \left\{ (-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} F_{x_i^{(k)}}\Bigg|_{t=t_1} \right\} \delta x_{i1}^{(j-1)} +$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=j}^{n} \left\{ (-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} \left\{ \overline{F}_{x_i^{(k)}(z_{ik})} (1-\varphi'_{ik}) \right\}\Bigg|_{t=t_1-\eta_1} \right\} \delta x_{\eta_{i1}}^{(j-1)}.$$

Then the following two cases arises:

(i) If all arbitrary functions are independent, then the fundamental necessary condition for an extremum is:

$$\left(F\Big|_{t=t_1} - \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{(j)} \sum_{k=j}^{n} \left\{ (-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} F_{x_i^{(k)}} \right\}\Bigg|_{t=t_1} - \right.$$

$$\left. \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^{(j)}(t_1-\eta_{i1}) \sum_{k=j}^{n} \left\{ (-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} \left\{ \overline{F}_{x_i^{(k)}(z_{ik})} (1-\varphi'_{ik}) \right\} \right\}\Bigg|_{t=t_1-\eta_1} \right) = 0,$$

$$\sum_{k=j}^{n} \left((-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} F_{x_i^{(k)}}\Bigg|_{t=t_1} \right) = 0$$

and

$$\sum_{k=j}^{n} \left((-1)^{k-j} \frac{d^{k-j}}{dt^{k-j}} \left\{ \overline{F}_{x_i^{(k)}(z_{ik})} (1-\varphi'_{ik}) \right\}\Bigg|_{t=t_1-\eta_1} \right) = 0;$$

where $z_{ik} = (t - \varphi'_{ik})$, for $(i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n)$.

(ii) If there is a relation between the arbitrary functions, such as:

$$x_{i1}^{(j)} = G_{ij}(t_1) \text{ and } x_{\eta_{i1}}^{(j)} = H_{ij}(t_1 - \eta_{i1}).$$

Implies that,

$$\delta x_{i1}^{(j)} = G'_{ij}(t_1)\delta t_1 \text{ and } \delta x_{\eta_{i1}}^{(j)} = H'_{ij}(t_1 - \eta_{i1})\delta t_1.$$

Then the transversility condition, takes the form:

$$\left(F\Big|_{t=t_1} + \sum_{i=1}^{m}\sum_{j=1}^{n}(G'_{ij} - x_i^{(j)})\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}F_{x_i^{(k)}}\right\}\Bigg|_{t=t_1} + \right.$$

$$\left. \sum_{i=1}^{m}\sum_{j=1}^{n}(H'_{ij} - x_i^{(j)})\sum_{k=j}^{n}\left\{(-1)^{k-j}\frac{d^{k-j}}{dt^{k-j}}\left\{\overline{F}_{x_i^{(k)}(z_{ik})}(1 - \varphi'_{ik})\right\}\right\}\Bigg|_{t=t_1-\eta_1} \right) = 0$$

where $z_{ik} = (t - \varphi_{ik})$, for $(i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n)$, all classes with the general Euler's equation (2.19). ■

2.3.3 The Curve-Point Problems of ODDEv's:

In this subsection, the extrema for a functional will be described subjected to the restriction that $(t_0; x_0)$ moving along certain curve $x = G(t)$, that is the end point $(t_0; x_0)$ is movable, and can move turning to $(t_0 + \delta t_0; x_0 + \delta x_0)$ and the other point $(t_1; x_1)$ is fixed.

The next theorem illustrates this case of variational problems:

Theorem (2.12):

Let F be a function with continuous first and second partial derivatives with respect to the argument t, and of the variable x(t), x(t − φ_i), x′(t) and x′(t − φ_i), for (i = 0, 1). Consider the functional (2.22), which satisfies the boundary conditions:

$$x(t_1) = x_1 \text{ and } x(t_1 - \eta_1) = x_{\eta_1},$$

with movable boundary at (t_0; x_0). Then the fundamental necessary condition for an extremum will satisfy Euler's equation (2.7) and the natural boundary condition:

$$F\big|_{t=t_0} = 0.$$

Proof:

In order to find the necessary condition for the (curve-point) problem, the linear part of the increment may of the functional is considered:

$$\delta v = \int_{t_0}^{t_1} F\,dt - \int_{t_0+\delta t_0}^{t_1} F\,dt$$

$$= \int_{t_0}^{t_0+\delta t_0} F\,dt,$$

which implies that:

$$\delta v = \int_{t_0}^{t_0+\delta t_0} F(t; g, g', g(t-\varphi_0), g'(t-\varphi_1))\,dt,$$

for x = g, on the interval $[t_0, t_0 + \delta t_0]$.

Applying the Mean Value theorem and using the continuity of the function F, yields to:

$$\delta v = F(t; g, g', g(t - \varphi_0), g'(t - \varphi_1)) \Big|_{t=t_0} \delta t_0.$$

Since δt_0 is arbitrary, the necessary condition for an extremum will satisfy Euler's equation (2.7) and the natural condition:

$$F\Big|_{t=t_0} = 0. \quad \blacksquare$$

2.3.4 The Curve-Curve Problems of ODDEv's:

Such problems are called mixed variational problems of ODDEv's, which has to examine the extrema for functionals that has two movable end points. Different kinds of such problems exist, and in the next theorems, we will discuss the fundamental necessary condition for two kinds of such problems.

Theorem (2.13):

Let F be a function with continuous first and second partial derivatives with respect to t and $x(t)$, $x(t - \varphi_{ij})$, $x'(t)$, $x'(t - \varphi_{ij})$, $(i = 0, 1)$, which satisfies the boundary conditions:

$$x(t_0) = x_0, \quad x(t_1) = x_1,$$

$$x(t_0 - \eta_0) = x_{\eta_0} \text{ and } x(t_1 - \eta_1) = x_{\eta_1},$$

with the additional condition that the admissible curve can pass through a certain domain D, bounded by a certain curve $\varphi(t; x) = 0$, then the fundamental necessary condition for an extremum of the functional (2.22), is given by:

$$\left(F\Big|_{t=t^*} - F(t;g,g',g(t-\varphi_0),g'(t-\varphi_1)\Big|_{t=t^*} + (G'-x')\overline{F}_{x'}\Big|_{t=t^*} + \right.$$

$$\left. (H'-x')\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\Big|_{t=t^*-\eta} \right) = 0,$$

with Euler's equations (2.7) to be satisfied.

Proof:

Consider the first variation δv for the functional v[x]:

$$v = \int_{t_0}^{t_1} F\,dt$$

$$= \int_{t_0}^{t^*} F\,dt + \int_{t^*}^{t_1} F\,dt;$$

and assumed that δv is determined by $\delta v = \delta v_1 + \delta v_2$, where $\delta v_1 = \int_{t_0}^{t^*} F\,dt$, is due to the variation of the functional has a movable upper point $(t^*; x^*)$, which can move freely along the boundary curve $\varphi(t; x) = 0$, and consequently, by theorem (2.10), the transversility condition for δv_1 is:

$$\delta v_1 = \left(F + (G'-x')F_{x'}\Big|_{t=t^*} + (H'-x')\,\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\Big|_{t=t^*-\eta} \right) = 0;$$

and $\delta v_2 = \int_{t^*}^{t_1} F\,dt$, is due to the variation of the functional has also a movable end point (t^*; x^*), thus, by theorem (2.12), the necessary condition for δv_2, is:

$$\delta v_2 = F\big|_{t=t^*}.$$

Whereas for δv the necessary condition is:

$$\delta v = \Big(F\big|_{t=t^*} + F(t; g, g', g(t-\varphi_0), g'(t-\varphi_1)\big|_{t=t^*} + (G' - x')F_{x'}\big|_{t=t^*} +$$

$$(H' - x')\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\big|_{t=t^*-\eta} \Big) = 0;$$

with Euler's equation (2.7) is satisfied. ■

<u>*Theorem (2.14):*</u>

Let F be a function with continuous first and second partial derivatives with respect to t and $x(t)$, $x(t-\varphi_i)$, $x'(t)$, $x'(t-\varphi_i)$, $(i = 0, 1)$, and their derivatives. Consider the functional:

$$v[x] = \int_{t_0}^{t_1} F(t; x, x', x(t-\varphi_0), x'(t-\varphi_1))\,dt + \Phi(t_0, t_1; x_0, x_1) \quad \text{....(2.28)}$$

where Φ is a constant, which has no influence on the extremal properties of the functional, with movable end points. Then the fundamental necessary condition for an extremum for the functional (2.28) satisfies Euler's equation (2.7), as well as the transverality necessary condition:

$$\left(F\Big|_{t=t_1} + (G' - x')F_{x'}\Big|_{t=t_1} + (H' - x')\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\Big|_{t=t_1-\eta_1} + \Phi_{t_1} + \Phi_{x_1}G' \right) = 0$$

and

$$\left(F\Big|_{t=t_0} + (T' - x')F_{x'}\Big|_{t=t_0} + (U' - x')\overline{F}_{x'(t-\varphi_1)}(1-\varphi_1')\Big|_{t=t_0-\eta_0} + \Phi_{t_0} + \Phi_{x_0}T' \right) = 0.$$

Proof:

To find the first variation δv, assume that:

$\delta v = \delta v_1 + \delta\Phi$,

where δv_1 is due to the variation of F with the two end points are movable, such that:

$$\delta v_1 = \left(\int_{t_0}^{t_1+\delta t_1} F(t; x+\delta x, x' + \delta x', x(t-\varphi_0) + \delta x(t-\varphi_0), x'(t-\varphi_1) + \delta x'(t-\varphi_1))\, dt - \int_{t_0}^{t_1} F\, dt \right) + \left(\int_{t_0+\delta t_0}^{t_1} F\, dt - \int_{t_0}^{t_1} F\, dt \right)$$

$$= \int_{t_1}^{t_1+\delta t_1} F\, dt - \int_{t_0}^{t_0+\delta t_0} F\, dt$$

or equivalently

$$\delta v_1 = \int_{t_1}^{t_1+\delta t_1} F\,dt - \int_{t_0}^{t_0+\delta t_0} F\,dt + \int_{t_0}^{t_1} \Delta F\,dt \quad\text{.................................(2.29)}$$

Applying the Mean Value theorem for the first two integrals of the right hand side of eq.(2.29), and integrating by parts the third integral, yields to:

$$\delta v_1 = F\big|_{t=t_1}\delta t_1 - F\big|_{t=t_0}\delta t_0 + F_{x'}\delta x\big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\left\{F_x - \frac{d}{dt}F_{x'}\right\}\delta x\,dt +$$

$$\left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\delta x(z_1)\right\}\Big|_{t_0-\eta_0}^{t_1-\eta_1} +$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1}\left\{\overline{F}_{x(z_0)}(1-\varphi_0')\delta x(z_0) - \frac{d}{dt}\left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\right\}\delta x(z_1)\right\}dt \quad\text{..(2.30)}$$

where $z_i = (t - \varphi_i)$, $(i = 0, 1)$, and

$$\int_{t_0}^{t_1}\left\{F_x - \frac{d}{dt}F_{x'}\right\}\delta x\,dt +$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1}\left\{\overline{F}_{x(z_0)}(1-\varphi_0')\delta x(z_0) - \frac{d}{dt}\left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\right\}\delta x(z_1)\right\}dt,$$

are the Euler's equations for the functional $v_1[x]$.

Now, since:

$$\left.\begin{aligned} \delta x\big|_{t=t_0} &= \delta x_0 - x'(t_0)\delta t_0, \\ \delta x\big|_{t=t_1} &= \delta x_1 - x'(t_1)\delta t_1, \\ \delta x\big|_{t=t_0-\eta_0} &= \delta x_{\eta_0} - x'(t_0-\eta_0)\delta t_0 \\ \delta x\big|_{t=t_1-\eta_1} &= \delta x_{\eta_1} - x'(t_1-\eta_1)\delta t_1 \end{aligned}\right\} \quad\text{...............................(2.30.a)}$$

Substituting (2.30.a) in eq.(2.30), one can get:

$$\delta v_1 = F\big|_{t=t_1} \delta t_1 - F\big|_{t=t_0} \delta t_0 + F_{x'}\big|_{t=t_1} (\delta x_1 - x'(t_1)\delta t_1) - F_{x'}\big|_{t=t_0} (\delta x_0 - x'(t_0)\delta t_0) + \left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\right\}\Big|_{t=t_1-\eta_1} (\delta x_{\eta_1} - x'(t_1 - \eta_1)\delta t_1) - \overline{F}_{x'(z_1)}(1-\varphi_1')\Big|_{t=t_0-\eta_0} (\delta x_{\eta_0} - x'(t_0 - \eta_0)\delta t_0).$$

Implies that,

$$\delta v_1 = \left\{\left(F\big|_{t=t_1} - x' F_{x'}\big|_{t=t_1} - x'(t_1 - \eta_1) F_{x'(z_1)}(1-\varphi_1')\Big|_{t=t_1-\eta_1} \right)\delta t_1 + F_{x'}\big|_{t=t_1} \delta x_1 + \left\{F_{x'(z_1)}(1-\varphi_1')\right\}\Big|_{t=t_1-\eta_1} \delta x_{\eta_1} \right\} - \left\{\left(F\big|_{t=t_0} - x' F_{x'}\big|_{t=t_0} - x'(t_0 - \eta_0) \overline{F}_{x'(z_1)}(1-\varphi_1')\Big|_{t=t_0-\eta_0} \right)\delta t_0 + F_{x'}\big|_{t=t_0} \delta x_0 + \left\{\overline{F}_{x'(z_1)}(1-\varphi_1')\right\}\Big|_{t=t_0-\eta_0} \delta x_{\eta_0} \right\} = 0.$$

From Taylor series expansion, $\delta\Phi$ may be written as:

$$\delta\Phi = \Phi(t_0 + \delta t_0, t_1 + \delta t_1; x_0 + \delta x_0, x_1 + \delta x_1) - \Phi(t_0, t_1; x_0, x_1)$$
$$= \Phi_{t_0}\delta t_0 + \Phi_{t_1}\delta t_1 + \Phi_{x_0}\delta x_0 + \Phi_{x_1}\delta x_1 .$$

Since the boundary points $(t_0; x_0)$, $(t_1; x_1)$, $(t_0 - \eta_0; x_{\eta_0})$ and $(t_1 - \eta_1; x_{\eta_1})$ can move freely along prescribed curves $x_0 = T(t_0)$, $x_{\eta_0} = U(t_0 - \eta_0)$, $x_1 = G(t_1)$ and $x_{\eta_1} = H(t_1 - \eta_1)$, implies that [El'sgol'c, 1964]:

$$\delta x_0 = T'(t_0)\delta t_0, \quad \delta x_1 = G'(t_1)\delta t_1,$$

$$\delta x_{\eta_0} = U'(t_0 - \eta_0)\delta t_0 \text{ and } \delta x_{\eta_1} = H'(t_1 - \eta_1)\delta t_1.$$

Then the transversility condition for $\delta v = \delta v_1 + \delta\Phi$ takes the form:

$$\left(F\big|_{t=t_1} + (G' - x')F_{x'}\big|_{t=t_1} + (H' - x')\overline{F}_{x'(z_1)}(1 - \varphi_1')\Big|_{t=t_1-\eta_1} + \Phi_{t_1} + \Phi_{x_1}G'\right)\delta t_1 -$$

$$\left\{F\big|_{t=t_0} + (T' - x')F_{x'}\big|_{t=t_0} + (U' - x')\overline{F}_{x'(z_1)}(1 - \varphi_1')\Big|_{t=t_0-\eta_0} + \Phi_{t_0} + \Phi_{x_0}T'\right\}\delta t_0 = 0$$

when δt_0 and δt_1 are independent, as may be easily shown, the necessary condition for δv to have an extremum is:

$$\left(F\big|_{t=t_1} + (G' - x')F_{x'}\big|_{t=t_1} + (H' - x')\overline{F}_{x'(z_1)}(1 - \varphi_1')\Big|_{t=t_1-\eta_1} + \Phi_{t_1} + \Phi_{x_1}G'\right) = 0$$

and

$$\left\{F\big|_{t=t_0} + (T' - x')F_{x'}\big|_{t=t_0} + (U' - x')\overline{F}_{x'(z_1)}(1 - \varphi_1')\Big|_{t=t_0-\eta_0} + \Phi_{t_0} + \Phi_{x_0}T'\right\}\delta t_0 = 0$$

where $z_1 = (t - \varphi_1)$, with Euler's equation (2.7) is satisfied. ■

2.4 The Sufficient Condition for an Extremum of the Variational Problems of ODDEv's

In the previous section, we established the most useful necessary conditions for a relative local minimum of the basic problems in calculus of variations and its variants. Among these conditions, the Euler-Lagrange equation which enable us to search for a minimum point of a small group of piecewise smooth functions called extremal. However, the fact that $x^*(t)$ satisfies the Euler equation does not necessarily make it the solution of the problem, [Wan, 1995], i.e., the condition that $\delta v[x^*] = 0$ does not necessarily imply that x^* is a local minimum for $v[x]$. Therefore, one must examine $\delta^2 v[x^*]$ to decide whether x^* is a minimum point, a maximum point or inflection point of the variational problem v[x].

In this section, the attention will be focused on the minimization problem unless the context of the specific problem requires to do otherwise. The basic problem is stated as:

$$\underset{x \in S}{\text{Min}}\, v[x \mid x(t_0) = x_0, x(t_1) = x_1, x(t_0 - \eta_0) = x_{\eta_0}, x(t_1 - \eta_1) = x_{\eta_1}],$$

$$v[x] = \int_{t_0}^{t_1} \{F(t; x, x', x(t - \varphi_0), x'(t - \varphi_1)\}\, dt \qquad \text{.........................(2.31)}$$

where φ_i, for (i = 0, 1) are the delay functions, $\eta_0 = \text{Max}\{\varphi_i(t_0; x(t_0), x'(t_0))\}$, $\eta_1 = \text{Max}\ \{\varphi_i(t_1; x(t_1), x'(t_1))\}$, (i = 0, 1) and S is the collection of all admissible functions to be specified for the basic problem; the necessary condition for v[x] to have a minimum at x* = x and to satisfy the inequality $v[x] \leq v[x^*]$, for all admissible x*, is that, the stationary condition $\delta v[x^*] = 0$ holds for all admissible δx^*. Also we need to determine the second variation $\delta^2 v[x^*]$, which is an integral that may be represented as a functional and called the accessory variational problem of the functional (2.31), [Leitmann, 1981], to decide whether x* is a minimum point.

The accessory variational problem obtained by:

$$\delta^2 v = \int_{t_0}^{t_1} \delta^2 F\, dt.$$

Then according to theorem (2.2), it is necessary for the functional v[x] to have a minimum at x = x*, is that the set of inequalities:

$$F_{x'x'} \geq 0, \quad \begin{vmatrix} F_{x'x'} & \overline{F}_{x'x'(t-\varphi_1)}(1-\varphi_1') \\ \overline{F}_{x'(t-\varphi_1)x'}(1-\varphi_1') & \overline{F}_{x'(t-\varphi_1)x'(t-\varphi_1)}(1-\varphi_1') \end{vmatrix} \geq 0 \text{.......(2.32)}$$

be satisfied at every point in the curve, and (2.32) is called the Legendre condition for the functional (2.31). But, Legendre attempted unsuccessfully to show that a sufficient condition for v[x] to have a minimum; therefore, Jacobi test plays an important role in the basic problem depending on whether that there is no points in $[t_0 - \eta_0, t_1]$ conjugate relative to x*, it suffices for x* to be minimum point for v[x]; therefore the following definition is necessary;

Definition (2.15), [Gelfand, 1963]:

The point $\tilde{a} \neq a$ is said to be conjugate to the point a if $\int_{t_0}^{t_1} \left(P\delta x'^2 + Q\delta x^2\right) dt$ has a solution which vanishes for x = a and $x = \tilde{a}$ but is not identically zero.

After these considerations, the sufficiency conditions can be stated as:

i) x*(t) is an (admissible) extremal (from Euler-Lagrange equation).

ii) For every $t \in [t_0 - \eta_0, t_1]$, the Legendre condition (2.32) is satisfied.

iii) x*(t) has no conjugate points to $t_0 - \eta_0$ in $[t_0 - \eta_0, t_1]$ (Jacobi's test), [Wan, 1995].

The expression:

$$\int_{t_0}^{t_1} (P_1\delta x^2 + P_2\delta x'^2 + Q\delta x)\, dt + \frac{1}{2}\int_{t_0-\eta_0}^{t_1-\eta_1} (\overline{P}_1\delta x^2(z_0) + \overline{P}_2\delta x'^2(z_1) + \overline{Q})^2\, dt,$$

where:

$$P_1 = F_{xx},\ P_2 = F_{x'x'},\ Q = -\frac{d}{dt}\{F_{xx'}\delta x\},\ \overline{P}_1 = (\overline{F}_{x(z_0)x(z_0)}(1-\varphi_0'),$$

$$\overline{P}_2 = \overline{F}_{x'(z_1)x'(z_1)}(1-\varphi_1') \text{ and}$$

$$\overline{Q} = \left(\overline{F}_{xx(z_0)}(1-\varphi_0')\delta x + \overline{F}_{x'x(z_0)}(1-\varphi_0')\delta x'\right)\delta x(z_0) -$$

$$\frac{d}{dt}\left(\overline{F}_{xx'(z_1)}(1-\varphi_1')\delta x + \overline{F}_{x'x'(z_1)}(1-\varphi_1')\delta x' +\right.$$

$$\left.\overline{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1')\delta x(z_0)\right)\delta x(z_1)$$

where $z_i = (t - \varphi_i)$, $(i = 0, 1)$, for the second variation of the functional, is called the Jacobi's condition.

Hereinafter, the expression of the second variation for the different types of functionals with the Legendre condition and Jacobi's equation of each type will be established, as it is shown in the next theorems.

Theorem (2.16):

If F is a function with continuous partial derivatives up to order three with respect to t, $x(t)$, $x(t - \varphi_i)$, $x'(t)$, $x'(t - \varphi_i)$, $(i = 0, 1)$. Consider the functional of the form (2.31), defined for curves $x = x(t)$ and satisfies the given boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$, $x(t_0 - \eta_0) = x_{\eta_0}$ and

$x(t_1 - \eta_1) = x_{\eta_1}$. Then the Legendre condition related to the functional (2.31) is the following set of inequalities:

$$F_{x'x'} \geq 0, \quad \begin{vmatrix} F_{x'x'} & \overline{F}_{x'x'(t-\varphi_1)}(1-\varphi_1') \\ \overline{F}_{x'(t-\varphi_1)x'}(1-\varphi_1') & \overline{F}_{x'(t-\varphi_1)x'(t-\varphi_1)}(1-\varphi_1') \end{vmatrix} \geq 0,$$

must be hold for all $t \in [t_0 - \eta_0, t_1]$.

Proof:

The accessory problem of the functional (2.31) for the second variation can be expressed as:

$$\delta^2 v = \int_{t_0}^{t_1} \delta^2 F \, dt,$$

where:

$$\begin{aligned}\delta^2 F = \frac{1}{2!}\{&F_{xx}\delta x^2 + F_{x'x'}\delta x'^2 + F_{x(t-\varphi_0)x(t-\varphi_0)}\delta x^2(t-\varphi_0) + \\ &F_{x'(t-\varphi_1)x'(t-\varphi_1)}\delta x'^2(t-\varphi_1) + 2F_{xx'}\delta x\delta x' + \\ &2F_{xx(t-\varphi_0)}\delta x\delta x(t-\varphi_0) + 2F_{xx'(t-\varphi_1)}\delta x\delta x'(t-\varphi_1) + \\ &2F_{x'x(t-\varphi_0)}\delta x'\delta x(t-\varphi_0) + 2F_{x'x'(t-\varphi_1)}\delta x'\delta x'(t-\varphi_1) + \\ &2F_{x'(t-\varphi_0)x'(t-\varphi_1)}\delta x(t-\varphi_0)\delta x(t-\varphi_1)\}.\end{aligned}$$

Thus, the second variation for eq.(2.31) can be rewritten as:

$$\delta^2 v = \frac{1}{2!}\int_{t_0}^{t_1} \{F_{xx}\delta x^2 + F_{x'x'}\delta x'^2 + 2F_{xx'}\delta x\delta x'\}\, dt +$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \{\bar{F}_{x(z_0)x(z_0)}(1-\varphi_0')\delta x^2(z_0) +$$

$$\bar{F}_{x'(z_1)x'(z_1)}(1-\varphi_1')\delta x'^2(z_1) + 2\bar{F}_{xx(z_0)}(1-\varphi_0')\delta x\delta x(z_0) +$$

$$2\bar{F}_{xx'(z_1)}(1-\varphi_1')\delta x\delta x'(z_1) + 2\bar{F}_{x'x(z_0)}(1-\varphi_0')\delta x'\delta x(z_0) +$$

$$2\bar{F}_{x'x'(z_1)}(1-\varphi_1')\delta x'\delta x'(z_1) +$$

$$2\bar{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1')\delta x(z_0)\delta x'(z_1)\}dt \quad\text{........(2.33)}$$

where $z_i = (t - \varphi_i)$, $(i = 0, 1)$, and $\bar{F}$ is a symbol for F with variable delay functions.

Hence, by Legendre theorem (2.2), the necessary condition for the functional (2.31) to have a minimum at $x = x^*$, is that the following set of inequalities must be hold:

$$F_{x'x'} \geq 0, \quad \begin{vmatrix} F_{x'x'} & \bar{F}_{x'x'(z_1)}(1-\varphi_1') \\ \bar{F}_{x'(z_1)x'}(1-\varphi_1') & \bar{F}_{x'(z_1)x'(z_1)}(1-\varphi_1') \end{vmatrix} \geq 0,$$

for every $t \in [t_0 - \eta_0, t_1]$, where $z_i = (t - \varphi_i)$, for $(i = 0, 1)$.

Now, in order to find Jacobi's equation for the functional (2.31), integrate by parts and taking into account of the boundary conditions, one obtains:

$$\int_{t_0}^{t_1} F_{xx'}\delta x\delta x'\, dt = \{F_{xx'}\delta x\}\delta x \Big|_{t_0}^{t_1} - \frac{d}{dt}\{F_{xx'}\delta x\}\,\delta x,$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \bar{F}_{xx'(z_1)}(1-\varphi_1')\delta x\delta x'(z_1)dt = \left\{\bar{F}_{xx'(z_1)}(1-\varphi_1')\delta x\right\}\delta x(z_1)\Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d}{dt}\left\{\bar{F}_{xx'(z_1)}(1-\varphi_1')\delta x\right\}\delta x(z_1),$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \bar{F}_{x'x'(z_1)}(1-\varphi_1')\delta x'\delta x'(z_1)dt = \left\{\bar{F}_{x'x'(z_1)}(1-\varphi_1')\delta x'\right\}\delta x'(z_1)\Big|_{t_0-\eta_0}^{t_1-\eta_1}$$

$$-\int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d}{dt}\bar{F}_{x'x'(z_1)}(1-\varphi_1')\delta x(z_1)$$

and

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \bar{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1')\delta x(z_0)\delta x'(z_1)dt =$$

$$\left\{\bar{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1')\delta x(z_0)\right\}\delta x(z_1)\Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d}{dt}\left\{\bar{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1')\delta x(z_0)\right\}\delta x(z_1).$$

Therefore, the second variation $\delta^2 v$ becomes:

$$\delta^2 v = \frac{1}{2!}\int_{t_0}^{t_1}\left\{ F_{xx}\delta x^2 + F_{x'x'}\delta x'^2 - \frac{d}{dt}\left\{F_{xx'}\delta x\right\}\delta x\right\} dt +$$

$$\frac{1}{2!}\int_{t_0-\eta_0}^{t_1-\eta_1}\left\{\bar{F}_{x(z_0)x(z_0)}(1-\varphi_0')\delta x^2(z_0) + \right.$$

$$\bar{F}_{x'(z_1)x'(z_1)}(1-\varphi_1')\delta x'^2(z_1) +$$

$$\left(\overline{F}_{xx(z_0)}(1-\varphi_0')\delta x + \overline{F}_{x'x(z_0)}(1-\varphi_0')\delta x'\right)\delta x(z_0) -$$

$$\frac{d}{dt}\left(\overline{F}_{x'x'(z_1)}(1-\varphi_1')\delta x' + \overline{F}_{xx'(z_1)}(1-\varphi_1')\delta x +\right.$$

$$\left.\left.\overline{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1')\delta x(z_0)\right)\delta x(z_1)\right\} dt\,.$$

Then Jacobi's equation takes the form:

$$\delta^2 v = \frac{1}{2}\int_{t_0}^{t_1} (P_1\delta x^2 + P_2\delta x'^2 + Q\delta x)\, dt + \frac{1}{2}\int_{t_0-\eta_0}^{t_1-\eta_1} (\overline{P}_1\,\delta x^2(z_0) +$$

$$\overline{P}_2\,\delta x'^2(z_1) + \overline{Q})\, dt,$$

where:

$$P_1 = F_{xx},\ P_2 = F_{x'x'},\ Q = -\frac{d}{dt}\left\{F_{xx'}\delta x\right\},$$

$$\overline{P}_1 = (\overline{F}_{x(z_0)x(z_0)}(1-\varphi_0'),\ \overline{P}_2 = \overline{F}_{x'(z_1)x'(z_1)}(1-\varphi_1') \text{ and}$$

$$\overline{Q} = \left(\overline{F}_{xx(z_0)}(1-\varphi_0')\delta x + \overline{F}_{x'x(z_1)}(1-\varphi_0')\delta x'\right)\delta x(z_0) -$$

$$\frac{d}{dt}\left(\overline{F}_{xx'(z_1)}(1-\varphi_1')\delta x + \overline{F}_{x'x'(z_1)}(1-\varphi_1')\delta x' +\right.$$

$$\left.\overline{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1')\delta x(z_0)\right)\delta x(z_1)$$

where $z_i = (t - \varphi_i)$, $(i = 0, 1)$. ■

Theorem (2.17):

If F is a function with continuous partial derivatives with respect to t, x(t), $x(t - \varphi_i)$ and their derivatives for $(i = 0, 1, \dots, n)$, then the necessary condition for the functional (2.13) defined for the curves x = x(t), and satisfies the boundary conditions (2.14), to have a minimum at x = x* is:

$$F_{x^{(n)}x^{(n)}} \geq 0, \begin{vmatrix} F_{x^{(n)}x^{(n)}} & \overline{F}_{x^{(n)}x^{(n)}(t-\varphi_n)}(1-\varphi'_n) \\ \overline{F}_{x^{(n)}(t-\varphi_n)x^{(n)}}(1-\varphi'_n) & \overline{F}_{x^{(n)}(t-\varphi_n)x^{(n)}(t-\varphi_n)}(1-\varphi'_n) \end{vmatrix} \geq 0,$$

must be hold for every $t \in [t_0 - \eta_0, t_1]$.

Proof:

The second variation of v is:

$$\delta^2 v = \int_{t_0}^{t_1} \delta^2 F \, dt,$$

where $\delta^2 F$ is obtained by using Taylor series expansion, and may be written in the form:

$$\delta^2 F = (\delta x \delta x' \ldots \delta x^{(n)} \delta x(z_0) \delta x'(z_1) \ldots \delta x^{(n)}(z_n)).A.\begin{pmatrix} \delta x \\ \delta x' \\ \vdots \\ \delta x^{(n)} \\ \delta x(z_0) \\ \delta x'(z_1) \\ \vdots \\ \delta x^{(n)}(z_n) \end{pmatrix},$$

where:

$$A = \begin{bmatrix} A1 & A2 \\ A3 & A4 \end{bmatrix};$$

and

$$A1=\begin{bmatrix} F_{xx} & F_{xx'} & \cdots & F_{xx^{(n)}} \\ F_{x'x} & F_{x'x'} & \cdots & F_{x'x^{(n)}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{x^{(n)}x} & F_{x^{(n)}x'} & \cdots & F_{x^{(n)}x^{(n)}} \end{bmatrix},$$

$$A2=\begin{bmatrix} \overline{F}_{xx(z_0)}(1-\varphi_0') & \overline{F}_{xx'(z_1)}(1-\varphi_1') & \cdots & \overline{F}_{xx^{(n)}(z_n)}(1-\varphi_n') \\ \overline{F}_{x'x(z_0)}(1-\varphi_0') & \overline{F}_{x'x'(z_1)}(1-\varphi_1') & \cdots & \overline{F}_{x'x^{(n)}(z_n)}(1-\varphi_n') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x^{(n)}x(z_0)}(1-\varphi_0') & \overline{F}_{x^{(n)}x'(z_1)}(1-\varphi_1') & \cdots & \overline{F}_{x^{(n)}x^{(n)}(z_n)}(1-\varphi_n') \end{bmatrix},$$

$$A3=\begin{bmatrix} \overline{F}_{x(z_0)x}(1-\varphi_0') & \overline{F}_{x(z_0)x'}(1-\varphi_0') & \cdots & \overline{F}_{x(z_0)x^{(n)}}(1-\varphi_0') \\ \overline{F}_{x'(z_1)x}(1-\varphi_1') & \overline{F}_{x'(z_1)x'}(1-\varphi_1') & \cdots & \overline{F}_{x'(z_1)x^{(n)}}(1-\varphi_1') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x^{(n)}(z_n)x}(1-\varphi_n') & \overline{F}_{x^{(n)}(z_n)x'}(1-\varphi_n') & \cdots & \overline{F}_{x^{(n)}(z_n)x^{(n)}}(1-\varphi_n') \end{bmatrix},$$

and

$$A4=\begin{bmatrix} \overline{F}_{x(z_0)x(z_0)}(1-\varphi_0') & \overline{F}_{x(z_0)x'(z_1)}(1-\varphi_0')(1-\varphi_1') & \cdots & \overline{F}_{x(z_0)x^{(n)}(z_n)}(1-\varphi_0')(1-\varphi_n') \\ \overline{F}_{x'(z_1)x(z_0)}(1-\varphi_1')(1-\varphi_0') & \overline{F}_{x'(z_1)x'(z_1)}(1-\varphi_1') & \cdots & \overline{F}_{x'(z_1)x^{(n)}(z_n)}(1-\varphi_1')(1-\varphi_n') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x^{(n)}(z_n)x(z_0)}(1-\varphi_n')(1-\varphi_0') & \overline{F}_{x^{(n)}(z_n)x'(z_1)}(1-\varphi_n')(1-\varphi_1') & \cdots & \overline{F}_{x^{(n)}(z_n)x^{(n)}(z_n)}(1-\varphi_n') \end{bmatrix}$$

where $z_i = (t - \varphi_i)$, $(i = 0, 1, \ldots, n)$.

Hence, the Legendre condition for the second variation to be nonnegative is that the following set of inequalities must be hold for every $t \in [t_0 - \eta_0, t_1]$:

$$F_{x^{(n)}x^{(n)}} \geq 0, \begin{vmatrix} F_{x^{(n)}x^{(n)}} & \overline{F}_{x^{(n)}x^{(n)}(z_n)}(1-\varphi_n') \\ \overline{F}_{x^{(n)}(z_n)x^{(n)}}(1-\varphi_n') & \overline{F}_{x^{(n)}(z_n)x^{(n)}(z_n)}(1-\varphi_n') \end{vmatrix} \geq 0.$$

Integrating by parts, yields to:

$$\int_{t_0}^{t_1} \left\{ F_{x^{(i)}x^{(j)}} \delta x^{(i)} \right\} \delta x^{(j)} \, dt = F_{x^{(i)}x^{(j)}} \delta x^{(i)} \delta x^{(j-1)} \Big|_{t_0}^{t_1} -$$

$$\frac{d}{dt} \left\{ F_{x^{(i)}x^{(j)}} \delta x^{(i)} \right\} \delta x^{(j-2)} \Big|_{t_0}^{t_1} + \ldots + (-1)^j \int_{t_0}^{t_1} \frac{d^j}{dt^j} \left\{ F_{x^{(i)}x^{(j)}} \delta x^{(i)} \right\} \delta x \, dt \ ,$$

for each (i = 0, 1, …, n – 1), j = i + 1, i + 2, …, n.

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \left\{ \overline{F}_{x^{(i)}x^{(j)}(z_j)} (1-\varphi_j') \delta x^{(i)} \right\} \delta x^{(j)}(z_j) \, dt =$$

$$\overline{F}_{x^{(i)}x^{(j)}(z_j)} (1-\varphi_j') \delta x^{(i)} \delta x^{(j-1)}(z_j) \Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\frac{d}{dt} \left\{ \overline{F}_{x^{(i)}x^{(j)}(z_j)} (1-\varphi_j') \delta x^{(i)} \right\} \delta x^{(j-2)}(z_j) \Big|_{t_0-\eta_0}^{t_1-\eta_1} + \ldots +$$

$$(-1)^j \int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d^j}{dt^j} \left\{ \overline{F}_{x^{(i)}x^{(j)}(z_j)} (1-\varphi_j') \delta x^{(i)} \right\} \delta x(z_j) \, dt \ ,$$

for each (i = 0, 1, …, n – 1), j = i + 1, i + 2, …, n and

$$\int_{t_0-\eta_0}^{t_1-\eta_1} \left\{ \overline{F}_{x^{(i)}(z_i)x^{(j)}(z_j)} (1-\varphi_i')(1-\varphi_j') \delta x^{(i)}(z_i) \right\} \delta x^{(j)}(z_j) \, dt =$$

$$\overline{F}_{x^{(i)}(z_i)x^{(j)}(z_j)} (1-\varphi_i')(1-\varphi_j') \delta x^{(i)}(z_i) \delta x^{(j-1)}(z_j) \Big|_{t_0-\eta_0}^{t_1-\eta_1} -$$

$$\frac{d}{dt} \left\{ \overline{F}_{x^{(i)}(z_i)^{(j)}(z_j)} (1-\varphi_i')(1-\varphi_j') \delta x^{(i)}(z_i) \right\} \delta x^{(j-2)}(z_j) \Big|_{t_0-\eta_0}^{t_1-\eta_1} + \ldots +$$

$$(-1)^j \int_{t_0-\eta_0}^{t_1-\eta_1} \frac{d^j}{dt^j} \left\{ \overline{F}_{x^{(i)}(z_i)x^{(j)}(z_j)} (1-\varphi_i')(1-\varphi_j') \delta x^{(i)}(z_i) \right\} \delta x(z_j) \, dt \ ,$$

for each (i = 0, 1, …, n – 1), j = i + 1, i + 2, …, n.

Taking the boundary conditions into account, Jacobi's equation can be written as:

$$\delta^2 v = \int_{t_0}^{t_1} \left(\sum_{i=0}^{n} P_i \delta x^{(i)^2} + Q\delta x \right) dt \ + \int_{t_0-\eta_0}^{t_1-\eta_1} \left(\sum_{i=0}^{n} P_i \delta x^{(i)^2}(z_i) + \overline{Q} \right) dt \ ,$$

where:

$$P_i = F_{x_i x_i} \ , \ Q = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (-1)^j \frac{d^j}{dt^j} \left\{ F_{x^{(i)} x^{(j)}} \delta x^{(j)} \right\},$$

$$\overline{P}_i = \overline{F}_{x_i(z_i) x_i(z_i)} (1 - \varphi_i') \text{ and}$$

$$\overline{Q} = \sum_{i=0}^{n} \overline{F}_{x^{(i)} x(z_0)} (1 - \varphi_0') \delta x^{(i)} \delta x(z_0) \ +$$
$$\sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (-1)^j \frac{d^j}{dt^j} \left\{ \overline{F}_{x^{(i)} x^{(j)}(z_j)} (1 - \varphi_j') \delta x^{(i)} + \right.$$
$$\left. \overline{F}_{x^{(i)}(z_i) x^{(j)}(z_j)} (1 - \varphi_i')(1 - \varphi_j') \delta x^{(i)}(z_i) \right\} \delta x(z_j),$$

where $z_i = (t - \varphi_i)$, $(i = 0, 1, \ldots, n)$. ■

Theorem (2.18):

Let F be a function with continuous partial derivatives up to order three with respect to t, $x_i(t)$, $x_i(t - \varphi_{ij})$ and their derivatives for ($i = 1, 2, \ldots, m$ and $j = 0, 1$). Then the necessary condition for the functional:

$$v[x_1, x_2, \ldots, x_m] = \int_{t_0}^{t_1} F(t; x_1, x_2, \ldots, x_m, x'_1, x'_2, \ldots, x'_m, x_1(t - \varphi_{10}),$$
$$x_2(t - \varphi_{20}), \ldots, x_m(t - \varphi_{m0}), x'_1(t - \varphi_{11}), x'_2(t - \varphi_{21}, .., x'_m(t - \varphi_{m1})) \, dt \ldots\ldots(2.34)$$

defined for the curves $x_i = x_i(t)$, for all (i = 1, 2, …, m) and satisfies the given boundary conditions:

$$x_i(t_0) = x_{i_0},\ x_i(t_1) = x_{i_1},\ x_i(t_0 - \eta_{i_0}) = x_{\eta_{i_0}} \text{ and } x_i(t_1 - \eta_{i_1}) = x_{\eta_{i_1}}$$

to have a minimum at $x_i = x_i^*$, is that; the following 4m×4m Hessian matrix is positive definite.

$$H = \begin{bmatrix} H1 & H2 \\ H3 & H4 \end{bmatrix},$$

where:

$$H1 = \begin{bmatrix} F_{x_1'x_1'} & F_{x_1'x_2'} & \cdots & F_{x_1'x_m'} \\ F_{x_2'x_1'} & F_{x_2'x_2'} & \cdots & F_{x_2'x_m'} \\ \vdots & \vdots & \ddots & \vdots \\ F_{x_m'x_1'} & F_{x_m'x_2'} & \cdots & F_{x_m'x_m'} \end{bmatrix},$$

$$H2 = \begin{bmatrix} \overline{F}_{x_1'x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_1'x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_1'x_m'(z_{m1})}(1-\varphi_{m1}') \\ \overline{F}_{x_2'x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_2'x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_2'x_m'(z_{m1})}(1-\varphi_{m1}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_m'x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_m'x_m'(z_{m1})}(1-\varphi_{m1}') \end{bmatrix}$$

$$H3 = \begin{bmatrix} \overline{F}_{x_1'(z_{11})x_1'}(1-\varphi_{11}') & \overline{F}_{x_1'(z_{11})x_2'}(1-\varphi_{11}') & \cdots & \overline{F}_{x_1'(z_{11})x_m'}(1-\varphi_{11}') \\ \overline{F}_{x_2'(z_{21})x_1'}(1-\varphi_{21}') & \overline{F}_{x_2'(z_{21})x_2'}(1-\varphi_{21}') & \cdots & \overline{F}_{x_2'(z_{21})x_m'}(1-\varphi_{21}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'(z_{m1})x_1'}(1-\varphi_{m1}') & \overline{F}_{x_m'(z_{m1})x_2'}(1-\varphi_{m1}') & \cdots & \overline{F}_{x_m'(z_{m1})x_m'}(1-\varphi_{m1}') \end{bmatrix}$$

and

$$H4 = \begin{bmatrix} \overline{F}_{x_1'(z_{11})x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_1'(z_{11})x_2'(z_{21})}(1-\varphi_{11}')(1-\varphi_{21}') & \cdots & \overline{F}_{x_1'(z_{11})x_m'(z_{m1})}(1-\varphi_{11}')(1-\varphi_{m1}') \\ \overline{F}_{x_2'(z_{21})x_1'(z_{11})}(1-\varphi_{21}')(1-\varphi_{11}') & \overline{F}_{x_2'(z_{21})x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_2'(z_{21})x_m'(z_{m1})}(1-\varphi_{21}')(1-\varphi_{m1}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'(z_{m1})x_1'(z_{11})}(1-\varphi_{m1}')(1-\varphi_{11}') & \overline{F}_{x_m'(z_{m1})x_2'(z_{21})}(1-\varphi_{m1}')(1-\varphi_{21}') & \cdots & \overline{F}_{x_m'(z_{m1})x_m'(z_{m1})}(1-\varphi_{m1}') \end{bmatrix}$$

Proof:

The expression of the second variation for the functional (2.34) can be stated as follows:

$$\delta^2 v = \int_{t_0}^{t_1} \delta^2 F \, dt,$$

where:

$$\delta^2 F = \frac{1}{2} (\delta x_1 \delta x_2 \ldots \delta x_m \;\; \delta' x_1 \delta' x_2 \ldots \delta' x_m \;\; \delta x_1(z_{10}) \delta x_2(z_{20}) \ldots \delta x_m(z_{m0})$$

$$\delta x_1'(z_{11}) \delta x_2'(z_{21}) \ldots \delta x_m'(z_{m1}) A \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_m \\ \delta' x_1 \\ \delta' x_2 \\ \vdots \\ \delta' x_m \\ \delta x_1(z_{10}) \\ \delta x_2(z_{20}) \\ \vdots \\ \delta x_m(z_{m0}) \\ \delta x_1'(z_{m1}) \\ \delta x_2'(z_{m2}) \\ \vdots \\ \delta x_m'(z_{mm}) \end{pmatrix}.$$

where:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

and

$$a_{11} = \begin{bmatrix} F_{x_1x_1} & F_{x_1x_2} & \cdots & F_{x_1x_m} \\ F_{x_2x_1} & F_{x_2x_2} & \cdots & F_{x_2x_m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{x_mx_1} & F_{x_mx_2} & \cdots & F_{x_mx_m} \end{bmatrix},$$

$$a_{12} = \begin{bmatrix} F_{x_1x'_1} & F_{x_1x'_2} & \cdots & F_{x_1x'_m} \\ F_{x_2x'_1} & F_{x_2x'_2} & \cdots & F_{x_2x'_m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{x_mx'_1} & F_{x_mx'_2} & \cdots & F_{x_mx'_m} \end{bmatrix},$$

$$a_{13} = \begin{bmatrix} \overline{F}_{x_1x_1(z_{10})}(1-\varphi'_{10}) & \overline{F}_{x_1x_2(z_{20})}(1-\varphi'_{20}) & \cdots & \overline{F}_{x_1x_m(z_{m0})}(1-\varphi'_{m0}) \\ \overline{F}_{x_2x_1(z_{10})}(1-\varphi'_{10}) & \overline{F}_{x_2x_2(z_{20})}(1-\varphi'_{20}) & \cdots & \overline{F}_{x_2x_m(z_{m0})}(1-\varphi'_{m0}) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_mx_1(z_{10})}(1-\varphi'_{10}) & \overline{F}_{x_mx_2(z_{20})}(1-\varphi'_{20}) & \cdots & \overline{F}_{x_mx_m(z_{m0})}(1-\varphi'_{m0}) \end{bmatrix},$$

$$a_{14} = \begin{bmatrix} \overline{F}_{x_1x_1(z_{11})}(1-\varphi'_{11}) & \overline{F}_{x_1x_2(z_{21})}(1-\varphi'_{21}) & \cdots & \overline{F}_{x_1x_m(z_{m1})}(1-\varphi'_{m1}) \\ \overline{F}_{x_2x_1(z_{11})}(1-\varphi'_{11}) & \overline{F}_{x_2x_2(z_{21})}(1-\varphi'_{21}) & \cdots & \overline{F}_{x_2x_m(z_{m1})}(1-\varphi'_{m1}) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_mx_1(z_{11})}(1-\varphi'_{11}) & \overline{F}_{x_mx_2(z_{21})}(1-\varphi'_{21}) & \cdots & \overline{F}_{x_mx_m(z_{m1})}(1-\varphi'_{m1}) \end{bmatrix},$$

$$a_{21} = \begin{bmatrix} F_{x'_1x_1} & F_{x'_1x_2} & \cdots & F_{x'_1x_m} \\ F_{x'_2x_1} & F_{x'_2x_2} & \cdots & F_{x'_2x_m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{x'_mx_1} & F_{x'_mx_2} & \cdots & F_{x'_mx_m} \end{bmatrix},$$

$$a_{22} = \begin{bmatrix} F_{x_1'x_1'} & F_{x_1'x_2'} & \cdots & F_{x_1'x_m'} \\ F_{x_2'x_1'} & F_{x_2'x_2'} & \cdots & F_{x_2'x_m'} \\ \vdots & \vdots & \ddots & \vdots \\ F_{x_m'x_1'} & F_{x_m'x_2'} & \cdots & F_{x_m'x_m'} \end{bmatrix},$$

$$a_{23} = \begin{bmatrix} \overline{F}_{x_1'x_1(z_{10})}(1-\varphi_{10}') & \overline{F}_{x_1'x_2(z_{20})}(1-\varphi_{20}') & \cdots & \overline{F}_{x_1'x_m(z_{m0})}(1-\varphi_{m0}') \\ \overline{F}_{x_2'x_1(z_{10})}(1-\varphi_{10}') & \overline{F}_{x_2'x_2(z_{20})}(1-\varphi_{20}') & \cdots & \overline{F}_{x_2'x_m(z_{m0})}(1-\varphi_{m0}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'x_1(z_{10})}(1-\varphi_{10}') & \overline{F}_{x_m'x_2(z_{20})}(1-\varphi_{20}') & \cdots & \overline{F}_{x_m'x_m(z_{m0})}(1-\varphi_{m0}') \end{bmatrix},$$

$$a_{24} = \begin{bmatrix} \overline{F}_{x_1'x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_1'x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_1'x_m'(z_{m1})}(1-\varphi_{m1}') \\ \overline{F}_{x_2'x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_2'x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_2'x_m'(z_{m1})}(1-\varphi_{m1}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_m'x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_m'x_m'(z_{m1})}(1-\varphi_{m1}') \end{bmatrix},$$

$$a_{31} = \begin{bmatrix} \overline{F}_{x_1(z_{10})x_1}(1-\varphi_{10}') & \overline{F}_{x_1(z_{10})x_2}(1-\varphi_{10}') & \cdots & \overline{F}_{x_1(z_{10})x_m}(1-\varphi_{10}') \\ \overline{F}_{x_2(z_{20})x_1}(1-\varphi_{20}') & \overline{F}_{x_2(z_{20})x_2}(1-\varphi_{20}') & \cdots & \overline{F}_{x_2(z_{20})x_m}(1-\varphi_{20}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m(z_{m0})x_1}(1-\varphi_{m0}') & \overline{F}_{x_m(z_{m0})x_2}(1-\varphi_{m0}') & \cdots & \overline{F}_{x_m(z_{m0})x_m}(1-\varphi_{m0}') \end{bmatrix}$$

$$a_{32} = \begin{bmatrix} \overline{F}_{x_1(z_{10})x_1'}(1-\varphi_{10}') & \overline{F}_{x_1(z_{10})x_2'}(1-\varphi_{10}') & \cdots & \overline{F}_{x_1(z_{10})x_m'}(1-\varphi_{10}') \\ \overline{F}_{x_2(z_{20})x_1'}(1-\varphi_{20}') & \overline{F}_{x_2(z_{20})x_2'}(1-\varphi_{20}') & \cdots & \overline{F}_{x_2(z_{20})x_m'}(1-\varphi_{20}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m(z_{m0})x_1'}(1-\varphi_{m0}') & \overline{F}_{x_m(z_{m0})x_2'}(1-\varphi_{m0}') & \cdots & \overline{F}_{x_m(z_{m0})x_m'}(1-\varphi_{m0}') \end{bmatrix}$$

$$a_{33} = \begin{bmatrix} \overline{F}_{x_1(z_{10})x_1(z_{10})}(1-\varphi_{10}') & \overline{F}_{x_1(z_{10})x_2(z_{20})}(1-\varphi_{10}')(1-\varphi_{20}') & \cdots & \overline{F}_{x_1(z_{10})x_m(z_{m0})}(1-\varphi_{10}')(1-\varphi_{m0}') \\ \overline{F}_{x_2(z_{20})x_1(z_{10})}(1-\varphi_{20}')(1-\varphi_{10}') & \overline{F}_{x_2(z_{20})x_2(z_{20})}(1-\varphi_{20}') & \cdots & \overline{F}_{x_2(z_{20})x_m(z_{m0})}(1-\varphi_{20}')(1-\varphi_{m0}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m(z_{m0})x_1(z_{10})}(1-\varphi_{m0}')(1-\varphi_{10}') & \overline{F}_{x_m(z_{m0})x_2(z_{20})}(1-\varphi_{m0}')(1-\varphi_{20}') & \cdots & \overline{F}_{x_m(z_{m0})x_m(z_{m0})}(1-\varphi_{m0}')(1-\varphi_{m0}') \end{bmatrix}$$

$$a_{34} = \begin{bmatrix} \overline{F}_{x_1(z_{10})x_1'(z_{11})}(1-\varphi_{10}') & \overline{F}_{x_1(z_{10})x_2'(z_{21})}(1-\varphi_{10}')(1-\varphi_{21}') & \cdots & \overline{F}_{x_1(z_{10})x_m'(z_{m1})}(1-\varphi_{10}')(1-\varphi_{m1}') \\ \overline{F}_{x_2(z_{20})x_2'(z_{11})}(1-\varphi_{20}')(1-\varphi_{11}') & \overline{F}_{x_2(z_{20})x_2'(z_{21})}(1-\varphi_{20}') & \cdots & \overline{F}_{x_2(z_{20})x_m'(z_{m1})}(1-\varphi_{20}')(1-\varphi_{m1}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m(z_{m0})x_1'(z_{11})}(1-\varphi_{m0}')(1-\varphi_{11}') & \overline{F}_{x_m(z_{m0})x_2'(z_{21})}(1-\varphi_{m0}')(1-\varphi_{21}') & \cdots & \overline{F}_{x_m(z_{m0})x_m'(z_{m1})}(1-\varphi_{m0}')(1-\varphi_{m1}') \end{bmatrix}$$

$$a_{41} = \begin{bmatrix} \overline{F}_{x_1'(z_{11})x_1}(1-\varphi_{11}') & \overline{F}_{x_1'(z_{11})x_2}(1-\varphi_{11}') & \cdots & \overline{F}_{x_1'(z_{11})x_m}(1-\varphi_{11}') \\ \overline{F}_{x_2'(z_{21})x_1}(1-\varphi_{21}') & \overline{F}_{x_2'(z_{21})x_2}(1-\varphi_{21}') & \cdots & \overline{F}_{x_2'(z_{21})x_m}(1-\varphi_{21}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'(z_{m1})x_1}(1-\varphi_{m1}') & \overline{F}_{x_m'(z_{m1})x_2}(1-\varphi_{m1}') & \cdots & \overline{F}_{x_m'(z_{m1})x_m}(1-\varphi_{m1}') \end{bmatrix}$$

$$a_{42} = \begin{bmatrix} \overline{F}_{x_1'(z_{11})x_1'}(1-\varphi_{11}') & \overline{F}_{x_1'(z_{11})x_2'}(1-\varphi_{11}') & \cdots & \overline{F}_{x_1'(z_{11})x_m'}(1-\varphi_{11}') \\ \overline{F}_{x_2'(z_{21})x_1'}(1-\varphi_{21}') & \overline{F}_{x_2'(z_{21})x_2'}(1-\varphi_{21}') & \cdots & \overline{F}_{x_2'(z_{21})x_m'}(1-\varphi_{21}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'(z_{m1})x_1'}(1-\varphi_{m1}') & \overline{F}_{x_m'(z_{m1})x_2'}(1-\varphi_{m1}') & \cdots & \overline{F}_{x_m'(z_{m1})x_m'}(1-\varphi_{m1}') \end{bmatrix}$$

$$a_{43} = \begin{bmatrix} \overline{F}_{x_1'(z_{11})x_1(z_{10})}(1-\varphi_{11}')(1-\varphi_{10}') & \overline{F}_{x_1'(z_{11})x_2(z_{20})}(1-\varphi_{11}')(1-\varphi_{20}') & \cdots & \overline{F}_{x_1'(z_{11})x_m(z_{m0})}(1-\varphi_{11}')(1-\varphi_{m0}') \\ \overline{F}_{x_2'(z_{21})x_2(z_{10})}(1-\varphi_{21}')(1-\varphi_{10}') & \overline{F}_{x_2'(z_{21})x_2(z_{20})}(1-\varphi_{21}')(1-\varphi_{20}') & \cdots & \overline{F}_{x_2'(z_{21})x_m(z_{m0})}(1-\varphi_{21}')(1-\varphi_{m0}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'(z_{m1})x_1(z_{10})}(1-\varphi_{m1}')(1-\varphi_{10}') & \overline{F}_{x_m'(z_{m1})x_2(z_{20})}(1-\varphi_{m1}')(1-\varphi_{20}') & \cdots & \overline{F}_{x_m'(z_{m1})x_m(z_{m0})}(1-\varphi_{m1}')(1-\varphi_{m0}') \end{bmatrix}$$

and

$$a_{44} = \begin{bmatrix} \overline{F}_{x_1'(z_{11})x_1'(z_{11})}(1-\varphi_{11}') & \overline{F}_{x_1'(z_{11})x_2'(z_{21})}(1-\varphi_{11}')(1-\varphi_{21}') & \cdots & \overline{F}_{x_1'(z_{11})x_m'(z_{m1})}(1-\varphi_{11}')(1-\varphi_{m1}') \\ \overline{F}_{x_2'(z_{21})x_2'(z_{11})}(1-\varphi_{21}')(1-\varphi_{11}') & \overline{F}_{x_2'(z_{21})x_2'(z_{21})}(1-\varphi_{21}') & \cdots & \overline{F}_{x_2'(z_{21})x_m'(z_{m1})}(1-\varphi_{21}')(1-\varphi_{m1}') \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F}_{x_m'(z_{m1})x_1'(z_{11})}(1-\varphi_{11}')(1-\varphi_{m1}') & \overline{F}_{x_m'(z_{m1})x_2'(z_{21})}(1-\varphi_{m1}')(1-\varphi_{21}') & \cdots & \overline{F}_{x_m'(z_{m1})x_m'(z_{m1})}(1-\varphi_{m1}') \end{bmatrix}$$

Hence, according to Legendre theorem (2.2), the Legendre condition of the functional (2.34) is the 4m×4m Hessian matrix H which is positive definite. ■

2.5 The Direct-Ritz Method

Problems with time lag, especially in differential equations, are of great importance; therefore, their solutions are also of great importance, [Marie, 2001].

Since the analytic solution of DDE's is so difficult to be evaluated, thus approximate solutions that minimize the functional are necessary, and therefore there exists two alternative approaches:

i) Solving the DDE by approximate methods (e.g., the finite difference method, the collocation method, etc.).

ii) Direct-Ritz method.

The direct-Ritz method is used to find the approximate solution of boundary and initial value problems, which were first presented by Lord Rayleigh in 1870 where the approximating subspace was of dimension one. In 1909, Ritz generalized the method to an arbitrary dimension.

Ritz method was popular before the invention of the computer, and remains so today, because it can yield accurate results for complex problems that are difficult to solve analytically. The idea of Ritz method is to reduce the variational problem on the set of all admissible functions S to the problem of minimizing the same functional on a finite (N-dimensional) subspace $\mathbb{R}^n$ of continuous functions that can approximate the solution, such as the set of polynomials of degree less than or equal to n. Then:

$$v[x] = \min_{x \in S} v[x] \leq \min_{x^* \in \mathbb{R}^n} v[x^*],$$

so that the approximate solution x* is a polynomial expression:

$$x_n^*(t) = x_0^*(t) + \sum_{k=1}^{n} a_k x_k^*(t).$$

where $x_0^*(t)$ satisfies the nonhomogeneous initial and boundary conditions while $x_k^*(t)$ satisfies the homogeneous conditions. Giving upper bound and satisfying the given boundary conditions, a_k, (k = 1, 2, …, n); which are referred to as the generalized coordinates.

Substitute this approximation in the functional, then integrate and minimize with respect to the unknown parameters a_k, for all (k = 1, 2, …, n), [Lensnic, 1999].

2.6 Illustrative Examples

For the sake of illustration and explanation of the inverse problem of ODDEv's, we will consider here some illustrative examples in both cases; constant and variable delays.

Although, some examples with known exact solutions will be taken for accuracy and comparison purpose.

__*Example (2.19):*__

Consider the retarded initial-boundary ODDE with constant delay:

$$x''(t) - x(t-1) = 0,\ 0 \le t \le 1,$$

with the boundary conditions,

$$x(0) = 0,\ x(1) = \frac{2}{3};$$

and the delay initial condition, $x(t) = t;\ t \in [-1, 0]$, [Marie, 2001].

Applying Magrie's approach with the shift operator for the delay terms:

$$Dx(t) = x(t-1),$$

which then implies:

$$x''(t) - x(t-1) = 0$$

$$\frac{d^2}{dt^2} x(t) - Dx(t) = 0$$

$$\left\{\frac{d^2}{dt^2} - D\right\} x(t) = 0.$$

Letting:

$$L = \left(\frac{d^2}{dt^2} - D\right).$$

Now, to ensure the symmetry, let <x, y> = (x, Ly), then the functional takes the form:

$$v[x] = \frac{1}{2} \langle Lx, x \rangle - \langle f, x \rangle$$

$$= \frac{1}{2}(Lx, Lx) - 0$$

$$= \frac{1}{2} \int_{t_0}^{t_1} (Lx)^2 \, dt$$

$$= \frac{1}{2} \int_{0}^{1} \{x''(t) - x(t-1)\}^2 \, dt \quad \text{...(2.35)}$$

Hence, using the direct-Ritz method with the following approximate solution:

$$x(t) = \frac{2}{3} t + t(t-1)\{a_0 + a_1 t + a_2 t^2\} \quad \text{...(2.36)}$$

where a_0, a_1, and $a_2 \in \mathbb{R}$.

Substituting eq.(2.36) in functional (2.35) and carrying out the minimization using the computer program (ODDE 1) written in MathCad software, one can get the following results for a_0, a_1 and a_2:

$$a_0 = -0.333,\ a_1 = 0.167,\ a_2 = -2.71\times10^{-9}.$$

Hence, the approximate solution is given by:

$$x(t) = \frac{2}{3}t + t(t-1)\{-0.333 + 0.167t + 2.71\times10^{-9}t^2\}.$$

Figure (2.1) presents a comparison between the approximate and the exact solutions, where the exact solution is given by:

$$x(t) = \frac{t^3}{6} - \frac{t^2}{2} + t,\ t \in [0, 1].$$

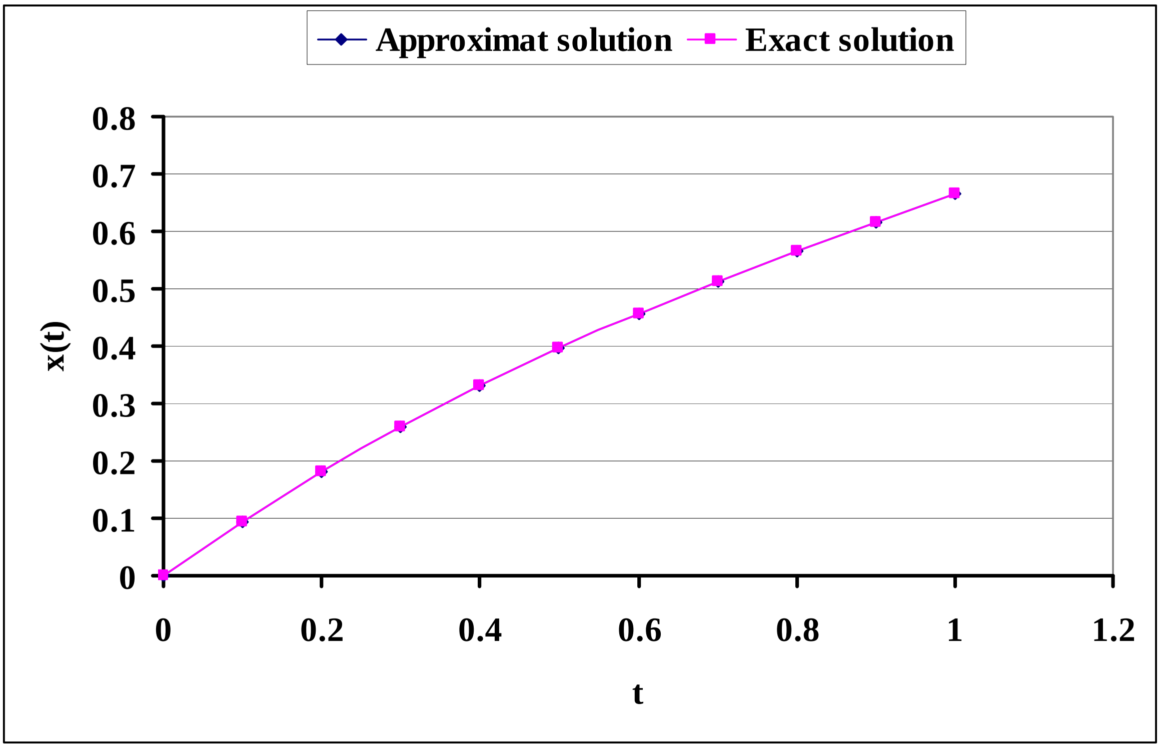

Fig.(2.1) Comparison between the approximate and exact solutions of example (2.19).

Example (2.20):

Consider the mixed initial-boundary ODDE with constant delay:

$$x''(t) = x(t-1) + 2x'(t-1),\ 0 \le t \le 1,$$

with the boundary conditions:

$$x(0) = 0,\ x(1) = \frac{5}{3};$$

and delay initial condition $x(t) = t$, $t \in [-1, 0]$, [Marie, 2001].

Hence:

$$x''(t) - x(t-1) - 2x'(t-1) = 0$$

$$\frac{d^2}{dt^2}x(t) - Dx(t) - 2\frac{d}{dt}Dx(t) = 0$$

$$\left\{\frac{d^2}{dt^2} - D - 2\frac{d}{dt}D\right\}x(t) = 0.$$

Then:

$$L = \left(\frac{d^2}{dt^2} - D - 2\frac{d}{dt}D\right).$$

Also, the symmetry is ensured by considering $<x, y> = (x, Ly)$; therefore:

$$v[x] = \frac{1}{2}<Lx, x> - <f, x>$$

$$= \frac{1}{2}(Lx, Lx) - (f, x)$$

$$= \frac{1}{2}(Lx, Lx)$$

$$= \frac{1}{2}\int_{t_0}^{t_1} (Lx)^2\, dt$$

$$= \frac{1}{2}\int_{0}^{1} \{x''(t) - (t-1) - 2\}^2\, dt \quad \text{.......................................(2.37)}$$

Now, using the direct-Ritz method with the following approximate solution:

$$x(t) = \frac{5}{3}t + t(t-1)\{a_0 + a_1 t + a_2 t^2\} \quad \text{.......................................(2.38)}$$

where $a_i \in \mathbb{R}$, for all (i = 0, 1, 2).

Substituting eq.(2.38) back into the functional (2.37) and carrying out the minimization using the computer program (ODDE 2) written in MathCad, one can get the following results for a_0, a_1 and a_2:

$a_0 = 0.667$, $a_1 = 0.167$, $a_2 = 1.478\times10^{-10}$.

Hence, the approximate solution is given by:

$$x(t) = \frac{5}{3}t + t(t-1)\{0.667 + 0.167t + 1.478\times10^{-10}t^2\}.$$

Figure (2.2) presents a comparison between the approximate and the exact solutions, where the exact solution is:

$$x(t) = \frac{t^3}{6} + \frac{t^2}{2} + t,\ t \in [0, 1].$$

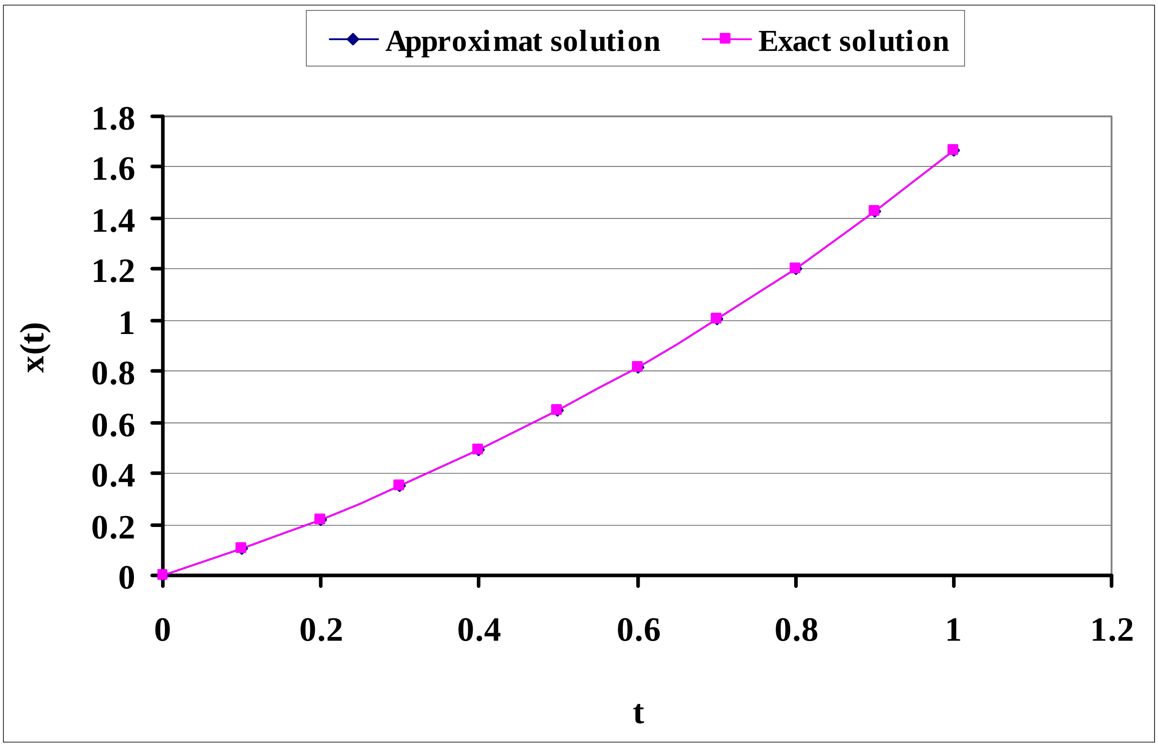

Fig.(2.2) Comparison between the approximate and exact solutions of example (2.20).

Example (2.21):

Consider the retarded initial-boundary functional-differential equation with variable delay:

$$x'(t) = 1 - x\left(e^{1-\frac{1}{t}}\right), \ t \in [0.1, 0.2] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.39)$$

with the boundary conditions

$$x(0.1) = -2.302, \ x(0.2) = -1.609;$$

and the delay initial condition $x(t) = \ln(t)$, $0 \le t \le 0.1$, [Al-Dafae'e, 2005].

The DDE (2.39) may be rewritten as:

$$x'(t) = 1 - x\left(t - \left(t - e^{1-\frac{1}{t}}\right)\right), \; 0.1 \le t \le 0.2,$$

with delay function $\varphi(t) = t - e^{1/t}$. It follows that:

$$x'(t) = 1 - x(t - \varphi(t))$$

$$x'(t) + x(t - \varphi(t)) = 1$$

$$\left\{\frac{d}{dt} + D\right\} x(t) = 1.$$

Hence:

$$L = \left(\frac{d}{dt} + D\right), \; f(x) = 1 \; .$$

Now by considering <x, y> = (x, Ly), then:

$$v[x] = \frac{1}{2} <Lx, x> - <f, x>$$

$$= \frac{1}{2}(Lx, Lx) - (f, Lx)$$

$$= \frac{1}{2}\int_{t_0}^{t_1} (Lx)^2 \, dt - \int_{t_0}^{t_1} f\,(Lx)\, dt$$

$$= \frac{1}{2}\int_{0.1}^{0.2} \{\{x'(t) + \ln(t - \varphi(t))\}^2 - 2\{x'(t) + \ln(t - \varphi(t))\}\}\, dt \quad \text{...(2.40)}$$

Using the direct-Ritz method with the following approximate solution:

$$x(t) = (6.941)t - (2.997) + (t - 0.1)(t - 0.2)\{a_0 + a_1 t\} \quad \text{...........(2.41)}$$

where $a_i \in \mathbb{R}$, i = 0, 1.

Substituting eq.(2.41) back into the functional (2.40) and carrying out the minimization using the computer program (ODDE 3) written in MathCad, one can get the following results for a_0 and a_1:

$$a_0 = -40.202,\ a_1 = 109.133.$$

Hence, the approximate solution is given by:

$$x(t) = (6.941)t - (2.997) + (t - 0.1)(t - 0.2)\{-40.202 + 109.133t\}.$$

Figure (2.3) presents a comparison between the approximate and the exact solutions, where the exact solution is:

$$x(t) = \ln(t),\ t \in [0.1, 0.2].$$

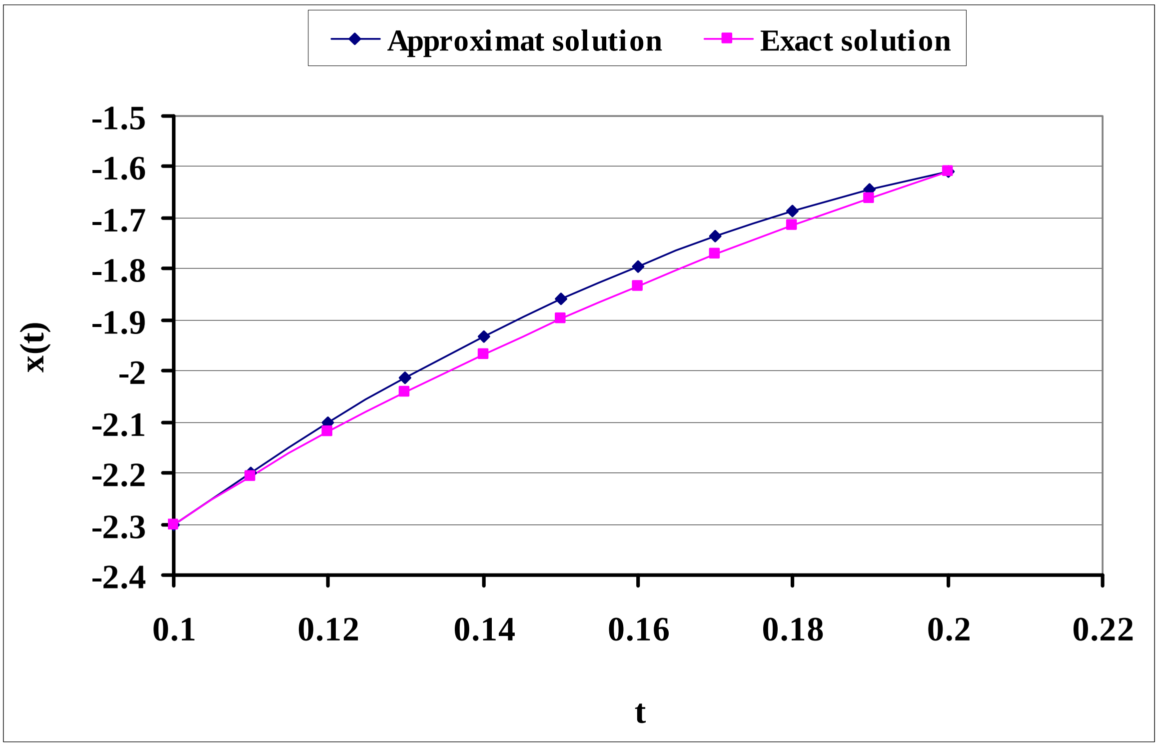

Fig.(2.3) Comparison between the approximate and exact solutions of example (2.21).

Example (2.22):

Consider the retarded initial-boundary ODDE with variable delay:

$$x'(t) = t + x(t) + x(t - 0.5tx(t)),\ t \in [0, 1],$$

with the boundary conditions:

$$x(0) = 0,\ x(1) = 0.892526,$$

and the delay initial condition x(t) = t, t < 0, [Al-Dafae'e, 2005].

Applying Magrie's approach, gives:

$$x'(t) - x(t) - x(t - \varphi(t)) = t,\ t \in [0, 1],\ \varphi(t) = 0.5tx(t)$$

$$\frac{d}{dt}x(t) - x(t) - Dx(t) = t$$

$$\left\{\frac{d}{dt} - 1 - D\right\}x(t) = t.$$

Hence:

$$L = \left(\frac{d}{dt} - 1 - D\right),\ f(x) = t.$$

By considering <x, y> = (x, Ly), then:

$$v[x] = \frac{1}{2}\langle Lx, x\rangle - \langle f, x\rangle$$

$$= \frac{1}{2}(Lx, Lx) - (f, Lx)$$

$$= \frac{1}{2}\int_0^1 \left\{\{x'(t) - x(t) - x(t - \varphi(t))\}^2 - 2t\{x'(t) - x(t) - x(t - \varphi(t))\}\right\} dt \quad \text{.......(2.42)}$$

Using the direct-Ritz method with the following approximate solution:

$$x(t) = (0.892526)\, t + t(t-1)\{a_0 + a_1 t\} \qquad (2.43)$$

where $a_i \in \mathbb{R}$, $i = 0, 1$.

Substituting eq.(2.43) back into the functional (2.41) and carrying out the minimization using the computer program (ODDE 4) written in MathCad, one can get the following results for a_0 and a_1:

$a_0 = 1.267$, $a_1 = 0.148$.

Hence, the approximate solution is given by:

$$x(t) = (0.892526)\, t + t(t-1)\{1.267 + 0.148t\}.$$

Table (2.1) presents the approximate results and the residue error.

Table (2.1)

Approximate results and the residue error of example (2.22).

x	*Approximate solution*	*Residue error*
0	0	0.14
0.1	–0.02611	0.103
0.2	–0.02895	0.08
0.3	-7.64×10^{-3}	0.066
0.4	0.038722	0.057
0.5	0.111013	0.052
0.6	0.210124	0.048
0.7	0.336942	0.044
0.8	0.492357	0.038
0.9	0.677255	0.03
1	0.892526	0.019

3 Variational Formulation of Partial Delay-Differential Equations with Variable Delays

Many problems in science and technology may be formulated as the variational problems of partial delay-differential equations, where certain functional whose domains consists of certain vector spaces of functions of several independent variables that the delay terms may occur in one or more of these variables, is required to be minimized, [Catillo, 2005].

In this chapter, the discussion will be restricted for PDDEv's in which the unknown function depends only on two variables, say x and t and the delay terms occur in x.

This chapter consists of five sections; In section one, the general form of the variational problem of PDDEv's is presented, the corresponding increment of the functional is also stated and the first and second variation are given for certain functionals. In section two, the necessary condition for the existence of an extremum is given. In section three; additional necessary conditions are discussed such as, Legendre and Jacobi's conditions. Some illustrative examples of constant and variable PDDE's are given in section four and solved by minimizing the related functional. Finally, in section five, real life problem is solved using variational formulation as an application for this subject in PDDE's.

3.1 The Variational Problem

The variational problem of PDDEv's are concerned with the functionals of the form:

$$v[u(x, y)] = \iint_D F\left(x, t; u(x,t), \frac{\partial}{\partial x}u(x,t), \frac{\partial}{\partial t}u(x,t), u(x-\varphi_0, t), \frac{\partial}{\partial x}u(x-\varphi_1^x, t), \frac{\partial}{\partial t}u(x-\varphi_1^t, t) \right) dxdt \quad \text{................(3.1)}$$

Here D is some given fixed region in the xt-plane, u = u(x, t) is a function defined for all points (x, t) ∈ D, which is assumed to be of class $C^n(D)$, and $\varphi_i^{x,t} = \varphi_i^{x,t}\left(x, t; u(x,t), \frac{\partial}{\partial x}u(x,t), \frac{\partial}{\partial t}u(x,t) \right)$, for (i = 0, 1); are the delay functions where i refers to the order of partial derivatives with respect to the independent variable x or t, [Lebedev, 2003].

Assume that u* = u*(x, t) to be the solution for the function (3.1) and that δu(x, t) is an dismissible function from $C_0^1(D)$; and satisfies $\delta u|_{\partial D} = 0$, where ∂D refers to the boundary of the region D.

Now, create the set of dismissible functions:

u*(x, t) = u(x, t) + εδu(x, t), for sufficiently small |ε|

Evaluate v[u(x, t)] at u = u*, to obtain the problem of minimizing the functional as [Russak, 2002]:

$$v[u] = \iint_D F\, dxdt$$

$$\leq \iint_D F(x, t; u + \delta u, \frac{\partial}{\partial x} u + \delta \frac{\partial}{\partial x} u, \frac{\partial}{\partial t} u + \delta \frac{\partial}{\partial t} u, u(x - \varphi_0, t)$$

$$+ \delta u(x - \varphi_0, t), \frac{\partial}{\partial x} u(x - \varphi_1^x, t) + \delta \frac{\partial}{\partial x} u(x - \varphi_1^x, t),$$

$$\frac{\partial}{\partial t} u(x - \varphi_1^t, t) + \delta \frac{\partial}{\partial t} u(x - \varphi_1^t, t))\, dxdt$$

$$= \iint_D F(x, t; u^*, \frac{\partial}{\partial x} u^*, \frac{\partial}{\partial t} u^*, u^*(x - \varphi_0, t), \frac{\partial}{\partial x} u^*(x - \varphi_1^x,$$

$$t), \frac{\partial}{\partial t} u^*(x - \varphi_1^t, t))\, dxdt$$

$$= v[u^*] \quad \ldots\ldots (3.2)$$

and when requires to maximize the functional (3.1), then just reverse the inequality (3.2).

Hence, the following generalization of the fundamental lemma of calculus of variation may be given:

Lemma (3.1), [Russak, 2002]:

If $\alpha(x, t)$ is continuous over the region D in the xt-plane, and if:

$$\iint_D \alpha(x, t)\, \delta u(x, t)\, dxdt = 0;$$

for every continuous function $\delta u(x, t)$ defined over D, and satisfying $\delta u = 0$, on the boundary of D, then $\alpha(x, t) \equiv 0$, for all $(x, t) \in D$.

Remark (3.2):

For easiness, the following will be symbolized as:

1. p and q denotes the partial derivatives $\frac{\partial}{\partial x}u$ and $\frac{\partial}{\partial t}u$, respectively.
2. The notion F, as shorthand for the general form of the integrand function $F(x, t; u, p, q, u(x - \varphi_0, t), p(x - \varphi_1^x, t), q(x - \varphi_1^t, t))$.
3. ∂D means the boundary of D, i.e., the boundary of the variational problem at the points; x_0, x_1, t_0, t_1, $x_0 - \eta_0$ and $x_1 - \eta_1$, where $\eta_0 = \operatorname{Max}\left\{\varphi_i^{x,t}(x_0, t; u(x_0, t), p(x_0, t), q(x_0, t))\right\}$, $\eta_1 = \operatorname{Max}\left\{\varphi_i^{x,t}(x_1, t; u(x_1, t), p(x_1, t), q(x_1, t))\right\}$, for each (i = 0, 1); and the boundary values being prescribed as:

   $u(x_0, t_0) = u_0$, $p(x_0, t_0) = p_0$, $q(x_0, t_0) = q_0$,

   $u(x_1, t_1) = u_1$, $p(x_1, t_1) = p_1$, $q(x_1, t_1) = q_1$,

   $u(x_0 - \eta_0, t_0) = u_{\eta_0}$, $p(x_0 - \eta_0, t_0) = p_{\eta_0}$, $q(x_0 - \eta_0, t_0) = q_{\eta_0}$

   and

   $u(x_1 - \eta_1, t_1) = u_{\eta_1}$, $p(x_1 - \eta_1, t_1) = p_{\eta_1}$, $q(x_1 - \eta_1, t_1) = q_{\eta_1}$.

In order to find the first and second variation of the functional (3.1) about an admissible curve u*, suppose that $F(x, t; u^*, p^*, q^*, u^*(x - \varphi_0, t), p^*(x - \varphi_1^x, t), q^*(x - \varphi_1^t, t))$, is a function whose n-th partial derivatives with respect to u*, p*, q* exist, and continuous in some domain D. Let ΔF refers to the linear part of the increment, i.e.,

$\Delta F = F(x, t; u + \delta u, p + \delta p, q + \delta q, u(x - \varphi_0, t) + \delta u(x, - \varphi_0, t),$

$p(x - \varphi_1^x, t) + \delta p(x - \varphi_1^x, t), q(x - \varphi_1^t, t) + \delta q(x - \varphi_1^t, t)) -$

$F|_{\text{linear part}}.$

Then if $(u + \delta u, p + \delta p, q + \delta q, u(x - \varphi_0, t) + \delta u(x, - \varphi_0, t), p(x - \varphi_1^x, t) + \delta p(x - \varphi_1^x, t), q(x - \varphi_1^t, t) + \delta q(x - \varphi_1^t, t))$ lies in D, then the Taylor series expansion of F is given by:

$$F(x, t; u^*, p^*, q^*, u^*(x - \varphi_0, t), p^*(x - \varphi_1^x, t), q^*(x - \varphi_1^t, t)) = F + \frac{1}{1!}\left(\delta u\frac{\partial}{\partial u} + \delta p\frac{\partial}{\partial p} + \delta q\frac{\partial}{\partial q} + \delta u(x-\varphi_0,t)\frac{\partial}{\partial u(x-\varphi_0,t)} + \delta p(x-\varphi_1^x,t)\frac{\partial}{\partial p(x-\varphi_1^x,t)} + \delta q(x-\varphi_1^t,t)\frac{\partial}{\partial q(x-\varphi_1^t,t)}\right)F + \frac{1}{2!}\left(\delta u\frac{\partial}{\partial u} + \delta p\frac{\partial}{\partial p} + \delta q\frac{\partial}{\partial q} + \delta u(x-\varphi_0,t)\frac{\partial}{\partial u(x-\varphi_0,t)} + \delta p(x-\varphi_1^x,t)\frac{\partial}{\partial p(x-\varphi_1^x,t)} + \delta q(x-\varphi_1^t,t)\frac{\partial}{\partial q(x-\varphi_1^t,t)}\right)^2 F + \ldots,$$

where the notation:

$$\left(\delta u\frac{\partial}{\partial u} + \delta p\frac{\partial}{\partial p} + \delta q\frac{\partial}{\partial q} + \delta u(x-\varphi_0,t)\frac{\partial}{\partial u(x-\varphi_0,t)} + \delta p(x-\varphi_1^x,t)\frac{\partial}{\partial p(x-\varphi_1^x,t)} + \delta q(x-\varphi_1^t,t)\frac{\partial}{\partial q(x-\varphi_1^t,t)}\right)^r F,$$

refers to the r-th variation of F and denoted by $\delta^r F$.

Therefore:

$$\Delta F = \delta F + \delta^2 F + \ldots + \delta^r F + \ldots$$

where:

$$\delta F = \left(\delta u \frac{\partial}{\partial u} + \delta p \frac{\partial}{\partial p} + \delta q \frac{\partial}{\partial q} + \delta u(x-\varphi_0,t) \frac{\partial}{\partial u(x-\varphi_0,t)} + \right.$$

$$\left. \delta p(x-\varphi_1^x,t) \frac{\partial}{\partial p(x-\varphi_1^x,t)} + \delta q(x-\varphi_1^t,t) \frac{\partial}{\partial q(x-\varphi_1^t,t)} \right) F$$

$$= F_u \delta u + F_p \delta p + F_q \delta q + F_{u(x-\varphi_0,t)} \delta u(x-\varphi_0,t) +$$

$$F_{p(x-\varphi_1^x,t)} \delta p(x-\varphi_1^x,t) + F_{q(x-\varphi_1^t,t)} \delta q(x-\varphi_1^t,t) \quad \text{.............(3.3)}$$

which refers to the first variation in F, and:

$$\delta^2 F = \frac{1}{2!} \left(\delta u \frac{\partial}{\partial u} + \delta p \frac{\partial}{\partial p} + \delta q \frac{\partial}{\partial q} + \delta u(x-\varphi_0,t) \frac{\partial}{\partial u(x-\varphi_0,t)} + \right.$$

$$\left. \delta p(x-\varphi_1^x,t) \frac{\partial}{\partial p(x-\varphi_1^x,t)} + \delta q(x-\varphi_1^t,t) \frac{\partial}{\partial q(x-\varphi_1^t,t)} \right)^2 F$$

$$= \frac{1}{2!} \left\{ F_{uu} \delta u^2 + F_{pp} \delta p^2 + F_{qq} \delta q^2 + F_{u(x-\varphi_0,t)u(x-\varphi_0,t)} \right.$$

$$\delta u^2(x-\varphi_0,t) + F_{p(x-\varphi_1^x,t)p(x-\varphi_1^x,t)} \delta p^2(x-\varphi_1^x,t) +$$

$$F_{q(x-\varphi_1^t,t)q(x-\varphi_1^t,t)} \delta q^2(x-\varphi_1^t,t) + 2F_{up} \delta u \delta p + 2F_{uq} \delta u \delta q$$

$$2F_{uu(x-\varphi_0,t)} \delta u \delta u(x-\varphi_0,t) + 2F_{up(x-\varphi_1^x,t)} \delta u \delta p(x-\varphi_1^x,t) +$$

$$2F_{uq(x-\varphi_1^t,t)} \delta u \delta q(x-\varphi_1^t,t) + 2F_{pq} \delta p \delta q +$$

$$2F_{pu(x-\varphi_0,t)} \delta p \delta u(x-\varphi_0,t) + 2F_{pp(x-\varphi_1^x,t)} \delta p \delta p(x-\varphi_1^x,t) +$$

$$2F_{pq(x-\varphi_1^t,t)} \delta p \delta q(x-\varphi_1^t,t) + 2F_{qu(x-\varphi_0,t)} \delta q \delta u(x-\varphi_0,t) +$$

$$2F_{qp(x-\varphi_1^x,t)}\delta q\delta p(x-\varphi_1^x,t)+2F_{qq(x-\varphi_1^t,t)}\delta q\delta q(x-\varphi_1^t,t)+$$

$$2F_{u(x-\varphi_0,t)p(x-\varphi_1^x,t)}\delta u(x-\varphi_0,t)\delta p(x-\varphi_1^x,t)+$$

$$2F_{u(x-\varphi_0,t)q(x-\varphi_1^t,t)}\delta u(x-\varphi_0,t)\delta q(x-\varphi_1^t,t)+$$

$$\left.2F_{p(x-\varphi_1^x,t)q(x-\varphi_1^t,t)}\delta p(x-\varphi_1^x,t)\delta q(x-\varphi_1^t,t)\right\} \qquad ....(3.4)$$

which refers to the second variation in F and so on for δ^3F, δ^4F, …, that are going to be more difficult to compute.

According to the last discussion, the first variation of the functional (3.1) about u*(x, t) corresponding to the first derivative of F about its argument along D, is:

$$\delta v = \iint_D \{F_u\delta u + F_p\delta p + F_q\delta q + F_{u(x-\varphi_0,t)}\delta u(x-\varphi_0,t) +$$

$$F_{p(x-\varphi_1^x,t)}\delta p(x-\varphi_1^x,t) + F_{q(x-\varphi_1^t,t)}\delta q(x-\varphi_1^t,t)\}\, dxdt$$

$$= \iint_D \delta F\, dxdt,$$

where δF as given in eq.(3.3).

The second variation of v about u*(x, t):

$$\delta^2 v = \iint_D \delta^2 F\, dxdt,$$

where δ^2F as is given in eq.(3.4).

3.2 The Fundamental Necessary Condition for an Extremum of the Variational Problems of PDDEv's

In this section, the fundamental necessary conditions for an extremum of PDDEv's are given in details. The general approach is based on similar steps that are followed to find the necessary conditions for the variational problem of ODDEv's, specifically:

i) Introduces a class of functions which will be needed to consider the problem of minimization.

ii) Depends on the fundamental lemma (3.1) of two independent variables.

iii) Recalls how to integrate by parts for two variables, [Berechtken-Mandrcheid, 1991].

The next theorems establish the necessary condition for an extremum of functionals of more than one independent variable.

Theorem (3.3):

Let $u = u(x, t) \in C^2(D)$, be an extremum for the functional:

$$v[u] = \iint_D F(x, t; u, p, q, u(x-\varphi_0, t), p(x-\varphi_1^x, t), q(x-\varphi_1^t, t))\, dxdt$$

.................................(3.5)

on the subset D of $\mathbb{R}^2$, consisting of those functions satisfying the boundary condition $\delta u\big|_{\partial D} = 0$. Then the necessary condition for an extremum is:

$$\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial x}F_q\right) + \left(\overline{F}_{u(x-\varphi_0,t)}(1-\varphi_0') - \frac{\partial}{\partial x}\left\{\overline{F}_{p(x-\varphi_1^x,t)}\right.\right.$$

$$\left.\left.(1-\varphi_1^{x'})\right\} - \frac{\partial}{\partial x}\left\{\overline{F}_{p(x-\varphi_1^t,t)}(1-\varphi_1^{t'})\right\}\right) = 0, \text{ for all } (x, t) \in D$$

where $\overline{F}$ is a symbol used for the function F with time-lag functions.

Proof:

The first variation for the simplest case of the functional v[u] is:

$$\delta v[u] = \iint_D \delta F \, dxdt$$

$$= \iint_D \{F_u\delta u + F_p\delta p + F_q\delta q + F_{u(x-\varphi_0,t)}\delta u(x-\varphi_0,t) +$$

$$F_{p(x-\varphi_1^x,t)}\delta p(x-\varphi_1^x,t) + F_{q(x-\varphi_1^t,t)}\delta q(x-\varphi_1^t,t)\} \, dxdt.$$

The necessary condition for a functional v to have an extremum, is that its first variation should be vanished, i.e., $\delta v = 0$, this condition is therefore yields to:

$$\iint_D \{F_u\delta u + F_p\delta p + F_q\delta q + F_{u(x-\varphi_0,t)}\delta u(x-\varphi_0,t) +$$

$$F_{p(x-\varphi_1^x,t)}\delta p(x-\varphi_1^x,t) + F_{q(x-\varphi_1^t,t)}\delta q(x-\varphi_1^t,t)\} \, dxdt = 0;$$

it follows that:

$$\iint_D \{F_u\delta u + F_p\delta p + F_q\delta q\} \, dxdt = -\iint_D \{F_{u(x-\varphi_0,t)}\delta u(x-\varphi_0,t) +$$

$$F_{p(x-\varphi_1^x,t)}\delta p(x-\varphi_1^x,t) + F_{q(x-\varphi_1^t,t)}\delta q(x-\varphi_1^t,t)\} \, dxdt \quad \text{...........(3.6)}$$

Assuming that $z_i^{x,t} = (x - \varphi_i^{x,t})$, i.e., $z_i^{x,t} = z_i^{x,t}(x, t; u(x, t), p(x, t), q(x, t))$, (i = 0, 1) or equivalently $x = (z_i^{x,t} + \varphi_i^{x,t})$, for (i = 0, 1), with the same index of z and φ in each equation, and let f_i be the inverse function of $z_i^{x,t}$, such that $x = f_i(z_i^{x,t})$, and hence:

$$dx = f'_i(z_i^{x,t})\, z_i^{x,t'}\, dz = (1 - \varphi_i^{x,t'})dz,$$

finally, producing a function $\overline{F}$ that satisfies all values of time, and can be written as:

$$\overline{F} = F\{t, x; u, p, q, u(z_0,t), p(z_1^x,t), q(z_1^t,t)\} \quad \text{.........................(3.6.a)}$$

Using the result (3.6.a) for the right hand side of eq.(3.6), yields to:

$$\iint_D \{F_{u(x-\varphi_0,t)}\delta u(x-\varphi_0,t) + F_{p(x-\varphi_1^x,t)}\delta p(x-\varphi_1^x,t) + F_{q(x-\varphi_1^t,t)}\delta q(x-\varphi_1^t,t)\}\, dxdt = \iint_D \{\overline{F}_{u(z_0,t)}(1-\varphi_0')\,\delta u(z_0,t) + \overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\,\delta p(z_1^x,t) + \overline{F}_{q(z_1^t,t)}(1-\varphi_1^t)\,\delta q(z_1^t,t)\}\, dxdt$$

...............................(3.6.b)

Substituting eq.(3.6.b) back in eq.(3.6), gives:

$$\iint_D \{F_u\delta u + F_p\delta p + F_q\delta q\}\, dxdt = -\iint_D \{\overline{F}_{u(z_0,t)}(1-\varphi_0')\,\delta u(z_0,t) + \overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\,\delta p(z_1^x,t) + \overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\,\delta q(z_1^t,t)\}\, dxdt \quad \text{......(3.7)}$$

and since:

$$\frac{\partial}{\partial x}\{F_p\delta u\} = \frac{\partial}{\partial x}\{F_p\}\delta u + F_p\delta p$$

and

$$\frac{\partial}{\partial t}\{F_q\delta u\} = \frac{\partial}{\partial t}\{F_q\}\delta u + F_q\delta q;$$

also similarly for the delay terms:

$$\frac{\partial}{\partial x}\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\delta u(z_1^x,t)\} = \frac{\partial}{\partial x}\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x})\}\delta u(z_1^x,t) + \overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\delta p(z_1^x,t)$$

and

$$\frac{\partial}{\partial t}\{\overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\delta u(z_1^t,t)\} = \frac{\partial}{\partial t}\{\overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t})\}\delta u(z_1^t,t) + \overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\delta q(z_1^t,t).$$

Then in eq.(3.7), the terms of partial derivatives may be rewritten as:

$$\iint_D \{F_p\delta p + F_q\delta q\}dxdt = \iint_D \left\{\frac{\partial}{\partial x}\{F_p\delta u\}+\frac{\partial}{\partial t}\{F_q\delta u\}\right\}dxdt - \iint_D \left\{\frac{\partial}{\partial x}\{F_p\} + \frac{\partial}{\partial t}\{F_q\}\right\}\delta u\, dxdt;$$

Also,

$$\iint_D \{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\delta p(z_1^x,t) + \overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\delta q(z_1^t,t)\}\, dxdt =$$

$$\iint_D\left\{\frac{\partial}{\partial x}\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\delta u(z_1^x,t)\}+\frac{\partial}{\partial t}\{\overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\delta u(z_1^t,t)\}\right\}dxdt -$$

$$\iint_D \left\{\frac{\partial}{\partial x}\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\}\delta u(z_1^x,t)+\frac{\partial}{\partial t}\{\overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\}\delta u(z_1^t,t)\right\}dxdt.$$

By using the Green's formula, one can have:

$$\iint\limits_D \left\{\frac{\partial}{\partial x}\{F_p\delta u\} + \frac{\partial}{\partial t}\{F_q\delta u\}\right\} dxdt = \int\limits_C \{F_p\, dt - F_q\, dx\}\delta u = 0$$

and

$$\iint\limits_D \left\{\frac{\partial}{\partial x}\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\delta u(z_1^x,t)\} + \frac{\partial}{\partial t}\{\overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\delta u(z_1^t,t)\}\right\} dxdt =$$

$$\int\limits_C \left\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\delta u(z_1^x,t)\, dt - \overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\delta u(z_1^t,t)\, dx\right\} = 0.$$

Consequently:

$$\iint\limits_D \{F_p\delta u + F_q\delta q\}\, dxdt = -\iint\limits_D \left\{\frac{\partial}{\partial x}\{F_p\}\delta u + \frac{\partial}{\partial t}\{F_q\}\delta u\right\} dxdt$$

and

$$\iint\limits_D \left\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\delta p(z_1^x,t) + \overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\delta q(z_1^t,t)\right\} dxdt =$$

$$-\iint\limits_D \left\{\frac{\partial}{\partial x}\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\}\delta u(z_1^x,t) + \frac{\partial}{\partial t}\{\overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\}\delta u(z_1^t,t)\right\} dxdt\,.$$

So, eq.(3.6) may be rewritten as:

$$\iint\limits_D \left\{F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial t}F_q\right\}\delta u\, dxdt = -\iint\limits_D \left\{\overline{F}_{u(z_0,t)}(1-\varphi_0')\delta u(z_0,t) - \right.$$

$$\left.\frac{\partial}{\partial x}\left\{\overline{F}_{p(z_1^x,t)}(1-\varphi_1^{x'})\right\}\delta u(z_1^x,t) - \frac{\partial}{\partial t}\left\{\overline{F}_{q(z_1^t,t)}(1-\varphi_1^{t'})\right\}\delta u(z_1^t,t)\right\} dxdt.$$

Then the necessary condition for an extremum is:

$$\iint_D \left\{ F_u - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial t} F_q \right\} \delta u \, dxdt \; + \iint_D \left\{ \overline{F}_{u(z_0,t)} (1 - \varphi_0') \delta u(z_0, t) - \right.$$

$$\left. \frac{\partial}{\partial x} \left\{ \overline{F}_{p(z_1^x,t)} (1 - \varphi_1^{x'}) \right\} \delta u(z_1^x, t) - \frac{\partial}{\partial t} \left\{ \overline{F}_{q(z_1^t,t)} (1 - \varphi_1^{t'}) \right\} \delta u(z_1^t, t) \right\} dxdt = 0.$$

Using the fundamental lemma (3.1), one have:

$$\left(F_u - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial x} F_q \right) + \left(\overline{F}_{u(z_0,t)} (1 - \varphi_0') - \frac{\partial}{\partial x} \left\{ \overline{F}_{p(z_1^x,t)} \; (1 - \varphi_1^{x'}) \right\} - \right.$$

$$\left. \frac{\partial}{\partial t} \left\{ \overline{F}_{q(z_1^t,t)} (1 - \varphi_1^{t'}) \right\} \right) = 0,$$

where $z_i^{x,t} = (x - \varphi_i^{x,t})$, for (i = 0, 1). ■

Theorem (3.4):

Let $u_i = u_i(x, t) \in C^2(D)$, for (i = 1, 2, …, m); be an extremum of the functional:

$$v[u_1, u_2, \ldots, u_m] = \iint_D F(x, t; u_1, u_2, \ldots, u_m, p_{11}, p_{12}, \ldots, p_{1n}, p_{21},$$

$$p_{22}, \ldots, p_{2n}, \ldots, p_{m1}, p_{m2}, \ldots, p_{mn}, q_{11}, q_{12}, \ldots, q_{1n}, q_{21}, q_{22}, \ldots,$$

$$q_{2n}, \ldots, q_{m1}, q_{m2}, \ldots, q_{mn}, w_{121}, w_{131}, w_{132}, \ldots, w_{1n(n-1)}, w_{221},$$

$$w_{231}, w_{232}, \ldots, w_{2n(n-1)}, \ldots, w_{m21}, w_{m31}, w_{m32}, \ldots, w_{mn(n-1)},$$

$$u_1(x - \varphi_{10}, t), u_2(x - \varphi_{20}, t), \ldots, u_m(x - \varphi_{m0}, t), p_{11}(x - \varphi_{11}^x, t),$$

$$p_{12}(x - \varphi_{12}^x, t), \ldots, p_{1n}(x - \varphi_{1n}^x, t), p_{21}(x - \varphi_{21}^x, t),$$

$$p_{22}(x - \varphi_{22}^x, t), \ldots, p_{2n}(x - \varphi_{2n}^x, t), \ldots, p_{m1}(x - \varphi_{m1}^x, t),$$

$$p_{m2}(x - \varphi_{m2}^x, t), \ldots, p_{mn}(x - \varphi_{mn}^x, t), q_{11}(x - \varphi_{11}^t, t), q_{12}(x -$$

φ_{12}^{t}, t), …, $q_{1n}(x - \varphi_{1n}^{t}$, t), $q_{21}(x-\varphi_{21}^{t}$, t), $q_{22}(x - \varphi_{22}^{t}$, t), …,

$q_{2n}(x - \varphi_{2n}^{t}$, t), …, $q_{m1}(x - \varphi_{m1}^{t}$,t), $q_{m2}(x - \varphi_{m2}^{t}$, t), …,

$q_{mn}(x - \varphi_{mn}^{t}$, t), $w_{121}(x - \varphi_{121}^{xt}$, t), $w_{131}(x-\varphi_{131}^{xt}$,t),

$w_{132}(x - \varphi_{132}^{xt}$,t), …, $w_{1n(n-1)}(x - \varphi_{1n(n-1)}^{xt}$,t), $w_{221}(x - \varphi_{221}^{xt}$, t),

$w_{231}(x - \varphi_{231}^{xt}$, t), $w_{232}(x - \varphi_{232}^{xt}$, t), …, $w_{2n(n-1)}(x - \varphi_{2n(n-1)}^{xt}$,

t), …, $w_{m21}(x - \varphi_{m21}^{xt}$, t), $w_{m31}(x - \varphi_{m31}^{xt}$, t), $w_{m32}(x - \varphi_{m32}^{xt}$,

t), …, $w_{mn(n-1)}(x - \varphi_{mn(n-1)}^{xt}$, t)) dxdt ....(3.8)

where $p_{ij} = \dfrac{\partial^j}{\partial x^j} u_i$, $q_{ij} = \dfrac{\partial^j}{\partial t^j} u_i$ and $w_{ijk} = \dfrac{\partial^j}{\partial x^{j-k} \partial t^k} u_i$, for (i = 1, 2, …, m, j = 1, 2, …, n and k = 1, 2, …, n − 1) and $\varphi_{ijk}^{x,t}$ are the delay functions, where j refers to the order of the partial derivative with respect to x or t of the i-th dependent variable, on the subset D of $\mathbb{R}^2$, consisting of those functions satisfying the boundary conditions:

$$\delta u_i \Big|_{\partial D} = \delta p_{ij} \Big|_{\partial D} = \delta q_{ij} \Big|_{\partial D} = \delta w_{ijk} \Big|_{\partial D} = 0,$$

for all (i = 1, 2, …, m, j = 1, 2, …, n and k = 1, 2, …, n − 1). Then u_i satisfy Euler's equations:

$$\left(F_{u_i} + \sum_{j=1}^{n} (-1)^j \left\{ \frac{\partial^j}{\partial x^j} F_{p_{ij}} + \frac{\partial^j}{\partial t^j} F_{q_{ij}} \right\} + \sum_{j=2}^{n} \sum_{k=1}^{j-1} (-1)^j \frac{\partial^j}{\partial x^{j-k} \partial t^k} F_{w_{ijk}} \right) +$$

$$\left(\overline{F}_{u_i(x-\varphi_{i0},t)} (1 - \varphi_{i0}) + \sum_{j=1}^{n} (-1)^j \left\{ \frac{\partial^j}{\partial x^j} \left\{ \overline{F}_{p_{ij}(x-\varphi_{ij}^{x},t)} (1 - \varphi_{ij}^{x'}) \right\} + \right. \right.$$

$$\frac{\partial^j}{\partial t^j}\left\{\overline{F}_{q_{ij}(x-\varphi_{ij}^t,t)}(1-\varphi_{ij}^{t'})\right\}\Bigg\}+$$

$$\left.\sum_{j=2}^{n}\sum_{k=1}^{j-1}(-1)^j\frac{\partial^j}{\partial x^{j-k}\partial t^k}\left\{\overline{F}_{w_{ijk}(x-\varphi_{ijk}^{xt},t)}(1-\varphi_{ijk}^{xt'})\right\}\right) = 0;$$

for all (i = 1, 2, …, m).

Proof:

In order to find the necessary conditions for the functional (3.8), whose domain of m-dependent variables with the higher order derivatives, vary only one of the dependent variables, and keeping the other fixed, so the first variation of u_i for some (i = 1, 2, …, m), may be expressed as:

$$v[u_i] = \iint_D \delta F\, dxdt$$

$$= \Bigg\{F_{u_i}\delta u_i + \sum_{j=1}^{n}\left\{F_{p_{ij}}\delta p_{ij} + F_{q_{ij}}\delta q_{ij}\right\} + \sum_{j=2}^{n}\sum_{k=1}^{j-1}F_{w_{ijk}}\delta w_{ijk} +$$

$$F_{u_i(x-\varphi_{i0},t)}(1-\varphi'_{i0})\delta u_i(x-\varphi_{i0},t) +$$

$$\sum_{j=1}^{n}\Bigg\{F_{p_{ij}(x-\varphi_{ij}^x,t)}(1-\varphi_{ij}^{x'})\delta p_{ij}(x-\varphi_{ij}^x,t) +$$

$$F_{q_{ij}(x-\varphi_{ij}^t,t)}(1-\varphi_{ij}^{t'})\delta q_{ij}(x-\varphi_{ij}^t,t)\Bigg\} +$$

$$\sum_{j=2}^{n}\sum_{k=1}^{j-1}\left\{F_{w_{ijk}(x-\varphi_{ijk}^{xt},t)}(1-\varphi_{ijk}^{xt'})\delta w_{ijk}(x-\varphi_{ijk}^{xt},t)\right\}\Bigg\}\, dxdt.$$

Applying $\delta v = 0$, yields to:

$$\iint_D \left\{ F_{u_i}\delta u_i + \sum_{j=1}^{n}\left\{ F_{p_{ij}}\delta p_{ij} + F_{q_{ij}}\delta q_{ij} \right\} + \sum_{j=2}^{n}\sum_{k=1}^{j-1} F_{w_{ijk}}\delta w_{ijk} \right\} dxdt =$$

$$-\iint_D \left\{ \bar{F}_{u_i(x-\varphi_{i0},t)}(1-\varphi'_{i0})\delta u_i(x-\varphi_{i0},t) + \right.$$

$$\sum_{j=1}^{n}\left\{ F_{p_{ij}(x-\varphi^{x}_{ij},t)}(1-\varphi^{x'}_{ij})\delta p_{ij}(x-\varphi^{x}_{ij},t) + \right.$$

$$\left. F_{q_{ij}(x-\varphi^{t}_{ij},t)}(1-\varphi^{t'}_{ij})\delta q_{ij}(x-\varphi^{t}_{ij},t) \right\} +$$

$$\left. \sum_{j=2}^{n}\sum_{k=1}^{j-1}\left\{ F_{w_{ijk}(x-\varphi^{xt}_{ijk},t)}(1-\varphi^{xt'}_{ijk})\delta w_{ijk}(x-\varphi^{xt}_{ijk},t) \right\} \right\} dxdt \qquad (3.9)$$

Now suppose that $z^{x,t}_{ijk} = (x-\varphi^{x,t}_{ijk})$,for all (j = 1, 2, …, n and k = 1, 2, …, j – 1) or equivalently $x = (z^{x,t}_{ijk} + \varphi^{x,t}_{ijk})$; with the same index of z and φ in each equation, and let f_{ijk} be the inverse function of $z^{x,t}_{ijk}$, such that $x = f_{ijk}(z^{x,t}_{ijk})$,hence $dx = f_{ijk}(z^{x,t}_{ijk})\, z^{x,t'}_{ijk}\, dz = (1-\varphi^{x,t'}_{ijk})dz$.

Thus, the right hand side of eq.(3.9) can be rewritten as:

$$\iint_D \left\{ F_{u_i(x-\varphi_{i0},t)}(1-\varphi'_{i0})\delta u_i(x-\varphi_{i0},t) + \right.$$

$$\sum_{j=1}^{n}\left\{ F_{p_{ij}(x-\varphi^{x}_{ij},t)}(1-\varphi^{x'}_{ij})\delta p_{ij}(x-\varphi^{x}_{ij},t) + \right.$$

$$\left. F_{q_{ij}(x-\varphi^{t}_{ij},t)}(1-\varphi^{t'}_{ij})\delta q_{ij}(x-\varphi^{t}_{ij},t) \right\} +$$

$$\left. \sum_{j=2}^{n}\sum_{k=1}^{j-1}\left\{ F_{w_{ijk}(x-\varphi^{xt}_{ijk},t)}(1-\varphi^{xt'}_{ijk})\delta w_{ijk}(x-\varphi^{xt}_{ijk},t) \right\} \right\} dxdt =$$

$$\iint\limits_D \Big\{ \overline{F}_{u_i(z_{i0},t)}(1-\varphi'_{i0})\delta u_i(z_{i0},t) + \sum_{j=1}^{n} \Big\{ \overline{F}_{p_{ij}(z^x_{ij},t)}(1-\varphi^{x'}_{ij})\delta p_{ij}(z^x_{ij},t) +$$

$$\overline{F}_{q_{ij}(z^t_{ij},t)}(1-\varphi^{t'}_{ij})\delta q_{ij}(z^t_{ij},t) \Big\} +$$

$$\sum_{j=2}^{n}\sum_{k=1}^{j-1} \Big\{ \overline{F}_{w_{ijk}(z^{xt}_{ijk},t)}(1-\varphi^{xt'}_{ijk})\delta w_{ijk}(z^{xt}_{ijk},t) \Big\} \Big\} \; dxdt;$$

where:

$$\frac{\partial^j}{\partial x^j}\Big\{F_{p_{ij}}\delta u_i\Big\} = (-1)^{j+1}\frac{\partial^j}{\partial x^j}\Big\{F_{p_{ij}}\Big\}\delta u_i + j\frac{\partial^{j-1}}{\partial x^{j-1}}\Big\{F_{p_{ij}}\Big\}\delta u'_{i_x} +$$

$$j\frac{\partial^{j-2}}{\partial x^{j-2}}\Big\{F_{p_{ij}}\Big\}\delta u''_{i_x} + \ldots + F_{p_{ij}}\delta p_{ij}; \text{ for all } (j=1,2,\ldots,n),$$

$$\frac{\partial^j}{\partial t^j}\Big\{F_{q_{ij}}\delta u_i\Big\} = (-1)^{j+1}\frac{\partial^j}{\partial t^j}\Big\{F_{q_{ij}}\Big\}\delta u_i + j\frac{\partial^{j-1}}{\partial t^{j-1}}\Big\{F_{q_{ij}}\Big\}\delta u'_{i_t} +$$

$$j\frac{\partial^{j-2}}{\partial t^{j-2}}\Big\{F_{q_{ij}}\Big\}\delta u''_{i_t} + \ldots + F_{q_{ij}}\delta q_{ij}; \text{ for all } (j = 1,2,\ldots,n)$$

and

$$\frac{\partial^j}{\partial x^{j-k}\partial t^k}\Big\{F_{w_{ijk}}\delta u_i\Big\} = (-1)^{j+1}\frac{\partial^j}{\partial x^{j-k}\partial t^k}\Big\{F_{w_{ijk}}\Big\}\delta u_i +$$

$$j\frac{\partial^{j-1}}{\partial x^{j-1-k}\partial t^k}\Big\{F_{w_{ijk}}\Big\}\delta u'_{i_x} + \ldots + F_{w_{ijk}}\delta w_{ijk};$$

for each $(j = 2, 3, \ldots, n)$, $k = 1, 2, \ldots, j-1$;

for the delay terms:

$$\frac{\partial^j}{\partial x^j}\left\{\overline{F}_{p_{ij}(z^x_{ij},t)}(1-\varphi^{x'}_{ij})\delta u_i(z^x_{ij},t)\right\} =$$

$$(-1)^{j+1}\frac{\partial^j}{\partial x^j}\left\{\overline{F}_{p_{ij}(z^x_{ij},t)}(1-\varphi^{x'}_{ij})\right\}\delta u_i(z^x_{ij},t) +$$

$$j\frac{\partial^{j-1}}{\partial x^{j-1}}\left\{\overline{F}_{p_{ij}(z^x_{ij},t)}(1-\varphi^{x'}_{ij})\right\}\delta u'_{i_x}(z^x_{ij},t) +$$

$$j\frac{\partial^{j-2}}{\partial x^{j-2}}\left\{\overline{F}_{p_{ij}(z^x_{ij},t)}(1-\varphi^{x'}_{ij})\right\}\delta u''_{i_x}(z^x_{ij},t)+\ldots+$$

$$\overline{F}_{p_{ij}(z^x_{ij},t)}(1-\varphi^{x'}_{ij})\delta p_{ij}(z^x_{ij},t);\ \text{for all } (j = 1, 2, \ldots, n),$$

$$\frac{\partial^j}{\partial t^j}\left\{\overline{F}_{q_{ij}(z^t_{ij},t)}(1-\varphi^{t'}_{ij})\delta u_i(z^t_{ij},t)\right\} =$$

$$(-1)^{j+1}\frac{\partial^j}{\partial t^j}\left\{\overline{F}_{q_{ij}(z^t_{ij},t)}(1-\varphi^{t'}_{ij})\right\}\delta u_i(z^t_{ij},t) +$$

$$j\frac{\partial^{j-1}}{\partial t^{j-1}}\left\{\overline{F}_{q_{ij}(z^t_{ij},t)}(1-\varphi^{t'}_{ij})\right\}\delta u'_{i_t}(z^t_{ij},t) +$$

$$j\frac{\partial^{j-2}}{\partial t^{j-2}}\left\{\overline{F}_{q_{ij}(z^t_{ij},t)}(1-\varphi^{t'}_{ij})\right\}\delta u''_{i_t}(z^t_{ij},t)+\ldots+$$

$$\overline{F}_{q_{ij}(z^t_{ij},t)}(1-\varphi^{t'}_{ij})\delta q_{ij}(z^t_{ij},t);\ \text{for all } (j = 1, 2, \ldots, n)$$

and

$$\frac{\partial^j}{\partial x^{j-k}\partial t^k}\left\{\overline{F}_{w_{ijk}(z^{xt}_{ijk},t)}(1-\varphi^{xt'}_{ijk})\delta u(z^{xt}_{ijk},t)\right\} =$$

$$(-1)^{j+1}\frac{\partial^j}{\partial x^{j-k}\partial t^k}\left\{\overline{F}_{w_{ijk}(z^{xt}_{ijk},t)}(1-\varphi^{xt'}_{ijk})\right\}\delta u_i(z^{xt}_{ijk},t) +$$

$$j\frac{\partial^{j-1}}{\partial x^{j-1-k}\partial t^k}\left\{\overline{F}_{w_{ijk}(z^{xt}_{ijk},t)}(1-\varphi^{xt'}_{ijk})\right\}\delta u'_{i_x}(z_{ijk},t) +$$

$$+\ldots+\overline{F}_{w_{ijk}(z^{xt}_{ijk},t)}(1-\varphi^{xt'}_{ijk})\delta w_{ijk}(z^{xt}_{ijk},t);$$

for each $(j = 2, 3, \ldots, n)$, $k = 1, 2, \ldots, j-1$.

Hence, from the divergence theorem, the terms of the arbitrary functions $\delta u_i', \delta u_i'', ..., \delta u_i^{j-1}$ will be vanished, one obtains:

$$\iint_D \left\{ F_{u_i}\delta u_i + \sum_{j=1}^{n}\left\{ F_{p_{ij}}\delta p_{ij} + F_{q_{ij}}\delta q_{ij} \right\} + \sum_{j=2}^{n}\sum_{k=1}^{j-1} F_{w_{ijk}}\delta w_{ijk} \right\} dxdt =$$

$$\iint_D \left\{ F_{u_i} + \sum_{j=1}^{n}(-1)^j \left\{ \frac{\partial^j}{\partial x^j}F_{p_{ij}} + \frac{\partial^j}{\partial t^j}F_{q_{ij}} \right\} + \right.$$

$$\left. \sum_{j=2}^{n}\sum_{k=1}^{j-1}(-1)^j \frac{\partial^j}{\partial x^{j-k}\partial t^k}\left\{ F_{w_{ijk}} \right\} \right\} \delta u_i \; dxdt \text{.................................}(3.9.a)$$

and:

$$\iint_D \left\{ \overline{F}_{u_i(z_{i0},t)}(1-\varphi_{i0}')\delta u_i(z_{i0},t) + \right.$$

$$\sum_{j=1}^{n}\left\{ \overline{F}_{p_{ij}(z_{ij}^x,t)}(1-\varphi_{ij}^{x'})\delta p_{ij}(z_{ij}^x,t) + \overline{F}_{q_{ij}(z_{ij}^t,t)}(1-\varphi_{ij}^{t'})\delta q_{ij}(z_{ij}^t,t) \right\} +$$

$$\left. \sum_{j=2}^{n}\sum_{k=1}^{j-1}\left\{ \overline{F}_{w_{ijk}(z_{ijk}^{xt},t)}(1-\varphi_{ijk}^{xt'})\delta w_{ijk}(z_{ijk}^{xt},t) \right\} \right\} dxdt =$$

$$\iint_D \left\{ \overline{F}_{u_i(z_{i0},t)}(1-\varphi_{i0}')\delta u_i(z_{i0},t) + \right.$$

$$\sum_{j=1}^{n}(-1)^j \left\{ \frac{\partial^j}{\partial x^j}\left\{ \overline{F}_{p_{ij}(z_{ij}^x,t)}(1-\varphi_{ij}^{x'}) \right\}\delta u_i(z_{ij}^x,t) + \right.$$

$$\left. \frac{\partial^j}{\partial t^j}\left\{ \overline{F}_{q_{ij}(z_{ij}^t,t)}(1-\varphi_{ij}^{t'}) \right\}\delta u_i(z_{ij}^t,t) \right\} +$$

$$\left. \sum_{j=2}^{n}\sum_{k=1}^{j-1}(-1)^j \frac{\partial^j}{\partial x^{j-k}\partial t^k}\left\{ \overline{F}_{w_{ijk}(z_{ijk}^{xt},t)}(1-\varphi_{ijk}^{xt'}) \right\}\delta u_i(z_{ijk}^{xt},t) \right\} dxdt$$

..............................(3.9.b)

Substituting the results (3.9.a) and (3.9.b) in (3.9), one can have:

$$\iint_D \left\{ F_{u_i} + \sum_{j=1}^{n} (-1)^j \left\{ \frac{\partial^j}{\partial x^j} F_{p_{ij}} + \frac{\partial^j}{\partial t^j} F_{q_{ij}} \right\} + \right.$$
$$\left. \sum_{j=2}^{n} \sum_{k=1}^{j-1} (-1)^j \frac{\partial^j}{\partial x^{j-k} \partial t^k} \left\{ F_{w_{ijk}} \right\} \right\} \delta u_i \, dxdt =$$
$$-\iint_D \left\{ \overline{F}_{u_i(z_{i0},t)} (1-\varphi'_{i0}) \delta u_i(z_{i0},t) + \right.$$
$$\sum_{j=1}^{n} (-1)^j \left\{ \frac{\partial^j}{\partial x^j} \left\{ \overline{F}_{p_{ij}(z_{ij}^x,t)} (1-\varphi_{ij}^{x'}) \right\} \delta u_i(z_{ij}^x,t) + \right.$$
$$\left. \frac{\partial^j}{\partial t^j} \left\{ \overline{F}_{q_{ij}(z_{ij}^t,t)} (1-\varphi_{ij}^{t'}) \right\} \delta u_i(z_{ij}^t,t) \right\} +$$
$$\left. \sum_{j=2}^{n} \sum_{k=1}^{j-1} (-1)^j \frac{\partial^j}{\partial x^{j-k} \partial t^k} \left\{ \overline{F}_{w_{ijk}(z_{ijk}^{xt},t)} (1-\varphi_{ijk}^{xt'}) \right\} \delta u_i(z_{ijk}^{xt},t) \right\} dxdt$$

...............................(3.10)

Applying eq.(3.10) for each dependent variable u_i, (i =1, 2, …, m), then the resulting system of m PDDE's given by eq.(3.10), for all (i = 1, 2, …, m) can be obtained, therefore Euler's equation for the functional (3.8), takes the form:

$$\left(F_{u_i} + \sum_{j=1}^{n} (-1)^j \left\{ \frac{\partial^j}{\partial x^j} F_{p_{ij}} + \frac{\partial^j}{\partial t^j} F_{q_{ij}} \right\} + \sum_{j=2}^{n} \sum_{k=1}^{j-1} (-1)^j \frac{\partial^j}{\partial x^{j-k} \partial t^k} F_{w_{ijk}} \right) +$$
$$\left(\overline{F}_{u_i(z_{i0},t)} (1-\varphi'_{i0}) + \sum_{j=1}^{n} (-1)^j \left\{ \frac{\partial^j}{\partial x^j} \left\{ \overline{F}_{p_{ij}(z_{ij}^x,t)} (1-\varphi_{ij}^{x'}) \right\} + \right. \right.$$
$$\left. \frac{\partial^j}{\partial t^j} \left\{ \overline{F}_{q_{ij}(z_{ij}^t,t)} (1-\varphi_{ij}^{t'}) \right\} \right\} +$$
$$\left. \sum_{j=2}^{n} \sum_{k=1}^{j-1} (-1)^j \frac{\partial^j}{\partial x^{j-k} \partial t^k} \left\{ \overline{F}_{w_{ijk}(z_{ijk}^{xt},t)} (1-\varphi_{ijk}^{xt'}) \right\} \right) = 0,$$

where $z_{ijk}^{x,t} = (x - \varphi_{ijk}^{x,t})$, for all (i =1, 2, …, m, j = 1, 2, …, n and k =1, 2, …, j – 1). ■

Theorem (3.5):

Let $u = u(x_1, x_2, \ldots, x_m) \in C^2(D)$, be an extremum for the functional:

$$v[u] = \iint\limits_D \cdots \int F(x_1, x_2, \ldots, x_m; u, p_1, p_2, \ldots, p_m, u(x_1 - \varphi_0, x_2, \ldots, x_m), p_1(x_1 - \varphi_1, x_2, \ldots, x_m), p_2(x_1 - \varphi_2, x_2, \ldots, x_m), \ldots, p_m(x_1 - \varphi_m, x_2, \ldots, x_m))\, dx_1 dx_2 \ldots dx_m \quad (3.11)$$

where $p_i = \frac{\partial}{\partial x_i} u(x_1, x_2, \ldots, x_m)$ on the subset of $C^1(D)$ consisting of

those functions satisfying the boundary condition $\delta u\big|_{\partial D} = 0$. Then the Euler's equation for the functional (3.11) is:

$$\left(F_u - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} F_{p_i} \right) + \left(\overline{F}_{u(x_1-\varphi_0, x_2, \ldots, x_m)} (1 - \varphi_0') - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left\{ \overline{F}_{p_i(x_1-\varphi_i, x_2, \ldots, x_m)} (1 - \varphi_i') \right\} \right) = 0.$$

Proof:

The functional v[u] given by eq.(3.11) consisting of function dependent on m independent variables, takes its minimum at:

$$\delta v[u] = \iint\limits_D \cdots \int \delta F\, dx_1 dx_2 \ldots dx_m$$

$$= \int\int\cdots\int_D \Big\{ F_u\delta u + \sum_{i=1}^{m} F_{p_i}\delta p_i +$$

$$F_{u(x_1-\varphi_0,x_2,\ldots,x_m)}\delta u(x_1-\varphi_0,x_2,\ldots,x_m) +$$

$$\sum_{i=1}^{m} F_{p_i(x_1-\varphi_i,x_2,\ldots,x_m)}\delta p_i(x_1-\varphi_i,x_2,\ldots,x_m)\Big\}\, dx_1dx_2\ldots dx_m.$$

Implies that:

$$\int\int\cdots\int_D \Big\{ F_u\delta u + \sum_{i=1}^{m} F_{p_i}\delta p_i \Big\}\, dx_1dx_2\ldots dx_m =$$

$$-\int\int\cdots\int_D \Big\{ F_{u(x_1-\varphi_0,x_2,\ldots,x_m)}\delta u(x_1-\varphi_0,x_2,\ldots,x_m) +$$

$$\sum_{i=1}^{m} F_{p_i(x_1-\varphi_i,x_2,\ldots,x_m)}\delta p_i(x_1-\varphi_i,x_2,\ldots,x_m)\Big\}\, dx_1dx_2\ldots dx_m$$

...............................(3.12)

Let $z_i = (x_1 - \varphi_i)$, or $x_1 = (z_i + \varphi_i)$ at the same index of z and φ in each equation, where $z_i = z_i(x_1, x_2, \ldots, x_m; u, p_1, p_2, \ldots, p_m)$, for $i = 1, 2, \ldots, m$ and $f_i(z_i)$ be the inverse function of z_i.

Hence $dx = f_i'(z_i)z'_i\, dz = (1 - \varphi'_i)dz$.

Since the corresponding right hand side of eq.(3.12) is given by:

$$\iint\cdots\int_D \Big\{ F_{u(x_1-\varphi_0,x_2,\ldots,x_m)}\delta u(x_1-\varphi_0,x_2,\ldots,x_m) +$$

$$\sum_{i=1}^{m} F_{p_i(x_1-\varphi_i,x_2,\ldots,x_m)}\delta p_i(x_1-\varphi_i,x_2,\ldots,x_m)\Big\}\, dx_1dx_2\ldots dx_m =$$

$$\iint\cdots\int_D \Big\{ \overline{F}_{u(z_0,x_2,\ldots,x_m)}(1-\varphi_0')\delta u(z_0,x_2,\ldots,x_m) +$$

$$\sum_{i=1}^{m} \overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\delta p_i(z_i,x_2,\ldots,x_m)\Big\}\, dx_1dx_2\ldots dx_m;$$

and since:

$$\frac{\partial}{\partial x_i}\left\{F_{p_i}\delta u\right\} = \frac{\partial}{\partial x_i}\left\{F_{p_i}\right\}\delta u + F_{p_i}\delta p_i$$

and

$$\frac{\partial}{\partial x_i}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\delta u(z_i,x_2,\ldots,x_m)\right\} =$$

$$\frac{\partial}{\partial x_i}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\right\}\delta u(z_i,x_2,\ldots,x_m) +$$

$$\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\delta p_i(z_i,x_2,\ldots,x_m)$$; for all (i=1,2,…,m).

This implies that:

$$\iint\cdots\int_D \sum_{i=1}^{m}\left\{F_{p_i}\delta p_i\right\}\, dx_1dx_2\ldots dx_m = \iint\cdots\int_D \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{F_{p_i}\delta u\right\}$$

$$dx_1dx_2\ldots dx_m - \iint\cdots\int_D \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{F_{p_i}\right\}\delta u\; dx_1dx_2\ldots dx_m$$

and

$$\iint\limits_{D}\cdots\int \sum_{i=1}^{m}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\delta p_i(z_i,x_2,\ldots,x_m)\right\}dx_1dx_2\ldots dx_m =$$

$$\iint\limits_{D}\cdots\int \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\delta u(z_i,x_2,\ldots,x_m)\right\}dx_1dx_2\ldots dx_m -$$

$$\iint\limits_{D}\cdots\int \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\right\}\delta u(z_i,x_2,\ldots,x_m)dx_1dx_2\ldots dx_m\,.$$

By using the Divergence theorem, it follows that:

$$\iint\limits_{D}\cdots\int \sum_{i=1}^{m}\left\{F_{p_i}\delta p_i\right\}\ dx_1dx_2\ldots dx_m = -\iint\limits_{\partial D}\cdots\int \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{F_{p_i}\right\}\delta u\ \ ds$$

and

$$\iint\limits_{D}\cdots\int \sum_{i=1}^{m}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\delta p_i(z_i,x_2,\ldots,x_m)\right\}dx_1dx_2\ldots dx_m =$$

$$-\iint\limits_{D}\cdots\int \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\right\}\ \delta u(z_i,x_2,\ldots,$$

$$x_m t)\,dx_1dx_2\ldots dx_m\,.$$

Hence, eq.(3.12) may be rewritten as:

$$\iint\limits_{D}\cdots\int\left\{\ F_u-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{F_{p_i}\right\}\right\}\delta u\ dx_1dx_2\ldots dx_m =$$

$$-\iint\limits_{D}\cdots\int\left\{\ \overline{F}_{u(z_0,x_2,\ldots,x_m)}(1-\varphi_0')\ \ \delta u(z_0,x_2,\ldots,x_m)\ -\right.$$

$$\left.\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{\overline{F}_{p_i(z_i,x_2,\ldots,x_m)}(1-\varphi_i')\right\}\delta u(z_i,x_2,\ldots,x_m)\right\}dx_1dx_2\ldots dx_m.$$

Thus, $\delta v = 0$ implies that the m-integrals vanish for any δu. According to the lemma (3.1), the Euler's equation takes the form:

$$\left(F_u - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} F_{p_i} \right) + \left(\overline{F}_{u(z_0, x_2, \ldots, x_m)} (1 - \varphi_0') - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left\{ \overline{F}_{p_i(z_i, x_2, \ldots, x_m)} (1 - \varphi_i') \right\} \right) = 0,$$

where $z_i = (t - \varphi_i)$, for (i = 1, 2, …, m). ■

3.3 The Sufficient Conditions for an Extremum of the Variational Problems of PDDEv's

In this section, the additional necessary conditions for a function u*(x, t) to render a local minimum of an integral v[u*(x, t)] on D are obtained by employing the condition $\delta^2 v[u^*(x, t)] \geq 0$. These conditions are the Legendre and Jacobie's conditions, which are also satisfied if u*(x, t) provides a relative maximum, an inflection point or generally a stationary point for v, where D is the class of piecewise smooth functions called extremums that are functions satisfying the Legendre and Jacobie's condition, [Lebedev, 2003], and the following set of three necessary conditions are sufficient for u = u*(x, t) to be a minimizer to the problem under consideration:

i) u satisfies Euler's equations (which were derived earlier).

ii) u satisfies Legendre condition (which will be discussed later).

iii) u satisfies Jacobie's condition, that is, $[x_0 - \eta_0, x_1]$ and $[t_0, t_1]$, do not contain points conjugate to $x_0 - \eta_0$ and t_0 with respect to v[u].

Theorem (3.6):

If on the extremal u = u(x, t), the functional (3.5) attains a minimum, then on that extremum, the following sets of inequalities are satisfied:

$$F_{pp} \geq 0, \quad \begin{vmatrix} F_{pp} & F_{pq} \\ F_{qp} & F_{qq} \end{vmatrix} \geq 0,$$

$$\begin{vmatrix} F_{pp} & F_{pq} & \overline{F}_{pp(x-\varphi_1^x,t)}(1-\varphi_1^{x'}) \\ F_{qp} & F_{qq} & \overline{F}_{qp(x-\varphi_1^x,t)}(1-\varphi_1^{x'}) \\ \overline{F}_{p(x-\varphi_1^x,t)p}(1-\varphi_1^{x'}) & \overline{F}_{p(x-\varphi_1^x,t)q}(1-\varphi_1^{x'}) & \overline{F}_{p(x-\varphi_1^x,t)p(x-\varphi_1^x,t)}(1-\varphi_1^{x'}) \end{vmatrix} \geq 0$$

and

$$\begin{vmatrix} F_{pp} & F_{pq} & \overline{F}_{pp(x-\varphi_1^x,t)}(1-\varphi_1^{x'}) & F_{pq(x-\varphi_1^t,t)}(1-\varphi_1^{t'}) \\ F_{qp} & F_{qq} & \overline{F}_{qp(x-\varphi_1^x,t)}(1-\varphi_1^{x'}) & F_{qq(x-\varphi_1^t,t)}(1-\varphi_1^{t'}) \\ \overline{F}_{p(x-\varphi_1^x,t)p}(1-\varphi_1^{x'}) & \overline{F}_{p(x-\varphi_1^x,t)q}(1-\varphi_1^{x'}) & \overline{F}_{p(x-\varphi_1^x,t)p(x-\varphi_1^x,t)}(1-\varphi_1^{x'}) & F_{p(x-\varphi_1^x,t)q(x-\varphi_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'}) \\ \overline{F}_{q(x-\varphi_1^t,t)p}(1-\varphi_1^{t'}) & \overline{F}_{q(x-\varphi_1^t,t)q}(1-\varphi_1^{t'}) & \overline{F}_{q(x-\varphi_1^t,t)p(x-\varphi_1^x,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'}) & F_{q(x-\varphi_1^t,t)q(x-\varphi_1^t,t)}(1-\varphi_1^{t'}) \end{vmatrix} \geq 0,$$

where:

$x_0 - \eta_0 \leq x \leq x_1$ and $t_0 \leq t \leq t_1$, $\eta_i = \text{Max}\left\{\varphi_i^{x,t}(x_0, t; u(x_0,t), p(x_0,t), q(x_0,t))\right\}$, for (i = 0, 1).

Proof:

The accessory problem for the functional (3.5) may be expressed as it is given in eq.(3.4):

Thus, the Legendre condition for the functional (3.5) to have a minimum value, is that:

$$F_{pp} \geq 0, \begin{vmatrix} F_{pp} & F_{pq} \\ F_{qp} & F_{qq} \end{vmatrix} \geq 0,$$

$$\begin{vmatrix} F_{pp} & F_{pq} & \overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'}) \\ F_{qp} & F_{qq} & \overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'}) \\ \overline{F}_{p(z_1^x,t)p}(1-\varphi_1^{x'}) & \overline{F}_{p(z_1^x,t)q}(1-\varphi_1^{x'}) & \overline{F}_{p(z_1^x,t)p(z_1^x,t)}(1-\varphi_1^{x'}) \end{vmatrix} \geq 0$$

and

$$\begin{vmatrix} F_{pp} & F_{pq} & \overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'}) & F_{pq(z_1^t,t)}(1-\varphi_1^{t'}) \\ F_{qp} & F_{qq} & \overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'}) & F_{qq(z_1^t,t)}(1-\varphi_1^{t'}) \\ \overline{F}_{p(z_1^x,t)p}(1-\varphi_1^{x'}) & \overline{F}_{p(z_1^x,t)q}(1-\varphi_1^{x'}) & \overline{F}_{p(z_1^x,t)p(z_1^x,t)}(1-\varphi_1^{x'}) & F_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{t'}) \\ \overline{F}_{q(z_1^t,t)p}(1-\varphi_1^{t'}) & \overline{F}_{q(z_1^t,t)q}(1-\varphi_1^{t'}) & \overline{F}_{q(z_1^t,t)p(z_1^x,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'}) & F_{q(z_1^t,t)q(z_1^t,t)}(1-\varphi_1^{t'}) \end{vmatrix} \geq 0,$$

where $z_i^{x,t} = (x - \varphi_i^{x,t})$, for (i = 0, 1).

In eq.(3.4), transforming the terms of the quadratic arbitrary functions with the partial derivatives:

$$\frac{\partial}{\partial t}\left\{F_{pq}\delta p\delta u\right\} = \frac{\partial}{\partial t}\left\{F_{pq}\delta p\right\}\delta u + F_{pq}\delta p\delta q,$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u\delta u(z_1^x,t)\right\} =$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u\right\}\delta u(z_1^x,t) + \overline{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u\delta p(z_1^x,t),$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u\delta u(z_1^t,t)\right\} = \frac{\partial}{\partial t}\left\{\overline{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u\right\}\delta u(z_1^t,t) +$$

$$\overline{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u\delta q(z_1^t,t),$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p\delta u(z_1^x,t)\right\} =$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p\right\}\delta u(z_1^x,t) + \overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p\delta p(z_1^x,t),$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p\delta u(z_1^t,t)\right\} = \frac{\partial}{\partial t}\left\{\overline{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p\right\}\delta u(z_1^t,t) +$$

$$\overline{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p\delta q(z_1^t,t),$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\delta u(z_1^x,t)\right\} =$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\right\}\delta u(z_1^x,t) + \overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\delta p(z_1^x,t),$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q\delta u(z_1^t,t)\right\} = \frac{\partial}{\partial t}\left\{\overline{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q\right\}\delta u(z_1^t,t) +$$

$$\overline{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q\delta q(z_1^t,t), \text{ and}$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'})\delta p(z_1^x,t)\delta u(z_1^t,t)\right\} =$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'})\delta p(z_1^x,t)\right\}\delta u(z_1^t,t) +$$

$$\overline{F}_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'})\delta p(z_1^x,t)\delta q(z_1^t,t).$$

It follows that:

$$\iint_D \Big\{ F_{pq}\delta p\delta q + \overline{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u\delta p(z_1^x,t) +$$

$$\overline{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u\delta q(z_1^t,t) + \overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p\delta p(z_1^x,t) +$$

$$\overline{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p\delta q(z_1^t,t) + \overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\delta p(z_1^x,t) +$$

$$\overline{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q\delta q(z_1^t,t) +$$

$$\overline{F}_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'})\delta p(z_1^x,t)\delta q(z_1^t,t) \} \, dxdt =$$

$$\iint_D \left\{ \frac{\partial}{\partial x}\left\{F_{pp}\delta p\delta u\right\} + \frac{\partial}{\partial x}\left\{\overline{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u\delta u(z_1^x,t)\right\} + \right.$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u\delta u(z_1^t,t)\right\} +$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p\delta u(z_1^x,t)\right\} +$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p\delta u(z_1^t,t)\right\} +$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\delta u(z_1^x,t)\right\} +$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q\delta u(z_1^t,t)\right\} +$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'})\delta p(z_1^x,t)\delta u(z_1^t,t)\right\} \, dxdt -$$

$$\iint_D \left\{ \frac{\partial}{\partial t}\left\{F_{pq}\delta p\right\}\delta u + \frac{\partial}{\partial x}\left\{\overline{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u\right\}\delta u(z_1^x,t) + \right.$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u\right\}\delta u(z_1^t,t) +$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p\right\}\delta u(z_1^x,t) +$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p\right\}\delta u(z_1^t,t) +$$

$$\frac{\partial}{\partial x}\left\{\overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\right\}\delta u(z_1^x,t) \; +$$

$$\frac{\partial}{\partial t}\left\{\overline{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q\right\}\delta u(z_1^t,t) \; +$$

$$\left.\frac{\partial}{\partial t}\left\{\overline{F}_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'})\delta p(z_1^x,t)\right\}\delta u(z_1^t,t)\right\} dxdt \text{ ......(3.13)}$$

According to the Divergence theorem, the first integral of the right hand side of eq.(3.13) will be vanished. Therefore, the accessory problem (3.4), becomes:

$$\delta^2 v = \frac{1}{2!}\iint_D \{F_{uu}\delta u^2 + F_{pp}\delta p^2 + F_{qq}\delta q^2 + \overline{F}_{u(z_0,t)u(z_0,t)}(1-\varphi_0')\delta u^2(z_0,t)$$

$$+ \; \overline{F}_{p(z_1^x,t)p(z_1^x,t)}(1-\varphi_1^{x'})\delta p^2(z_1^x,t) +$$

$$\overline{F}_{q(z_1^t,t)q(z_1^t,t)}(1-\varphi_1^{t'})\delta q^2(z_1^t,t)\} \; dxdt \; +$$

$$\iint_D \left(F_{up}\delta p + F_{uq}\delta q - \frac{\partial}{\partial t}\left\{F_{pq}\delta p\right\}\right)\delta u +$$

$$\iint_D \left\{ \left(\overline{F}_{uu(z_0,t)}(1-\varphi_0')\delta u + \overline{F}_{pu(z_0,t)}(1-\varphi_0')\delta p + \right.\right.$$

$$\overline{F}_{qu(z_0,t)}(1-\varphi_0')\delta q + \; \overline{F}_{u(z_0,t)p(z_1^x,t)}(1-\varphi_0')(1-\varphi_1^{x'})\delta p(z_1^x,t) +$$

$$\left.\overline{F}_{u(z_0,t)q(z_1^t,t)}(1-\varphi_0')(1-\varphi_1^{t'})\delta q(z_1^t,t)\right)\delta u(z_0,t) \; -$$

$$\left(\frac{\partial}{\partial x}\left\{\overline{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u + \overline{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p + \right.\right.$$

$$\left.\left.\overline{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\right\}\right)\delta u(z_1^x,t) \; -$$

$$\left(\frac{\partial}{\partial t}\left\{\overline{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u+\overline{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p+\right.\right.$$

$$\overline{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q+\overline{F}_{p(z_1^x,t)q(z_1^t,t)}(1-\varphi_1^{x'})$$

$$\left.\left.(1-\varphi_1^{t'})\delta p(z_1^x,t)\right)\delta u(z_1^t,t)\right\}dxdt \quad \text{.................................}(3.14)$$

Then, the Jacobie's equation of the accessory problem (3.4), takes the form

$$\delta^2 v=\frac{1}{2!}\iint_D (P_1\delta u^2+P_2\delta p^2+P_3\delta q^2+2Q\delta u)\,dxdt+$$

$$\frac{1}{2!}\iint_D (\overline{P}_1\delta u^2(z_0,t)+\overline{P}_2\delta p^2(z_1^x,t)+\overline{P}_3\delta q^2(z_1^t,t)+$$

$$2\overline{Q})\,dxdt$$

where:

$$P_1=F_{uu},\ P_2=F_{pp},\ P_3=F_{qq},$$

$$Q=\left(F_{up}\delta p+F_{uq}\delta q-\frac{\partial}{\partial x}\{F_{pq}\delta p\}\right),$$

$$\overline{P}_1=\overline{F}_{u(z_0,t)u(z_0,t)}(1-\varphi_0'),\ \overline{P}_2=\overline{F}_{p(z_1^x,t)p(z_1^x,t)}(1-\varphi_1^{x'}),$$

$$\overline{P}_3=\overline{F}_{q(z_1^t,t)q(z_1^t,t)}(1-\varphi_1^{t'}),$$

and

$$\overline{Q}=\left(\overline{F}_{uu(z_0,t)}(1-\varphi_0')\delta u+\overline{F}_{pu(z_0,t)}(1-\varphi_0')\delta p+\right.$$

$$\overline{F}_{qu(z_0,t)}(1-\varphi_0')\delta q+\overline{F}_{u(z_0,t)p(z_1^x,t)}(1-\varphi_0')(1-\varphi_1^{x'})\delta p(z_1^x,t)+$$

$$\bar{F}_{u(z_0,t)q(z_1^t,t)}(1-\varphi_0')(1-\varphi_1^{t'})\delta q(z_1^t,t)\Big)\delta u(z_0,t) -$$

$$\left(\frac{\partial}{\partial x}\Big\{\bar{F}_{up(z_1^x,t)}(1-\varphi_1^{x'})\delta u + \bar{F}_{pp(z_1^x,t)}(1-\varphi_1^{x'})\delta p +\right.$$

$$\left.\bar{F}_{qp(z_1^x,t)}(1-\varphi_1^{x'})\delta q\Big\}\right)\delta u(z_1^x,t) -$$

$$\left(\frac{\partial}{\partial t}\Big\{\bar{F}_{uq(z_1^t,t)}(1-\varphi_1^{t'})\delta u + \bar{F}_{pq(z_1^t,t)}(1-\varphi_1^{t'})\delta p +\right.$$

$$\bar{F}_{qq(z_1^t,t)}(1-\varphi_1^{t'})\delta q +$$

$$\bar{F}_{p(z_1^x,t)p(z_1^t,t)}(1-\varphi_1^{x'})(1-\varphi_1^{t'})\delta p(z_1^x,t)\Big)\delta u(z_1^t,t),$$

where $z_i^{x,t} = (t - \varphi_i^{x,t})$, for (i = 0, 1). ■

Theorem (3.7):

If on the extremal $u_i = u_i(x, t)$, (i = 1, 2, …, m); the functional (3.8) attains its minimum, if the following matrix H is positive definite, where $x_0 - \eta_0 \le x \le x_1$ and $t_0 \le t \le t_1$:

$$H = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1m} & H_{1\bar{1}} & H_{1\bar{2}} & \cdots & H_{1\bar{m}} \\ H_{21} & H_{22} & \cdots & H_{2m} & H_{2\bar{1}} & H_{2\bar{2}} & \cdots & H_{2\bar{m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{m1} & H_{m2} & \cdots & H_{mm} & H_{m\bar{1}} & H_{m\bar{2}} & \cdots & H_{m\bar{m}} \\ H_{\bar{1}1} & H_{\bar{1}2} & \cdots & H_{\bar{1}m} & H_{\overline{11}} & H_{\overline{12}} & \cdots & H_{\overline{1m}} \\ H_{\bar{2}1} & H_{\bar{2}2} & \cdots & H_{\bar{2}m} & H_{\overline{21}} & H_{\overline{22}} & \cdots & H_{\overline{2m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{\bar{m}1} & H_{\bar{m}2} & \cdots & H_{\bar{m}m} & H_{\overline{m1}} & H_{\overline{m2}} & \cdots & H_{\overline{mm}} \end{bmatrix},$$

where:

$$H_{ij} = \begin{bmatrix} F_{p_{in}p_{jn}} & F_{p_{in}q_{jn}} & F_{p_{in}w_{jn(n-1)}} \\ F_{q_{in}p_{jn}} & F_{q_{in}q_{jn}} & F_{q_{in}w_{jn(n-1)}} \\ F_{w_{in(n-1)}p_{jn}} & F_{w_{in(n-1)}q_{jn}} & F_{w_{in(n-1)}w_{jn(n-1)}} \end{bmatrix},$$

$$H_{i\bar{j}} = \begin{bmatrix} \bar{F}_{p_{in}p_{jn}(z^{x}_{jn},t)}(1-\varphi^{x'}_{jn}) & \bar{F}_{p_{in}q_{jn}(z^{t}_{jn},t)}(1-\varphi^{t'}_{jn}) & \bar{F}_{p_{in}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{q_{in}p_{jn}(z^{x}_{jn},t)}(1-\varphi^{x'}_{jn}) & \bar{F}_{q_{in}q_{jn}(z^{t}_{jn},t)}(1-\varphi^{t'}_{jn}) & \bar{F}_{q_{in}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{w_{in(n-1)}p_{jn}(z^{x}_{jn},t)}(1-\varphi^{x'}_{jn}) & \bar{F}_{w_{in(n-1)}q_{jn}(z^{t}_{jn},t)}(1-\varphi^{t'}_{jn}) & \bar{F}_{w_{in(n-1)}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \end{bmatrix},$$

$$H_{\bar{i}j} = \begin{bmatrix} \bar{F}_{p_{in}(z^{x}_{in},t)p_{jn}}(1-\varphi^{x'}_{in}) & \bar{F}_{p_{in}(z^{x}_{in},t)q_{jn}}(1-\varphi^{x'}_{in}) & \bar{F}_{p_{in}(z^{x}_{in},t)w_{jn(n-1)}}(1-\varphi^{x'}_{in}) \\ \bar{F}_{q_{in}(z^{t}_{in},t)p_{jn}}(1-\varphi^{t'}_{in}) & \bar{F}_{q_{in}(z^{t}_{in},t)q_{jn}}(1-\varphi^{t'}_{in}) & \bar{F}_{q_{in}(z^{t}_{in},t)w_{jn(n-1)}}(1-\varphi^{t'}_{in}) \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)p_{jn}}(1-\varphi^{xt'}_{in(n-1)}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{jn}}(1-\varphi^{xt'}_{in(n-1)}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{jn(n-1)}}(1-\varphi^{xt'}_{in(n-1)}) \end{bmatrix}$$

and

$$H_{\bar{i}\bar{j}} = \begin{bmatrix} \bar{F}_{p_{in}(z^{x}_{in},t)p_{jn}(z^{t}_{jn},t)}(1-\varphi^{x'}_{in})(1-\varphi^{x'}_{jn}) & \bar{F}_{p_{in}(z^{x}_{in},t)q_{jn}(z^{t}_{jn},t)}(1-\varphi^{x'}_{in})(1-\varphi^{t'}_{jn}) & \bar{F}_{p_{in}(z^{x}_{in},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{x'}_{in})(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{q_{in}(z^{t}_{in},t)p_{jn}(z^{x}_{jn},t)}(1-\varphi^{t'}_{in})(1-\varphi^{x'}_{jn}) & \bar{F}_{q_{in}(z^{t}_{in},t)q_{jn}(z^{t}_{jn},t)}(1-\varphi^{t'}_{in})(1-\varphi^{t'}_{jn}) & \bar{F}_{q_{in}(z^{t}_{in},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{t'}_{in})(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)p_{jn}(z^{x}_{jn},t)}(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{x'}_{jn}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{jn}(z^{t}_{jn},t)}(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{t'}_{jn}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{xt'}_{jn(n-1)}) \end{bmatrix}$$

for all (i = 1, 2, .., m and j = 1, 2, …, m), and $z^{x,t}_{ij} = (t - \varphi^{x,t}_{ij})$.

Proof:

The accessory problem for the functional (3.8), which is functional depending on m-dependent variables and consisting derivatives of higher order, can be stated as follows:

$$\delta^2 v = \iint_D \delta^2 F \, dxdt$$

where

$\delta^2 F = \big(\delta u_1, \delta u_2, \ldots, \delta u_2, \delta p_{11}, \delta p_{12}, \ldots, \delta p_{1n}, \delta p_{21}, \delta p_{22}, \ldots, \delta p_{2n}, \ldots,$

$\delta p_{m1}, \delta p_{m2}, \ldots, \delta p_{mn}, \delta q_{11}, \delta q_{12}, \ldots, \delta q_{1m}, \delta q_{21}, \delta q_{22}, \ldots,$

$\delta q_{2m}, \ldots, \delta q_{m1}, \delta q_{m2}, \ldots, \delta q_{mn}, \delta w_{121}, \delta w_{131}, \delta w_{132}, \ldots,$

$\delta w_{1n(n-1)}, \delta w_{221}, \delta w_{231}, \delta w_{232}, \ldots, \delta w_{2n(n-1)}, \ldots, \delta w_{m21},$

$\delta w_{m31}, \delta w_{m32}, \ldots, \delta w_{mn(n-1)}, \delta u_1(z_{10}, t), \delta u_2(z_{20}, t), \ldots,$

$\delta u_m(z_{m0}, t), \delta p_{11}(z^x_{11}, t), \delta p_{12}(z^x_{12}, t), \ldots, \delta p_{12}(z^x_{1n}, t),$

$\delta p_{21}(z^x_{21}, t), \delta p_{22}(z^x_{22}, t), \ldots, \delta p_{2n}(z^x_{2n}, t), \ldots, \delta p_{m1}(z^x_{m1}, t),$

$\delta p_{m2}(z^x_{m2}, t), \ldots, \delta p_{mn}(z^x_{mn}, t), \delta q_{11}(z^t_{11}, t), \delta q_{12}(z^t_{12}, t), \ldots,$

$\delta q_{1n}(z^t_{1n}, t), \delta q_{21}(z^t_{21}, t), \delta q_{22}(z^t_{22}, t), \ldots, \delta q_{2n}(z^t_{2n}, t), \ldots,$

$\delta q_{m1}(z^t_{m1}, t), \delta q_{m2}(z^t_{m2}, t), \ldots, \delta q_{mm}(z^t_{mm}, t), \delta w_{131}(z^{xt}_{131},$

$t), \delta w_{132}(z^{xt}_{132}, t), \ldots, \delta w_{1n(n-1)}(z^{xt}_{1n(n-1)}, t), \delta w_{231}(z^{xt}_{231}, t),$

$\delta w_{232}(z^{xt}_{232}, t), \ldots, \delta w_{2n(n-1)}(z^{xt}_{2n(n-1)}, t), \ldots, \delta w_{m21}(z^{xt}_{m21}, t),$

$\delta w_{m31}(z^{xt}_{m31}, t), \delta w_{m32}(z^{xt}_{m32}, t), \ldots, \delta w_{mn(n-1)}(z^{xt}_{mn(n-1)},$

$t)\big).A.\ (\delta u_1, \delta u_2, \ldots, \delta u_2, \delta p_{11}, \delta p_{12}, \ldots, \delta p_{1n}, \delta p_{21}, \delta p_{22},$

$\ldots, \delta p_{2n}, \ldots, \delta p_{m1}, \delta p_{m2}, \ldots, \delta p_{mn}, \delta q_{11}, \delta q_{12}, \ldots, \delta q_{1m}, \delta q_{21},$

$\delta q_{22}, \ldots, \delta q_{2m}, \ldots, \delta q_{m1}, \delta q_{m2}, \ldots, \delta q_{mn}, \delta w_{121}, \delta w_{131}, \delta w_{132},$

$\ldots, \delta w_{1n(n-1)}, \delta w_{221}, \delta w_{231}, \delta w_{232}, \ldots, \delta w_{2n(n-1)}, \ldots, \delta w_{m21},$

$\delta w_{m31}, \delta w_{m32}, \ldots, \delta w_{mn(n-1)}, \delta u_1(z_{10}, t), \delta u_2(z_{20}, t), \ldots,$

$\delta u_m(z_{m0}, t), \delta p_{11}(z^x_{11}, t), \delta p_{12}(z^x_{12}, t), \ldots, \delta p_{12}(z^x_{1n}, t),$

$\delta p_{21}(z^x_{21}, t), \delta p_{22}(z^x_{22}, t), \ldots, \delta p_{2n}(z^x_{2n}, t), \ldots, \delta p_{m1}(z^x_{m1}, t),$

$\delta p_{m2}(z^x_{m2}, t), \ldots, \delta p_{mn}(z^x_{mn}, t), \delta q_{11}(z^t_{11}, t), \delta q_{12}(z^t_{12}, t), \ldots,$

$\delta q_{1n}(z^t_{1n}, t), \delta q_{21}(z^t_{21}, t), \delta q_{22}(z^t_{22}, t), \ldots, \delta q_{2n}(z^t_{2n}, t), \ldots,$

$\delta q_{m1}(z^{t}_{m1}, t), \delta q_{m2}(z^{t}_{m2}, t), \ldots, \delta q_{mm}(z^{t}_{mm}, t), \delta w_{131}(z^{xt}_{131},$

$t), \delta w_{132}(z^{xt}_{132}, t), \ldots, \delta w_{1n(n-1)}(z^{xt}_{1n(n-1)}, t), \delta w_{231}(z^{xt}_{231}, t),$

$\delta w_{232}(z^{xt}_{232}, t), \ldots, \delta w_{2n(n-1)}(z^{xt}_{2n(n-1)}, t), \ldots, \delta w_{m21}(z^{xt}_{m21}, t),$

$\delta w_{m31}(z^{xt}_{m31}, t), \delta w_{m32}(z^{xt}_{m32}, t), \ldots, \delta w_{mn(n-1)}(z^{xt}_{mn(n-1)}, t)\big)^{T},$

where A is the m×m Hussian matrix consisting of m-dependent variables with higher derivatives of order n, which is defined as:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} & A_{1\bar{1}} & A_{1\bar{2}} & \cdots & A_{1\bar{m}} \\ A_{21} & A_{22} & \cdots & A_{2m} & A_{2\bar{1}} & A_{2\bar{2}} & \cdots & A_{2\bar{m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} & A_{m\bar{1}} & A_{m\bar{2}} & \cdots & A_{m\bar{m}} \\ A_{\bar{1}1} & A_{\bar{1}2} & \cdots & A_{\bar{1}m} & A_{\overline{11}} & A_{\overline{12}} & \cdots & A_{\overline{1m}} \\ A_{\bar{2}1} & A_{\bar{2}2} & \cdots & A_{\bar{2}m} & A_{\overline{21}} & A_{\overline{22}} & \cdots & A_{\overline{2m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{\bar{m}1} & A_{\bar{m}2} & \cdots & A_{\bar{m}m} & A_{\overline{m1}} & A_{\overline{m2}} & \cdots & A_{\overline{mm}} \end{bmatrix}$$

where:

$$A_{ij} = \begin{bmatrix} a_{p_i p_j} & a_{p_i q_j} & a_{p_i w_j} \\ a_{q_i p_j} & a_{q_i q_j} & a_{q_i w_j} \\ a_{w_i p_j} & a_{w_i q_j} & a_{w_i w_j} \end{bmatrix}$$

Here:

$$a_{p_i p_j} = \begin{bmatrix} F_{u_i u_j} & F_{u_i p_{j1}} & F_{u_i p_{j2}} & \cdots & F_{u_i p_{jn}} \\ F_{p_{i1} u_j} & F_{p_{i1} p_{j1}} & F_{p_{i1} p_{j2}} & \cdots & F_{p_{i1} p_{jn}} \\ F_{p_{i2} u_j} & F_{p_{i2} p_{j1}} & F_{p_{i2} p_{j2}} & \cdots & F_{p_{i2} p_{jn}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{p_{in} u_j} & F_{p_{in} p_{j1}} & F_{p_{in} p_{j2}} & \cdots & F_{p_{in} p_{jn}} \end{bmatrix},$$

$$a_{p_i q_j} = \begin{bmatrix} F_{u_i q_{j1}} & F_{u_i q_{j2}} & \cdots & F_{u_i q_{jn}} \\ F_{p_{i1} q_{j1}} & F_{p_{i1} q_{j2}} & \cdots & F_{p_{i1} q_{jn}} \\ F_{p_{i2} q_{j1}} & F_{p_{i2} q_{j2}} & \cdots & F_{p_{i2} q_{jn}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{p_{in} q_{j1}} & F_{p_{in} q_{j2}} & \cdots & F_{p_{in} q_{jn}} \end{bmatrix}$$

$$a_{p_i w_j} = \begin{bmatrix} F_{u_i w_{j21}} & F_{u_i w_{j31}} & F_{u_i w_{j32}} & \cdots & F_{u_i w_{jn(n-1)}} \\ F_{p_{i1} w_{j21}} & F_{p_{i1} w_{j31}} & F_{p_{i1} w_{j32}} & \cdots & F_{p_{i1} w_{jn(n-1)}} \\ F_{p_{i2} w_{j21}} & F_{p_{i2} w_{j31}} & F_{p_{i2} w_{j32}} & \cdots & F_{p_{i2} w_{jn(n-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{p_{in} w_{j21}} & F_{p_{in} w_{j31}} & F_{p_{in} w_{j32}} & \cdots & F_{p_{in} w_{jn(n-1)}} \end{bmatrix},$$

$$a_{q_i p_j} = \begin{bmatrix} F_{q_{i1} u_j} & F_{q_{i1} p_{j1}} & F_{q_{i1} p_{j2}} & \cdots & F_{q_{i1} p_{jn}} \\ F_{q_{i2} u_j} & F_{q_{i2} p_{j1}} & F_{q_{i2} p_{j2}} & \cdots & F_{q_{i2} p_{jn}} \\ F_{q_{i3} u_j} & F_{q_{i3} p_{j1}} & F_{q_{i3} p_{j2}} & \cdots & F_{q_{i3} p_{jn}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{q_{in} u_j} & F_{q_{in} p_{j1}} & F_{q_{in} p_{j2}} & \cdots & F_{q_{in} p_{jn}} \end{bmatrix},$$

$$a_{q_i q_j} = \begin{bmatrix} F_{q_{i1}q_{j1}} & F_{q_{i1}q_{j2}} & \cdots & F_{q_{i1}q_{jn}} \\ F_{q_{i2}q_{j1}} & F_{q_{i2}q_{j2}} & \cdots & F_{q_{i2}q_{jn}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{q_{in}q_{j1}} & F_{q_{in}q_{j2}} & \cdots & F_{q_{in}q_{jn}} \end{bmatrix},$$

$$a_{q_i w_j} = \begin{bmatrix} F_{q_{i1}w_{j21}} & F_{q_{i1}w_{j31}} & F_{q_{i1}w_{j32}} & \cdots & F_{q_{i1}w_{jn(n-1)}} \\ F_{q_{i2}w_{j21}} & F_{q_{i2}w_{j31}} & F_{q_{i2}w_{j32}} & \cdots & F_{q_{i2}w_{jn(n-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{q_{in}w_{j21}} & F_{q_{in}w_{j31}} & F_{q_{in}w_{j32}} & \cdots & F_{q_{in}w_{jn(n-1)}} \end{bmatrix},$$

$$a_{w_i p_j} = \begin{bmatrix} F_{w_{i21}u_{j1}} & F_{w_{i21}p_{j1}} & F_{w_{i21}p_{j2}} & \cdots & F_{w_{i21}p_{jn}} \\ F_{w_{i31}u_{j1}} & F_{w_{i31}p_{j1}} & F_{w_{i31}p_{j2}} & \cdots & F_{w_{i31}p_{jn}} \\ F_{w_{i32}u_{j1}} & F_{w_{i32}p_{j1}} & F_{w_{i32}p_{j2}} & \cdots & F_{w_{i32}p_{jn}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{w_{in(n-1)}u_{j1}} & F_{w_{in(n-1)}p_{j1}} & F_{w_{in(n-1)}p_{j2}} & \cdots & F_{w_{in(n-1)}p_{jn}} \end{bmatrix},$$

$$a_{w_i q_j} = \begin{bmatrix} F_{w_{i21}q_{j1}} & F_{w_{i21}q_{j2}} & \cdots & F_{w_{i21}q_{jn}} \\ F_{w_{i31}q_{j1}} & F_{w_{i31}q_{j2}} & \cdots & F_{w_{i31}q_{jn}} \\ F_{w_{i32}q_{j1}} & F_{w_{i32}q_{j2}} & \cdots & F_{w_{i32}q_{jn}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{w_{in(n-1)}q_{j1}} & F_{w_{in(n-1)}q_{j2}} & \cdots & F_{w_{in(n-1)}q_{jn}} \end{bmatrix}$$

and

$$a_{w_i w_j} = \begin{bmatrix} F_{w_{i21}w_{j21}} & F_{w_{i21}w_{j31}} & F_{w_{i21}w_{j32}} & \cdots & F_{w_{i21}w_{jn(n-1)}} \\ F_{w_{i31}w_{j21}} & F_{w_{i31}w_{j31}} & F_{w_{i31}w_{j32}} & \cdots & F_{w_{i31}w_{jn(n-1)}} \\ F_{w_{i32}w_{j21}} & F_{w_{i32}w_{j31}} & F_{w_{i32}w_{j32}} & \cdots & F_{w_{i32}w_{jn(n-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{w_{in(n-1)}w_{j21}} & F_{w_{in(n-1)}w_{j31}} & F_{w_{in(n-1)}w_{j32}} & \cdots & F_{w_{in(n-1)}w_{jn(n-1)}} \end{bmatrix}.$$

$$A_{i\bar{j}} = \begin{bmatrix} a_{p_i\bar{p}_j} & a_{p_i\bar{q}_j} & a_{p_i\bar{w}_j} \\ a_{q_i\bar{p}_j} & a_{q_i\bar{q}_j} & a_{q_i\bar{w}_j} \\ a_{w_i\bar{p}_j} & a_{w_i\bar{q}_j} & a_{w_i\bar{w}_j} \end{bmatrix}$$

where:

$$a_{p_i\bar{p}_j} = \begin{bmatrix} \bar{F}_{u_iu_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{u_ip_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{u_ip_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{u_ip_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \bar{F}_{p_{i1}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{p_{i1}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{p_{i1}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{p_{i1}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \bar{F}_{p_{i2}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{p_{i2}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{p_{i2}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{p_{i2}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{p_{in}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{p_{in}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{p_{in}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{p_{in}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \end{bmatrix},$$

$$a_{p_i\bar{q}_j} = \begin{bmatrix} \bar{F}_{u_iq_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{u_iq_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \cdots & \bar{F}_{u_iq_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \bar{F}_{p_{i1}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{p_{i1}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \cdots & \bar{F}_{p_{i1}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \bar{F}_{p_{i2}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{p_{i2}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \cdots & \bar{F}_{p_{i2}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{p_{in}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{p_{in}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \cdots & \bar{F}_{p_{in}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \end{bmatrix},$$

$$a_{p_i\bar{w}_j} = \begin{bmatrix} \bar{F}_{u_iw_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{u_iw_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{u_iw_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{u_iw_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{p_{i1}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{p_{i1}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{p_{i1}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{p_{i1}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{p_{i2}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{p_{i2}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{p_{i1}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{p_{i1}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{p_{in}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{p_{in}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{p_{in}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{p_{in}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \end{bmatrix}$$

$$a_{q_i\bar{p}_j} = \begin{bmatrix} \bar{F}_{q_{i1}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{q_{i1}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{q_{i1}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{q_{i1}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \bar{F}_{q_{i2}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{q_{i2}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{q_{i2}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{q_{i2}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{q_{in}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{q_{in}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{q_{in}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{q_{in}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \end{bmatrix}$$

$$a_{q_i\bar{q}_j}=\begin{bmatrix} \bar{F}_{q_{i1}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{q_{i1}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \cdots & \bar{F}_{q_{i1}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \bar{F}_{q_{i2}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{q_{i2}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \cdots & \bar{F}_{q_{i2}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{q_{in}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{q_{in}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \cdots & \bar{F}_{q_{in}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \end{bmatrix},$$

$$a_{q_i\bar{w}_j}=\begin{bmatrix} \bar{F}_{q_{i1}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{q_{i1}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{q_{i1}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{q_{i1}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{q_{i2}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{q_{i2}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{q_{i2}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{q_{i2}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{q_{in}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{q_{in}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{q_{in}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{q_{in}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \end{bmatrix}$$

$$a_{w_i\bar{p}_j}=\begin{bmatrix} \bar{F}_{w_{i21}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{w_{i21}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{w_{i21}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{w_{i21}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \bar{F}_{w_{i31}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{w_{i31}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{w_{i21}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{w_{i31}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \bar{F}_{w_{i32}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{w_{i32}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{w_{i32}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{w_{i32}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}u_j(z_{j0},t)}(1-\varphi'_{j0}) & \bar{F}_{w_{in(n-1)}p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{j1}) & \bar{F}_{w_{in(n-1)}p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{j2}) & \cdots & \bar{F}_{w_{in(n-1)}p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{jn}) \end{bmatrix}$$

$$a_{w_i\bar{q}_j}=\begin{bmatrix} \bar{F}_{w_{i21}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{w_{i21}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \bar{F}_{w_{i21}q_{j3}(z^t_{j3},t)}(1-\varphi^{t'}_{j3}) & \cdots & \bar{F}_{w_{i21}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \bar{F}_{w_{i31}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{w_{i31}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \bar{F}_{w_{i31}q_{j3}(z^t_{j3},t)}(1-\varphi^{t'}_{j3}) & \cdots & \bar{F}_{w_{i31}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \bar{F}_{w_{i32}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{w_{i32}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \bar{F}_{w_{i32}q_{j3}(z^t_{j3},t)}(1-\varphi^{t'}_{j3}) & \cdots & \bar{F}_{w_{i32}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{j1}) & \bar{F}_{w_{in(n-1)}q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{j2}) & \bar{F}_{w_{in(n-1)}q_{j3}(z^t_{j3},t)}(1-\varphi^{t'}_{j3}) & \cdots & \bar{F}_{w_{in(n-1)}q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{jn}) \end{bmatrix}$$

and

$$a_{w_i\bar{w}_j}=\begin{bmatrix} \bar{F}_{w_{i21}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{w_{i21}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{w_{i21}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{w_{i21}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{w_{i31}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{w_{i31}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{w_{i31}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{w_{i31}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \bar{F}_{w_{i32}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{w_{i32}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{w_{i32}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{w_{i32}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{xt'}_{j21}) & \bar{F}_{w_{in(n-1)}w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{xt'}_{j31}) & \bar{F}_{w_{in(n-1)}w_{j32}(z^{xt}_{j32},t)}(1-\varphi^{xt'}_{j32}) & \cdots & \bar{F}_{w_{in(n-1)}w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{xt'}_{jn(n-1)}) \end{bmatrix}$$

$$A_{\bar{i}\bar{j}}=\begin{bmatrix} a_{\bar{p}_ip_j} & a_{\bar{p}_iq_j} & a_{\bar{p}_iw_j} \\ a_{\bar{q}_ip_j} & a_{\bar{q}_iq_j} & a_{\bar{q}_iw_j} \\ a_{\bar{w}_ip_j} & a_{\bar{w}_iq_j} & a_{\bar{w}_iw_j} \end{bmatrix}$$

where:

$$a_{\bar{p}_i p_j}=\begin{bmatrix} \bar{F}_{u_i(z_{i0},t)u_j}(1-\varphi'_{i0}) & \bar{F}_{u_i(z_{i0},t)p_{j1}}(1-\varphi'_{i0}) & \bar{F}_{u_i(z_{i0},t)p_{j2}}(1-\varphi'_{i0}) & \cdots & \bar{F}_{u_i(z_{i0},t)p_{jn}}(1-\varphi'_{i0}) \\ \bar{F}_{p_{i1}(z^x_{i1},t)u_j}(1-\varphi^{x'}_{i1}) & \bar{F}_{p_{i1}(z^x_{i1},t)p_{j1}}(1-\varphi^{x'}_{i1}) & \bar{F}_{p_{i1}(z^x_{i1},t)p_{j2}}(1-\varphi^{x'}_{i1}) & \cdots & \bar{F}_{p_{i1}(z^x_{i1},t)p_{jn}}(1-\varphi^{x'}_{i1}) \\ \bar{F}_{p_{i2}(z^x_{i2},t)u_j}(1-\varphi^{x'}_{i2}) & \bar{F}_{p_{i2}(z^x_{i2},t)p_{j1}}(1-\varphi^{x'}_{i2}) & \bar{F}_{p_{i2}(z^x_{i2},t)p_{j2}}(1-\varphi^{x'}_{i2}) & \cdots & \bar{F}_{p_{i2}(z^x_{i2},t)p_{jn}}(1-\varphi^{x'}_{i2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{p_{in}(z^x_{in},t)u_j}(1-\varphi^{x'}_{in}) & \bar{F}_{p_{in}(z^x_{in},t)p_{j1}}(1-\varphi^{x'}_{in}) & \bar{F}_{p_{in}(z^x_{in},t)p_{j2}}(1-\varphi^{x'}_{in}) & \cdots & \bar{F}_{p_{in}(z^x_{in},t)p_{jn}}(1-\varphi^{x'}_{in}) \end{bmatrix}$$

$$a_{\bar{p}_i q_j}=\begin{bmatrix} \bar{F}_{u_i(z_{i0},t)q_{j1}}(1-\varphi'_{i0}) & \bar{F}_{u_i(z_{i0},t)q_{j2}}(1-\varphi'_{i0}) & \cdots & \bar{F}_{u_i(z_{i0},t)q_{jn}}(1-\varphi'_{i0}) \\ \bar{F}_{p_{i1}(z^x_{i1},t)q_{j1}}(1-\varphi^{x'}_{i1}) & \bar{F}_{p_{i1}(z^x_{i1},t)q_{j2}}(1-\varphi^{x'}_{i1}) & \cdots & \bar{F}_{p_{i1}(z^x_{i1},t)q_{jn}}(1-\varphi^{x'}_{i1}) \\ \bar{F}_{p_{i2}(z^x_{i2},t)q_{j1}}(1-\varphi^{x'}_{i2}) & \bar{F}_{p_{i2}(z^x_{i2},t)q_{j2}}(1-\varphi^{x'}_{i2}) & \cdots & \bar{F}_{p_{i2}(z^x_{i2},t)q_{jn}}(1-\varphi^{x'}_{i2}) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{p_{in}(z^x_{in},t)q_{j1}}(1-\varphi^{x'}_{in}) & \bar{F}_{p_{in}(z^x_{in},t)q_{j2}}(1-\varphi^{x'}_{in}) & \cdots & \bar{F}_{p_{in}(z^x_{in},t)q_{jn}}(1-\varphi^{x'}_{in}) \end{bmatrix}$$

$$a_{\bar{p}_i w_j}=\begin{bmatrix} \bar{F}_{u_i(z_{i0},t)w_{j21}}(1-\varphi'_{i0}) & \bar{F}_{u_i(z_{i0},t)w_{j31}}(1-\varphi'_{i0}) & \bar{F}_{u_i(z_{i0},t)w_{j32}}(1-\varphi'_{i0}) & \cdots & \bar{F}_{u_i(z_{i0},t)w_{jn(n-1)}}(1-\varphi'_{i0}) \\ \bar{F}_{p_{i1}(z^x_{i1},t)w_{j21}}(1-\varphi^{x'}_{i1}) & \bar{F}_{p_{i1}(z^x_{i1},t)w_{j31}}(1-\varphi^{x'}_{i1}) & \bar{F}_{p_{i1}(z^x_{i1},t)w_{j32}}(1-\varphi^{x'}_{i1}) & \cdots & \bar{F}_{p_{i1}(z^x_{i1},t)w_{jn(n-1)}}(1-\varphi^{x'}_{i1}) \\ \bar{F}_{p_{i2}(z^x_{i2},t)w_{j21}}(1-\varphi^{x'}_{i2}) & \bar{F}_{p_{i2}(z^x_{i2},t)w_{j31}}(1-\varphi^{x'}_{i2}) & \bar{F}_{p_{i1}(z^x_{i2},t)w_{j32}}(1-\varphi^{x'}_{i2}) & \cdots & \bar{F}_{p_{i1}(z^x_{i2},t)w_{jn(n-1)}}(1-\varphi^{x'}_{i2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{p_{in}(z^x_{in},t)w_{j21}}(1-\varphi^{x'}_{in}) & \bar{F}_{p_{in}(z^x_{in},t)w_{j31}}(1-\varphi^{x'}_{in}) & \bar{F}_{p_{in}(z^x_{in},t)w_{j32}}(1-\varphi^{x'}_{in}) & \cdots & \bar{F}_{p_{in}(z^x_{in},t)w_{jn(n-1)}}(1-\varphi^{x'}_{in}) \end{bmatrix}$$

$$a_{\bar{q}_i p_j}=\begin{bmatrix} \bar{F}_{q_{i1}(z^t_{i1},t)u_j}(1-\varphi^{t'}_{i1}) & \bar{F}_{q_{i1}(z^t_{i1},t)p_{j1}}(1-\varphi^{t'}_{i1}) & \bar{F}_{q_{i1}(z^t_{i1},t)p_{j2}}(1-\varphi^{t'}_{i1}) & \cdots & \bar{F}_{q_{i1}(z^t_{i1},t)p_{jn}}(1-\varphi^{t'}_{i1}) \\ \bar{F}_{q_{i2}(z^t_{i2},t)u_j}(1-\varphi^{t'}_{i2}) & \bar{F}_{q_{i2}(z^t_{i2},t)p_{j1}}(1-\varphi^{t'}_{i2}) & \bar{F}_{q_{i2}(z^t_{i2},t)p_{j2}}(1-\varphi^{t'}_{i2}) & \cdots & \bar{F}_{q_{i2}(z^t_{i2},t)p_{jn}}(1-\varphi^{t'}_{i2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{q_{in}(z^t_{in},t)u_j}(1-\varphi^{t'}_{in}) & \bar{F}_{q_{in}(z^t_{in},t)p_{j1}}(1-\varphi^{t'}_{in}) & \bar{F}_{q_{in}(z^t_{in},t)p_{j2}}(1-\varphi^{t'}_{in}) & \cdots & \bar{F}_{q_{in}(z^t_{in},t)p_{jn}}(1-\varphi^{t'}_{in}) \end{bmatrix}$$

$$a_{\bar{q}_i q_j}=\begin{bmatrix} \bar{F}_{q_{i1}(z^t_{i1},t)q_{j1}}(1-\varphi^{t'}_{i1}) & \bar{F}_{q_{i1}(z^t_{i1},t)q_{j2}}(1-\varphi^{t'}_{i1}) & \cdots & \bar{F}_{q_{i1}(z^t_{i1},t)q_{jn}}(1-\varphi^{t'}_{i1}) \\ \bar{F}_{q_{i2}(z^t_{i2},t)q_{j1}}(1-\varphi^{t'}_{i2}) & \bar{F}_{q_{i2}(z^t_{i2},t)q_{j2}}(1-\varphi^{t'}_{i2}) & \cdots & \bar{F}_{q_{i2}(z^t_{i2},t)q_{jn}}(1-\varphi^{t'}_{i2}) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{q_{in}(z^t_{in},t)q_{j1}}(1-\varphi^{t'}_{in}) & \bar{F}_{q_{in}(z^t_{in},t)q_{j2}}(1-\varphi^{t'}_{in}) & \cdots & \bar{F}_{q_{in}(z^t_{in},t)q_{jn}}(1-\varphi^{t'}_{in}) \end{bmatrix}$$

$$a_{\bar{q}_i w_j}=\begin{bmatrix} \bar{F}_{q_{i1}(z^{t}_{i1},t)w_{j21}}(1-\varphi^{t'}_{i1}) & \bar{F}_{q_{i1}(z^{t}_{i1},t)w_{j31}}(1-\varphi^{t'}_{i1}) & \bar{F}_{q_{i1}(z^{t}_{i1},t)w_{j32}}(1-\varphi^{t'}_{i1}) & \cdots & \bar{F}_{q_{i1}(z^{t}_{i1},t)w_{jn(n-1)}}(1-\varphi^{t'}_{i1}) \\ \bar{F}_{q_{i2}(z^{t}_{i2},t)w_{j21}}(1-\varphi^{t'}_{i2}) & \bar{F}_{q_{i2}(z^{t}_{i2},t)w_{j31}}(1-\varphi^{t'}_{i2}) & \bar{F}_{q_{i2}(z^{t}_{i2},t)w_{j32}}(1-\varphi^{t'}_{i2}) & \cdots & \bar{F}_{q_{i2}(z^{t}_{i2},t)w_{jn(n-1)}}(1-\varphi^{t'}_{i2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{q_{in}(z^{t}_{in},t)w_{j21}}(1-\varphi^{t'}_{in}) & \bar{F}_{q_{in}(z^{t}_{in},t)w_{j31}}(1-\varphi^{t'}_{in}) & \bar{F}_{q_{in}(z^{t}_{in},t)w_{j32}}(1-\varphi^{t'}_{in}) & \cdots & \bar{F}_{q_{in}(z^{t}_{in},t)w_{jn(n-1)}}(1-\varphi^{t'}_{in}) \end{bmatrix}$$

$$a_{\bar{w}_i p_j}=\begin{bmatrix} \bar{F}_{w_{i21}(z^{xt}_{i21},t)u_j}(1-\varphi^{xt'}_{i21}) & \bar{F}_{w_{i21}(z^{xt}_{i21},t)p_{j1}}(1-\varphi^{xt'}_{i21}) & \bar{F}_{w_{i21}(z^{xt}_{i21},t)p_{j2}}(1-\varphi^{xt'}_{i21}) & \cdots & \bar{F}_{w_{i21}(z^{xt}_{i21},t)p_{jn}}(1-\varphi^{xt'}_{i21}) \\ \bar{F}_{w_{i31}(z^{xt}_{i31},t)u_j}(1-\varphi^{xt'}_{i31}) & \bar{F}_{w_{i31}(z^{xt}_{i31},t)p_{j1}}(1-\varphi^{xt'}_{i31}) & \bar{F}_{w_{i21}(z^{xt}_{i31},t)p_{j2}}(1-\varphi^{xt'}_{i31}) & \cdots & \bar{F}_{w_{i31}(z^{xt}_{i31},t)p_{jn}}(1-\varphi^{xt'}_{i31}) \\ \bar{F}_{w_{i32}(z^{xt}_{i32},t)u_j}(1-\varphi^{xt'}_{i32}) & \bar{F}_{w_{i32}(z^{xt}_{i32},t)p_{j1}}(1-\varphi^{xt'}_{i32}) & \bar{F}_{w_{i32}(z^{xt}_{i32},t)p_{j2}}(1-\varphi^{xt'}_{i32}) & \cdots & \bar{F}_{w_{i32}(z^{xt}_{i32},t)p_{jn})}(1-\varphi^{xt'}_{i32}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)u_j}(1-\varphi^{xt'}_{in(n-1)}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)p_{j1}}(1-\varphi^{xt'}_{in(n-1)}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)p_{j2}}(1-\varphi^{xt'}_{in(n-1)}) & \cdots & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)p_{jn}}(1-\varphi^{xt'}_{in(n-1)}) \end{bmatrix}$$

$$a_{\bar{w}_i q_j}=\begin{bmatrix} \bar{F}_{w_{i21}(z^{xt}_{i21},t)q_{j1}}(1-\varphi^{xt'}_{i21}) & \bar{F}_{w_{i21}(z^{xt}_{i21},t)q_{j2}}(1-\varphi^{xt'}_{i21}) & \cdots & \bar{F}_{w_{i21}(z^{xt}_{i21},t)q_{jn}}(1-\varphi^{xt'}_{i21}) \\ \bar{F}_{w_{i31}(z^{xt}_{i31},t)q_{j1}}(1-\varphi^{xt'}_{i31}) & \bar{F}_{w_{i31}(z^{xt}_{i31},t)q_{j2}}(1-\varphi^{xt'}_{i31}) & \cdots & \bar{F}_{w_{i31}(z^{xt}_{i31},t)q_{jn}}(1-\varphi^{xt'}_{i31}) \\ \bar{F}_{w_{i32}(z^{xt}_{i32},t)q_{j1}}(1-\varphi^{xt'}_{i32}) & \bar{F}_{w_{i32}(z^{xt}_{i32},t)q_{j2}}(1-\varphi^{xt'}_{i32}) & \cdots & \bar{F}_{w_{i32}(z^{xt}_{i32},t)q_{jn}}(1-\varphi^{xt'}_{i32}) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{j1}}(1-\varphi^{xt'}_{in(n-1)}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{j2}}(1-\varphi^{xt'}_{in(n-1)}) & \cdots & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{jn}}(1-\varphi^{xt'}_{in(n-1)}) \end{bmatrix}$$

and

$$a_{\bar{w}_i x_j}=\begin{bmatrix} \bar{F}_{w_{i21}(z^{xt}_{i21},t)w_{j21}}(1-\varphi^{xt'}_{i21}) & \bar{F}_{w_{i21}(z^{xt}_{i21},t)w_{j31}}(1-\varphi^{xt'}_{i21}) & \bar{F}_{w_{i21}(z^{xt}_{i21},t)w_{j32}}(1-\varphi^{xt'}_{i21}) & \cdots & \bar{F}_{w_{i21}(z^{xt}_{i21},t)w_{jn(n-1)}}(1-\varphi^{xt'}_{i21}) \\ \bar{F}_{w_{i31}(z^{xt}_{i31},t)w_{j21})}(1-\varphi^{xt'}_{i31}) & \bar{F}_{w_{i31}(z^{xt}_{i31},t)w_{j31}}(1-\varphi^{xt'}_{i31}) & \bar{F}_{w_{i31}(z^{xt}_{i31},t)w_{j32}}(1-\varphi^{xt'}_{i31}) & \cdots & \bar{F}_{w_{i31}(z^{xt}_{i31},t)w_{j31}}(1-\varphi^{xt'}_{i31}) \\ \bar{F}_{w_{i32}(z^{xt}_{i32},t)w_{j21}}(1-\varphi^{xt'}_{i32}) & \bar{F}_{w_{i32}(z^{xt}_{i32},t)w_{j31}}(1-\varphi^{xt'}_{i32}) & \bar{F}_{w_{i32}(z^{xt}_{i32},t)w_{j32}}(1-\varphi^{xt'}_{i32}) & \cdots & \bar{F}_{w_{i32}(z^{xt}_{i32},t)w_{jn(n-1)}}(1-\varphi^{xt'}_{i32}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{j21}}(1-\varphi^{xt'}_{in(n-1)}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{j31}}(1-\varphi^{xt'}_{in(n-1)}) & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{j32}}(1-\varphi^{xt'}_{in(n-1)}) & \cdots & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{jn(n-1)}}(1-\varphi^{xt'}_{in(n-1)}) \end{bmatrix}$$

and

$$A_{\bar{ij}}=\begin{bmatrix} a_{\overline{p_i p_j}} & a_{\overline{p_i q_j}} & a_{\overline{p_i w_j}} \\ a_{\overline{q_i p_j}} & a_{\overline{q_i q_j}} & a_{\overline{q_i w_j}} \\ a_{\overline{w_i p_j}} & a_{\overline{w_i q_j}} & a_{\overline{w_i w_j}} \end{bmatrix}$$

where

$$a_{\overline{p_i p_j}}=\begin{bmatrix}
\overline{F}_{u_i(z_{i0},t)u_j(z_{j0},t)}(1-\varphi'_{i0})(1-\varphi'_{j0}) & \overline{F}_{u_i(z_{i0},t)p_{j1}(z^x_{j1},t)}(1-\varphi'_{i0})(1-\varphi^{x'}_{j1}) & \overline{F}_{u_i(z_{i0},t)p_{j2}(z^x_{j2},t)}(1-\varphi'_{i0})(1-\varphi^{x'}_{j2}) & \cdots & \overline{F}_{u_i(z_{i0},t)p_{jn}(z^x_{jn},t)}(1-\varphi'_{i0})(1-\varphi^{x'}_{jn}) \\
\overline{F}_{p_{i1}(z^x_{i1},t)u_j(z_{j0},t)}(1-\varphi^{x'}_{i1})(1-\varphi'_{j0}) & \overline{F}_{p_{i1}(z^x_{i1},t)p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{x'}_{j1}) & \overline{F}_{p_{i1}(z^x_{i1},t)p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{x'}_{j2}) & \cdots & \overline{F}_{p_{i1}(z^x_{i1},t)p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{x'}_{jn}) \\
\overline{F}_{p_{i2}(z^x_{i2},t)u_j(z_{j0},t)}(1-\varphi^{x'}_{i2})(1-\varphi'_{j0}) & \overline{F}_{p_{i2}(z^x_{i2},t)p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{x'}_{j1}) & \overline{F}_{p_{i2}(z^x_{i2},t)p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{x'}_{j2}) & \cdots & \overline{F}_{p_{i2}(z^x_{i2},t)p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{x'}_{jn}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\overline{F}_{p_{in}(z^x_{in},t)u_j(z_{j0},t)}(1-\varphi^{x'}_{in})(1-\varphi'_{j0}) & \overline{F}_{p_{in}(z^x_{in},t)p_{j1}(z^x_{j1},t)}(1-\varphi^{x'}_{in})(1-\varphi^{x'}_{j1}) & \overline{F}_{p_{in}(z^x_{in},t)p_{j2}(z^x_{j2},t)}(1-\varphi^{x'}_{in})(1-\varphi^{x'}_{j2}) & \cdots & \overline{F}_{p_{in}(z^x_{in},t)p_{jn}(z^x_{jn},t)}(1-\varphi^{x'}_{in})(1-\varphi^{x'}_{jn})
\end{bmatrix}$$

$$a_{\overline{p_i q_j}}=\begin{bmatrix}
\overline{F}_{u_i(z_{i0},t)q_{j1}(z^t_{j1},t)}(1-\varphi'_{i0})(1-\varphi^{t'}_{j1}) & \overline{F}_{u_i(z_{i0},t)q_{j2}(z^t_{j2},t)}(1-\varphi'_{i0})(1-\varphi^{t'}_{j2}) & \cdots & \overline{F}_{u_i(z_{i0},t)q_{jn}(z^t_{jn},t)}(1-\varphi'_{i0})(1-\varphi^{t'}_{jn}) \\
\overline{F}_{p_{i1}(z^x_{i1},t)q_{j1}(z^t_{j1},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{t'}_{j1}) & \overline{F}_{p_{i1}(z^x_{i1},t)q_{j2}(z^t_{j2},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{t'}_{j2}) & \cdots & \overline{F}_{p_{i1}(z^x_{i1},t)q_{jn}(z^t_{jn},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{t'}_{jn}) \\
\overline{F}_{p_{i2}(z^x_{i2},t)q_{j1}(z^t_{j1},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{t'}_{j1}) & \overline{F}_{p_{i2}(z^x_{i2},t)q_{j2}(z^t_{j2},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{t'}_{j2}) & \cdots & \overline{F}_{p_{i2}(z^x_{i2},t)q_{jn}(z^t_{jn},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{t'}_{jn}) \\
\vdots & \vdots & \ddots & \vdots \\
\overline{F}_{p_{in}(z^x_{in},t)q_{j1}(z^t_{j1},t)}(1-\varphi^{x'}_{in})(1-\varphi^{t'}_{j1}) & \overline{F}_{p_{in}(z^x_{in},t)q_{j2}(z^t_{j2},t)}(1-\varphi^{x'}_{in})(1-\varphi^{t'}_{j2}) & \cdots & \overline{F}_{p_{in}(z^x_{in},t)q_{jn}(z^t_{jn},t)}(1-\varphi^{x'}_{in})(1-\varphi^{t'}_{jn})
\end{bmatrix},$$

$$a_{\overline{p_i w_j}}=\begin{bmatrix}
\overline{F}_{u_i(z_{i0},t)w_{j21}(z^{xt}_{j21},t)}(1-\varphi'_{i0})(1-\varphi^{xt'}_{j21}) & \overline{F}_{u_i(z_{i0},t)w_{j31}(z^{xt}_{j31},t)}(1-\varphi'_{i0})(1-\varphi^{xt'}_{j31}) & \cdots & \overline{F}_{u_i(z_{i0},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi'_{i0})(1-\varphi^{xt'}_{jn(n-1)}) \\
\overline{F}_{p_{i1}(z^x_{i1},t)w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{xt'}_{j21}) & \overline{F}_{p_{i1}(z^x_{i1},t)w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{xt'}_{j31}) & \cdots & \overline{F}_{p_{i1}(z^x_{i1},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{x'}_{i1})(1-\varphi^{xt'}_{jn(n-1)}) \\
\overline{F}_{p_{i2}(z^x_{i2},t)w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{xt'}_{j21}) & \overline{F}_{p_{i2}(z^x_{i2},t)w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{xt'}_{j31}) & \cdots & \overline{F}_{p_{i2}(z^x_{i2},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{x'}_{i2})(1-\varphi^{xt'}_{jn(n-1)}) \\
\vdots & \vdots & \ddots & \vdots \\
\overline{F}_{p_{in}(z^x_{in},t)w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{x'}_{in})(1-\varphi^{xt'}_{j21}) & \overline{F}_{p_{in}(z^x_{in},t)w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{x'}_{in})(1-\varphi^{xt'}_{j31}) & \cdots & \overline{F}_{p_{in}(z^x_{in},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{x'}_{in})(1-\varphi^{xt'}_{jn(n-1)})
\end{bmatrix}$$

$$a_{\overline{q_i p_j}}=\begin{bmatrix}
\overline{F}_{q_{i1}(z^t_{i1},t)u_j(z_{j0},t)}(1-\varphi^{t'}_{i1})(1-\varphi'_{j0}) & \overline{F}_{q_{i1}(z^t_{i1},t)p_{j1}(z^x_{j1},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{x'}_{j1}) & \cdots & \overline{F}_{q_{i1}(z^t_{i1},t)p_{jn}(z^x_{jn},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{x'}_{jn}) \\
\overline{F}_{q_{i2}(z^t_{i2},t)u_j(z_{j0},t)}(1-\varphi^{t'}_{i2})(1-\varphi'_{j0}) & \overline{F}_{q_{i2}(z^t_{i2},t)p_{j1}(z^x_{j1},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{x'}_{j1}) & \cdots & \overline{F}_{q_{i2}(z^t_{i2},t)p_{jn}(z^x_{jn},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{x'}_{jn}) \\
\vdots & \vdots & \ddots & \vdots \\
\overline{F}_{q_{in}(z^t_{in},t)u_j(z_{j0},t)}(1-\varphi^{t'}_{in})(1-\varphi'_{j0}) & \overline{F}_{q_{in}(z^t_{in},t)p_{j1}(z^x_{j1},t)}(1-\varphi^{t'}_{in})(1-\varphi^{x'}_{j1}) & \cdots & \overline{F}_{q_{in}(z^t_{in},t)p_{jn}(z^x_{jn},t)}(1-\varphi^{t'}_{in})(1-\varphi^{x'}_{jn})
\end{bmatrix}$$

$$a_{\overline{q_i q_j}}=\begin{bmatrix}
\overline{F}_{q_{i1}(z^t_{i1},t)q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{t'}_{j1}) & \overline{F}_{q_{i1}(z^t_{i1},t)q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{t'}_{j2}) & \cdots & \overline{F}_{q_{i1}(z^t_{i1},t)q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{t'}_{jn}) \\
\overline{F}_{q_{i2}(z^t_{i2},t)q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{t'}_{j1}) & \overline{F}_{q_{i2}(z^t_{i2},t)q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{t'}_{j2}) & \cdots & \overline{F}_{q_{i2}(z^t_{i2},t)q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{t'}_{jn}) \\
\vdots & \vdots & \ddots & \vdots \\
\overline{F}_{q_{in}(z^t_{in},t)q_{j1}(z^t_{j1},t)}(1-\varphi^{t'}_{in})(1-\varphi^{t'}_{j1}) & \overline{F}_{q_{in}(z^t_{in},t)q_{j2}(z^t_{j2},t)}(1-\varphi^{t'}_{in})(1-\varphi^{t'}_{j2}) & \cdots & \overline{F}_{q_{in}(z^t_{in},t)q_{jn}(z^t_{jn},t)}(1-\varphi^{t'}_{in})(1-\varphi^{t'}_{jn})
\end{bmatrix}$$

$$a_{\overline{q_i w_j}}=\begin{bmatrix}
\overline{F}_{q_{i1}(z^t_{i1},t)w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{xt'}_{j21}) & \overline{F}_{q_{i1}(z^t_{i1},t)w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{xt'}_{j31}) & \cdots & \overline{F}_{q_{i1}(z^t_{i1},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{t'}_{i1})(1-\varphi^{xt'}_{jn(n-1)}) \\
\overline{F}_{q_{i2}(z^t_{i2},t)w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{xt'}_{j21}) & \overline{F}_{q_{i2}(z^t_{i2},t)w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{xt'}_{j31}) & \cdots & \overline{F}_{q_{i2}(z^t_{i2},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{t'}_{i2})(1-\varphi^{xt'}_{jn(n-1)}) \\
\vdots & \vdots & \ddots & \vdots \\
\overline{F}_{q_{in}(z^t_{in},t)w_{j21}(z^{xt}_{j21},t)}(1-\varphi^{t'}_{in})(1-\varphi^{xt'}_{j21}) & \overline{F}_{q_{in}(z^t_{in},t)w_{j31}(z^{xt}_{j31},t)}(1-\varphi^{t'}_{in})(1-\varphi^{xt'}_{j31}) & \cdots & \overline{F}_{q_{in}(z^t_{in},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}(1-\varphi^{t'}_{in})(1-\varphi^{xt'}_{jn(n-1)})
\end{bmatrix}$$

$$a_{\overline{w_i p_j}} = \begin{bmatrix} \bar{F}_{w_{i21}(z^{xt}_{i21},t)u_j(z_{j0},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi'_{j0})} & \bar{F}_{w_{i21}(z^{xt}_{i21},t)p_{j1}(z^{x}_{j1},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{x'}_{j1})} & \cdots & \bar{F}_{w_{i21}(z^{xt}_{i21},t)p_{jn}(z^{x}_{jn},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{x'}_{jn})} \\ \bar{F}_{w_{i31}(z^{xt}_{i21},t)u_j(z_{j0},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi'_{j0})} & \bar{F}_{w_{i31}(z^{xt}_{i31},t)p_{j1}(z^{x}_{j1},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{x'}_{j1})} & \cdots & \bar{F}_{w_{i31}(z^{xt}_{i31},t)p_{jn}(z^{x}_{jn},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{x'}_{jn})} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{i21},t)u_j(z_{j0},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi'_{j0})} & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)p_{j1}(z^{x}_{j1},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{x'}_{j1})} & \cdots & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)p_{jn}(z^{x}_{jn},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{x'}_{jn})} \end{bmatrix}$$

$$a_{\overline{w_i q_j}} = \begin{bmatrix} \bar{F}_{w_{i21}(z^{xt}_{i21},t)q_{j1}(z^{t}_{j1},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{t'}_{j1})} & \bar{F}_{w_{i21}(z^{xt}_{i21},t)q_{j2}(z^{t}_{j2},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{t'}_{j2})} & \cdots & \bar{F}_{w_{i21}(z^{xt}_{i21},t)q_{jn}(z^{t}_{jn},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{t'}_{jn})} \\ \bar{F}_{w_{i31}(z^{xt}_{i31},t)q_{j1}(z^{t}_{j1},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{t'}_{j1})} & \bar{F}_{w_{i31}(z^{xt}_{i31},t)q_{j2}(z^{t}_{j2},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{t'}_{j2})} & \cdots & \bar{F}_{w_{i31}(z^{xt}_{i31},t)q_{jn}(z^{t}_{jn},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{t'}_{jn})} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{j1}(z^{t}_{j1},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{t'}_{j1})} & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{j2}(z^{t}_{j2},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{t'}_{j2})} & \cdots & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)q_{jn}(z^{t}_{jn},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{t'}_{jn})} \end{bmatrix}$$

and

$$a_{\overline{w_i w_j}} = \begin{bmatrix} \bar{F}_{w_{i21}(z^{xt}_{i21},t)w_{j21}(z^{xt}_{j21},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{xt'}_{j21})} & \bar{F}_{w_{i21}(z^{xt}_{i21},t)w_{j31}(z^{xt}_{j31},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{xt'}_{j31})} & \cdots & \bar{F}_{w_{i21}(z^{xt}_{i21},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}^{(1-\varphi^{xt'}_{i21})(1-\varphi^{xt'}_{jn(n-1)})} \\ \bar{F}_{w_{i31}(z^{xt}_{i31},t)w_{j21}(z^{xt}_{j21},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{xt'}_{j21})} & \bar{F}_{w_{i31}(z^{xt}_{i31},t)w_{j31}(z^{xt}_{j31},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{xt'}_{j31})} & \cdots & \bar{F}_{w_{i31}(z^{xt}_{i31},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}^{(1-\varphi^{xt'}_{i31})(1-\varphi^{xt'}_{jn(n-1)})} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{j21}(z^{xt}_{j21},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{xt'}_{j21})} & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{j31}(z^{xt}_{j31},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{xt'}_{j31})} & \cdots & \bar{F}_{w_{in(n-1)}(z^{xt}_{in(n-1)},t)w_{jn(n-1)}(z^{xt}_{jn(n-1)},t)}^{(1-\varphi^{xt'}_{in(n-1)})(1-\varphi^{xt'}_{jn(n-1)})} \end{bmatrix}.$$

Thus, the Legendere condition for the functional (3.8), is that the matrix H is positive definite, and the Jacobi's equation may be given as:

$$\delta^2 v = \frac{1}{2!}\iint\limits_{D} (\delta u_1 \delta u_2 \ldots \delta w_{mn(n-1)}(z^{xt}_{mn(n-1)},t)) \begin{pmatrix} P_{11} & Q_{12} & \cdots & Q_{(1)m(1+2n+\frac{n(n-1)}{2})} \\ Q_{21} & P_{22} & \cdots & Q_{(2)m(1+2n+\frac{n(n-1)}{2})} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m(1+2n+\frac{n(n-1)}{2})(1)} & Q_{m(1+2n+\frac{n(n-1)}{2})(2)} & \cdots & P_{m(1+2n+\frac{n(n-1)}{2})m(1+2n+\frac{n(n-1)}{2})} \end{pmatrix} \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \vdots \\ \delta w_{mn(n-1)}(z^{xt}_{mn(n-1)},t) \end{pmatrix} dxdt$$

where:

$$P_{11} = F_{u_1u_1},\ P_{22} = F_{p_{11}p_{11}},\ \ldots,$$

$$P_{m\left(1+2n+\frac{n(n-1)}{2}\right)m\left(1+2n+\frac{n(n-1)}{2}\right)} = \overline{F}_{w_{mn(n-1)}(z^{xt}_{mn(n-1)},t)w_{mn(n-1)}(z^{xt}_{mn(n-1)},t)}(1-\varphi^{xt\prime}_{mn(n-1)}),$$

$$Q_{12} = F_{u_1p_{11}},\ldots,Q_{(1)m\left(1+2n+\frac{n(n-1)}{2}\right)} = \overline{F}_{u_1w_{mn(n-1)}(z^{xt}_{mn(n-1)},t)}(1-\varphi^{xt\prime}_{mn(n-1)}),$$

$$Q_{21} = F_{p_{11}u_1},\ldots,Q_{(2)m\left(1+2n+\frac{n(n-1)}{2}\right)} = \overline{F}_{p_{11}w_{mn(n-1)}(z^{xt}_{mn(n-1)},t)}(1-\varphi^{xt\prime}_{mn(n-1)}),$$

$$\vdots$$

$$Q_{m(1+2n+\frac{n(n-1)}{2})(1)} = \overline{F}_{w_{mn(n-1)}(z^{xt}_{mn(n-1)},t)u_1}(1-\varphi^{xt\prime}_{mn(n-1)}),\ \ldots,$$

and

$$Q_{m\left(1+2n+\frac{n(n-1)}{2}\right)(2)} = \overline{F}_{w_{mn(n-1)}(z^{xt}_{mn(n-1)},t)p_{11}}(1-\varphi^{xt\prime}_{mn(n-1)})$$

where $z_i^{x,t} = (x - \varphi_i^{x,t})$, for (i = 1, 2, …, m; j = 0, 1, …, n). ■

3.4 Illustrative Examples

The improved Magri's approach for evaluating the variational formulation of the linear PDDEv's, is completely similar to that approach discussed in chapter two for the linear ODDEv's, but with some modifications in the linear operator L, which is based on partial derivatives of arguments.

This approach may be explained easily and without loose of generality by the following examples:

Example (3.8):

Consider the RPDDE with constant delay:

$$u_{xx}(x, t) - u_t(x, t) - u(x - 1, t) = f(x, t) \qquad (3.15)$$

where $f(x, t) = 2(1 - t) + (x - 1)^3 t - (x - 1)^2$, subject to the boundary conditions:

$$u(0, t) = t^2,\ u(1, t) = 1 + t^2,\ t \in [0, 1],$$

and initial condition

$$u(x, 0) = x^2,\ x \in [0, 1],$$

with the delay initial condition:

$$u(x, t) = x^3 t + x^2,\ x \in [-1, 0] \text{ and } t \in [0, 1],$$ [Buite, 2004].

In order to find the variational formulation of the RPDDEv (3.15) by using Magrie's approach with the shift operator:

$$Du(x, t) = u(x - 1, t).$$

Let:

$$u_{xx}(x, t) - u_t(x, t) - u(x- 1, t) = f(x, t),$$

$$\frac{\partial^2}{\partial x^2} u(x, t) - \frac{\partial}{\partial t} u(x, t) - Du(x, t) = f(x, t)$$

or:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} - D\right] u(x, t) = f(x, t),$$

i.e., Lu = f, where:

$$L \equiv \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} - D \right).$$

Thus:

$$v[u] = \frac{1}{2}\langle Lu, u\rangle - \langle f, u\rangle$$

$$= \frac{1}{2}(Lu, Lu) - (f, Lu)$$

$$= \frac{1}{2}\iint_D \{(Lu)^2 - 2f\,Lu\}\, dxdt$$

$$= \frac{1}{2}\iint_D \{(u_{xx}(x, t) - u_t(x, t) - u(x - 1, t))^2 - 2\{f(u_{xx}(x, t) - u_t(x, t) - u(x - 1, t))\}\}\, dxdt \quad \text{................................(3.16)}$$

However, applying direct-Ritz method with assumption:

$$u(x, t) = \psi(x, t) + \sum_{i=1}^{n} a_i\varphi_i(x, t),$$

where $\psi(x, t)$ satisfies the non-homogeneous conditions and $\varphi_i(x, t)$ which are called the "coordinate functions" with constant coefficients a_i, $(i = 1, 2, \ldots, n)$; satisfy the homogeneous conditions. For simplicity purpose, eq.(3.15) may be written as:

$$u(x, t) = \psi(x, t) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\varphi_i(x)\varphi_j(x).$$

Consequently, the approximate solution for the problem, where:

$\psi(x, t) = x^2 + t^2,$

can be represented as:

$$u(x,t) = (x^2 + t^2) + x(x-1)t\{a_0 + a_1x + a_2t + a_3x^2 + a_4xt + a_5t^2\} \quad ...(3.17)$$

where $a_i \in \mathbb{R}$, for i = 0, 1, …, 5.

Substituting eq.(3.17) back into functional (3.16) and using the computer program (PDDE 1) written in MathCad, one can get the following results for a_i, i = 0, 1, …, 5:

$a_0 = 4.131\times10^{-6}$, $a_1 = -4.828\times10^{-7}$, $a_2 = 2.421\times10^{-7}$,

$a_3 = 2.144\times10^{-6}$, $a_4 = -3.36\times10^{-6}$ and $a_5 = -4.293\times10^{-6}$.

Hence:

$$u(x, t) = (x^2 + t^2) + x(x-1)t\{(4.131\times10^{-6}) + (-4.828\times10^{-7})x + (2.421\times10^{-7})t + (2.144\times10^{-6})x^2 - (3.36\times10^{-6})xt - (4.293\times10^{-6})t^2\}.$$

The obtained results are presented in table (3.1) and the accuracy of the results is given in the residue errors from 0 to 1for x and t, which is also given in this table.

Table (3.1)

The approximate results and residue error of example (3.8).

(x_i , t_i)	*Approximate solution*	*Residue error*
(0,0)	0	0
(0.1,0.7)	0.5	2.424×10^{-11}
(0.2,1)	1.04	2.311×10^{-14}
(0.3,0.6)	0.45	5.578×10^{-12}
(0.4,0.4)	0.32	6.004×10^{-12}
(0.5,0.3)	0.34	5.974×10^{-12}
(0.6,0.2)	0.4	5.548×10^{-12}
(0.7,0.1)	0.5	3.625×10^{-12}
(0.8,1)	1.64	4.161×10^{-11}
(0.9,0.6)	1.17	1.23×10^{-11}
(1,0.5)	1.25	2.792×10^{-11}

Example (3.9):

Consider the PDDE with the variable retarded argument in x:

$$u_{xx}(x, t) + u_{tt}(x, t) + 2u_{xt}(x - xt, t) = f(x, t) \qquad (3.18)$$

where $f(x, t) = 6xt + 2 + 6(x - xt)^2$; with the following boundary conditions:

$$u(0, t) = 0,\ u(1, t) = t + 1, \text{ for } t \in [0, 1]$$

and

$$u(x, 0) = x^2,\ u(x, 1) = x^3 + x^2, \text{ for } x \in [0, 1]$$

of the delay initial condition:

$u(x, t) = x^3t + x^2$, for $0 < t \leq 1, 0 < x < 1 - t$.

Let the shift operator:

$$Du(x, t) = u(x - xt, t).$$

then:

$$u_{xx}(x, t) + u_{tt}(x, t) + 2u_{xt}(x - xt, t) = f(x, t),$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial x \partial t} D\right) u(x, t) = f(x, t).$$

Therefore, the operator L related to eq.(3.18) is given by:

$$L \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial x \partial t} D\right).$$

Hence, the variational formulation can be written as:

$$v[u] = \frac{1}{2} \langle Lu, u \rangle - \langle f, u \rangle$$

$$= \frac{1}{2}(Lu, Lu) - (f, Lu)$$

$$= \frac{1}{2} \iint_D \{(Lu)^2 - 2f\, Lu\}\, dxdt$$

$$= \frac{1}{2} \iint_D \{(u_{xx}(x, t) + u_{tt}(x, t) + 2u_{xt}(x - xt, t) - 2\{f(x, t)$$

$$(u_{xx}(x, t) + u_{tt}(x, t) + 2u_{xt}(x - xt, t)\}\}\, dxdt \quad \text{..........(3.19)}$$

Using the direct-Ritz method with the following approximate solution:

$$u(x,t) = (x^3t+x^2) + x(x-1)t(t-1)\{a_0 + a_1x + a_2t + a_3x^2 + a_4xt + a_5t^2\} \quad \text{...............................(3.20)}$$

where $a_i \in \mathbb{R}$, for (i = 0, 1, …, 5).

Substituting eq.(3.20) back into functional (3.19) and using the computer program (PDDE 2) written in MathCad, one can get the following results for a_i, i = 0, 1, …, 5:

$a_0 = 4.516\times10^{-7}$, $a_1 = 1.151\times10^{-6}$, $a_2 = -3.294\times10^{-7}$,

$a_3 = 1.98\times10^{-6}$, $a_4 = 1.853\times10^{-7}$, $a_5 = -1.861\times10^{-6}$.

and the results are presented in table (3.2) in which the accuracy of the results are obtained by evaluating the residue error function from 0 to 1, for x and t.

Table (3.2)

The approximate results and residue error of example (3.9).

(x_i , t_i)	***Approximate solution***	***Residue error***
(0,0)	0	0
(0.1,0.7)	0.011	1.218×10^{-12}
(0.1,1)	0.11	1.022×10^{-12}
(0.2,0.3)	0.042	1.849×10^{-15}
(0.4,0.5)	0.192	1.063×10^{-13}
(0.4,0.8)	0.211	1.582×10^{-12}
(0.6,0.2)	0.403	3.555×10^{-12}
(0.7,0.4)	0.627	6.563×10^{-12}
(0.8,0.9)	1.101	3.819×10^{-15}
(0.9,1)	1.539	2.57×10^{-13}
(1,1)	2	0

3.5 Real Life Problem of PDDE (The Simple Food Web Problem)

Delay differential equations arise in many areas of mathematical modeling, for example; population dynamics (taking into account the gestation times), [Wangersky, 1957], infection diseases (accounting for the incubation periods), [Gourley, 2008], physiological and pharmaceutical kinetics modeling (e.g., the body's reaction to CO_2, etc., reactants), [Mackey, 1977], [Beuter, 1993] and [Milton, 1989], the navigational control of ships and aircrafts (with respectively large and short lags), [Grush, 2003], the neutral network models (interactions of the neurons are delayed), [Campbell, 1999], and more general control problems, [Craig, 1986] and [Insperger, 2004].

Mathematical models of increasing complexity is the models of describing the mathematical analysis of material recycling in closed ecosystem, which are constructed by, [Nisbet, 1983], [Ulanowicz, 1972] and the most common models in this field; the dynamic models of food network.

So, in this section, we will discuss and simulate the application of real life problem of simple food web problem.

3.5.1 The Model:

The model that we treat has 2n + 1 components consisting of the n + 1 type living organisms (zooplankton, phytoplankton and microorganisms) and the n-type dissolved organic and inorganic nutrients and detritus.

The nutrient recycling beginning with the microorganisms x_i, such that these organisms pass through the some living levels, in these levels the microorganisms produce the nutrients y_i, i = 1, 2, …, n. The nutrients y_i, i = 2, 3, …, n – 1, which are the metabolic product of the microorganisms x_i, are assumed to be the one primarily responsible for limiting the x_i production for i = 1, 2, …, n. The dissolved organic nutrient concentration y_1 is a result of the partial decomposition of the dead organisms. The phytoplankton x_n, which assimilated the metabolic product y_n, excretes the dissolved organic nutrient and limits zooplankton growth (z). Zooplankton excretes dissolved organic matter too. The n + 1 living organism's levels, detritus and the n nutrients are modeled in terms of their nitrogen content N (see Fig.(3.1)).

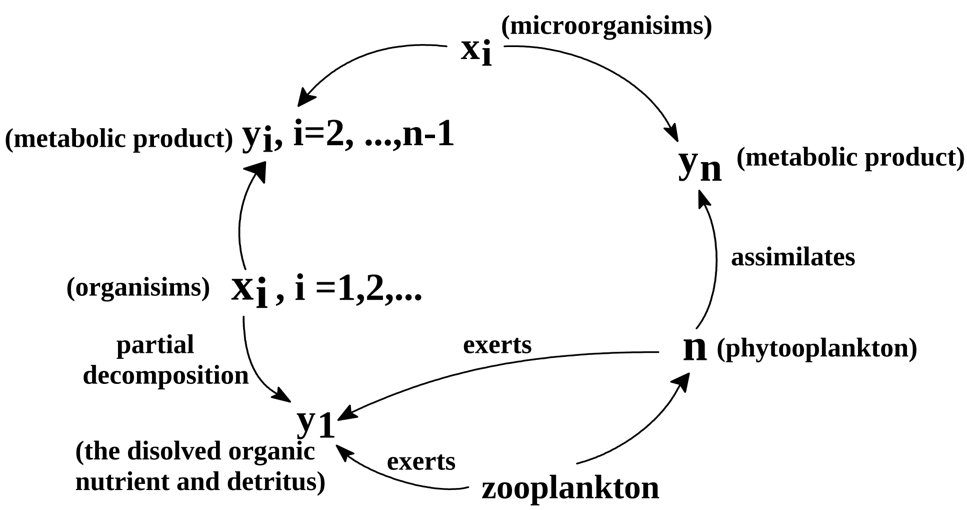

Fig.(3.1) The cycling of simple food web.

The quadratic assumptions describe the asymptotic behavior of the similar trophic chain.

From the biological viewpoint, the dynamics of a simple food chain is governed by the following functional-differential equation:

$$\frac{\partial d}{\partial t} = \alpha \frac{\partial^2 d}{\partial p^2} + \sum_{i=1}^{n} x_i M_i(z_i) + zM_z(x_n) - K_1 d(t-1, p) \quad \text{.............(3.21)}$$

with the boundary condition:

$$\frac{\partial d}{\partial t}(0, p) = 0,\ 0 \le p \le 1,$$

and initial conditions:

$$\frac{\partial d}{\partial p}(t, 0) = \frac{\partial d}{\partial p}(t, 1) = 0,\ 0 \le t \le 1,$$

of delay initial condition $\varphi = \varphi_{2n+1}(p) = p(p-1) \ge 0$, $0 \le p \le 1$, $-1 \le t \le 0$, where $x_i(t, p)$, $y_i(t, p)$, $i = 1, 2, \ldots, n$, $d(t, p)$ and $z(t, p)$ are the concentration of recycling matter in microorganisms, the available nutrients, detritus and zooplankton, respectively. The delay term is a scalar factor of time uptake and excretion of the nutrient, also decomposition of detritus and $K \ge \pi/2$, [Kmet, 2007].

In this study, the mathematical model of the problem under consideration for finding the concentration of the recycling material in microorganisms, the available nutrients, detritus and zooplankton, respectively. The considered problem is to find the recycling material in detritus using the discussed direct-Ritz method. Consider:

$$\frac{\partial d}{\partial t} - \alpha \frac{\partial^2 d}{\partial p^2} + K_1 d(t-1, p) = \sum_{i=1}^{n} x_i M_i(y_i) + z M_z(x_n),$$

or equivalently:

$$\frac{\partial d}{\partial t} - \alpha \frac{\partial^2 d}{\partial p^2} + K_1 d(t-1, p) = f(x, y) \qquad (3.22)$$

where:

$$f(x, y) = \sum_{i=1}^{n} x_i M_i(y_i) + z M_z(x_n).$$

Now, using Magri's approach with the shift operator Dd(t, p) = d(t – 1, p), one have:

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial p^2} + K_1 D \right) d(t, p) = f(x, y)$$

or

$$Ld = f.$$

where L = $\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial p^2} + KD \right)$, hence the variational formulation corresponding to eq.(3.22) may be written as:

$$v[d(t, p)] = \frac{1}{2} \langle Ld, d \rangle - \langle f, d \rangle$$

$$= \frac{1}{2} (Ld, Ld) - (f, Ld)$$

$$= \frac{1}{2}\int_0^1\int_0^1 \left\{ \left(\frac{\partial d}{\partial t} - \alpha \frac{\partial^2 d}{\partial p^2} + Kd(t-1,p) \right)^2 - 2f(x,y)\left(\frac{\partial d}{\partial t} - \right.\right.$$

$$\left.\left. \alpha \frac{\partial^2 d}{\partial p^2} + Kd(t-1,p) \right) \right\} dtdp \text{ ....(3.23)}$$

Using the direct-Ritz method with the following approximation:

$d(t, p) = tp(p - 1)\{a_0 + a_1 t + a_2 p + a_3 t^2 + a_4 tp + a_5 p^2\}$ ....(3.24)

where $a_i \in \mathbb{R}$, for all (i = 0, 1, …, 5).

Substituting eq.(3.24) in eq.(3.23) and carrying out the minimization, one get the following results for a_i's, i = 0, 1, …, 5:

$a_0 = -1.544$, $a_1 = 2.637$, $a_2 = -0.376$,

$a_3 = -1.354$, $a_4 = -1.075\times10^{-15}$, $a_5 = 0.376$.

These results are obtained upon carrying the computer program (PDDE 3) which is presented in table (3.3) and the accuracy of the results is obtained by evaluating the residue error function from 0 to 1, for different values of t and p.

Table (3.3)

The approximate results and residue error of dynamic of food network problem.

(t_i , p_i)	**Approximate solution**	**Residue error**
(0, 0)	0	0
(0.1, 0.1)	0.012	1.561×10^{-3}
(0.1, 0.4)	0.033	0.035
(0.1, 0.7)	0.029	0.02
(0.1, 1)	0	0.034
(0.2, 0.1)	0.02	0.019
(0.2, 0.4)	0.056	6.682×10^{-3}
(0.2, 0.8)	0.036	1.111×10^{-3}
(0.3, 0)	0	0.09
(0.3, 0.3)	0.06	7.821×10^{-5}
(0.3, 0.9)	0.025	0.029
(0.4, 0.1)	0.027	0.024
(0.4, 0.4)	0.076	2.362×10^{-4}
(0.4, 0.7)	0.066	2.948×10^{-4}
(0.5, 0.8)	0.05	1.877×10^{-3}
(0.6, 0.3)	0.067	4.087×10^{-4}
(0.6, 0.9)	0.026	1.406×10^{-3}
(1, 0.5)	0.089	0.048
(1,1)	0	0.053

Conclusions and Recommendations

The following conclusions may be drown from this work:

1. Partial delay differential equations could not be solved analytically; therefore, in most cases, numerical and approximate methods in general, and variational methods in particular are recommended.
2. The variational formulation of ODDEv's and PDDEv's, especially with multidelay are so difficult to be considered.
3. In comparison of the results, the residue error in some examples has been used, since the exact solution to the undertaken examples is not given in advance.
4. When comparison is made between the approximate results which were presented in this thesis with the results obtained from the exact solution, one can see the accuracy of the results obtained from the variational approach.

Also, from the present study, one can recommend the following problems for future work:

1. Solving integral-delay differential equations with variable delays using variational approach.
2. Using Magrie's approach to find the variational formulation for a system of linear ODDE's with constant or variable delays.
3. Solving nonlinear DDEv's using the variational approach.

4. Establish the necessary and sufficient condition for stability or instability of ODDE's or PDDE’s with constant or variable delay.
5. Deriving the necessary and sufficient conditions for an extremum of nonlinear DDE's.
6. Deriving the necessary and sufficient conditions for moving and free boundary value problems with deviating arguments in PDDEv’s.
7. Deriving the necessary and sufficient conditions for the mixed boundary-value problems in PDDEv’s.

References

[1] Al-Dafae'e Z. A., “Numerical Solution of Delay Differential Equations with Small Delays Using Linear Multistep Methods”, M.Sc. Thesis, Department of Mathematics, College of Education, Ibn Al-Haitham, Baghdad University, 2005.

[2] Al-Daynee S. K., “On Solution of Delay Boundary Value Problems”, M.Sc. Thesis, Department of Mathematics, Al-Nahrain University, Baghdad, Iraq, 2002.

[3] Al-Esawi I. S., "Magnitude Estimation of Solutions for Special Types of Delay Differential Equations", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, 2006.

[4] Al-Kubeisy S., "Numerical Solution of Delay Differential Equations Using Linear Multistep Methods", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2004.

[5] Al-Saady A. S., “Cubic Spline Technique in Solving Delay Differential Equations”, M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2000.

[6] Asl F. M. and Ulsoy A. G., "Analysis of a System of Linear Delay Differential Equations", Journal of Dynamic Systems, Measurement and Control, Vol.125, pp.215-222, 2003.

[7] Baker C. T. H., Paul C. A. H. and Will'e D. R., "Issues in the Numerical Solution of Evolutionary Delay Differential Equations", J, Adv. Comp. Math.,Vol.3, pp.171-196, 1994.

[8] Bellman R. and Cook K., "Differential Difference Equations", Academic Press, Inc., New York, 1963.

[9] Berechtken-Mandrcheid V., "Introduction to the Calculus of Variations", P. G. Engstrom, CRC Press, 1991.

[10] Beuter A., Bélair J. and Labrie, "Multi-Scale Analysis of Noise-Sensitivity Neur A Bifurcation", J. Bull. Math. Biol., Vol.55, pp. 525-541, 1993.

[11] Bica A. and Mureasan S., "Periodic Solution for a Delay Integro-Differential Equation in Biomathematices", J.R.G.M.I.A. Research Report Collection, Vol.6, No.4, pp.755-761, 2003.

[12] Bica A.M. and Muresan S., "Smooth Dependence by Lag of the Solution of a Delay Integro-Differential Equation from Biomathematic", J. Mathematical Analysis, Vol.1, No.1, pp.64-74, 2006.

[13] Boiza O., "Lectures on the Calculus of Variations", American Mathematical Society Bookstore, 2001.

[14] Bouzahir H., "Existence and Regularity of Local Solutions to Partial Neutral Functional Differential Equations with Infinite Delay", Electronic Journal of Differential Equations, Vol.2006, No.88, pp.1-16, 2006.

[15] Brauer F. and Nohel J. A., "Ordinary Differential Equations", W. A. Benjamin, Inc., 1973.

[16] Buite H. M., "Variational Formulation of Partial Delay Differential Equations", M. Sc. Thesis, Department of Mathematics, College of Education, Al-Mustansiriyah University, Baghdad, Iraq, 2004.

[17] Campbell S. A., Ruan S. and Wei J., "Qualitative Analysis of a Neural Network Model with Multiple Time Delays", International Journal of Bifurcation and Chaos, Vol.9, No.8, pp.1585-1595, 1999.

[18] Catillo E., Luceno A. and Pedregal P., "Composition Functionals in Calculus of Variations: Application to Products and Quotients", J. Optimization Methods Variationals, University of Castilla-La Mancha, Spain, Vol.11, pp.1-22, 2005.

[19] Cherkaev A. and Cherkaev E., "Calculus of Variations and Applications", Lectures Notes, 2003.

[20] Cooke K. L., Hale J. L. and Lasalle J. P., "Functional-Differential Equations: Some Models and Perturbation Problems, in Differential Equations and Dynamical Systems", Academic Press, New York, pp.167-183, 1967.

[21] Craig J. J. , "Introduction to Robot Mechanics and Control", Addison-Wesley, 1986.

[22] Dacorogna B. "Introduction to the Calculus of Variations", Imperial College Press, 2004.

[23] Desch W., Gramer R. and Schappacsher W., "Well Posedness and Wave Propagation for a Class of Integro-Differential Equations in Banach Space", J. Differential Equations, Vol.74, pp.391-441, 1988.

[24] Drager L. D. and Layton W., "Initial Value Problems for Nonlinear Nonresonant Delay Differential Equations with Possibly Infinite Delay", Electronic Journal of Differential Equations, Vol.1997, No.24, pp.1-20, 1997.

[25] Driver R. D. E., “Ordinary and Delay Differential Equations”, Springier Verlag Inc., New York, 1977.

[26] El’sgol’ts L. E. and Norkin S. B., “Introduction to the Theory and Application of Differential Equations with Deviating Arguments”, Academic Press, New York, 1973.

[27] Elsgolc L. E., “Calculus of Variations”, Pergamon Press, Inc., New York, 1962.

[28] Elsgolc L. E., “Qualitative Methods in Mathematical Analysis”, Trans. Math. Mono., American Mathematical Society, Vol.12, 1964.

[29] Falbo C. E., “Analytic and Numerical Solutions to the Delay Differential Equations”, Joint Meeting of the Norhern and southern California Sections of the MAA, San Luis Opisco CA., 1995.

[30] Forde J. E., “Delay Differential Equations Models in Mathematical Biology”, ph. D. Thesis, University of Michigan, 2006.

[31] Fox C., "An Introduction to the Calculus of Variations", Courier Dover Publications, 1987.

[32] Gelfand I. M. and Fomin S. V., "Calculus of Variations", Prentic-Hall, Inc., New Jersey, 1963.

[33] Gluesing-Luerssen H., "Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach", Springer, New York, 2002.

[34] Gourley S. A., Kuang Y. and Nagy J. D., "Dynamics of a Delay Differential Equation Model of Hepatitis B Virus Infection", J. Biological Dynamics, Vol.2, No.2, pp.140-153, 2008.

[35] Grush R., "In Defense of Some Cartesian, Assumptions Concerning the Brain and Its Operation", J. Biology and Philosophy, Kluwor Academic Publishers, Netherlands, Vol.18, pp.53-93, 2003.

[36] Gu K., Kharitonov V. L. and Chen J., "Stability of Time-Delay Systems", MA: Birkhauser, Boston, 2003.

[37] Guglielmi N. and Hairer E., "Order stars and Stability for Delay Differential Equations", J. Numbar. Math, Vol.83, No.3, pp.371-383, 1999.

[38] Guillouzic S., L'Heureux I. and Longtin A., "Small Delay Approximation of Stochastic Delay Differential Equations", The American Physical Society, Vol.59, No.4, pp.3970-3982, 1999.

[39] Halanay A., "Differential Equations, Stability, Oscillations, Time Lags", New York, Academic, 1966.

[40] Hale J. K. and Lunel S. M. V., "Introduction to Functional Differential Equations", Applied Mathematics Sciences, Springer-Verlag, New York, Vol.99, 1993.

[41] Hale J. K., "Theory of Functional Differential Equations", Springer, Heidelberg, New York, 1977.

[42] Hern'andez E., Sakthivel R. and Aki S. T., "Existence Results for Impulsive Evaluation Differential Equations with State-Dependent Delay", Electronic Journal of Differential Equations, Vol.2008, No.28, pp.1-11, 2008.

[43] Insperger T. and Stépán G., "Updated Semi-Discretization Method for Periodic Delay-Differential Equation with Discrete Delay", Int. J. Numer. Math. Engng., Vol.61, pp.177-241, 2004.

[44] Insperger T. and Stépán G., "Stability Improvements of Robot Control by Periodic Variation of the Gain Parameters", China Machinery Press, China, 2004.

[45] Kaiser C., "Integrated Semigroups and Linear Partial Differential Equations with Delay", J. Mathematics Subject Classification, Vol.2000, 2000.

[46] Kalmanovskii V.B. and Nosov V.R., "Stability of Functional Differential Equations", Academic Press, London, 1986.

[47] Kmet T. and Kmetova, "Dynamic Model of Food Network", J. Selye University, Slovakia, Vol.01, No.74, pp.945-949, 2007.

[48] Kolmanovskii V. and Myshkis A., "Applied Theory of Functional Differential Equations", Kluwer Academic Publishers, Springer, New York, 1992.

[49] Ladde G. S., Zhang B. G. and Lakshmikantham V., "Oscillation Theory of Differential Equations with Deviating Arguments", Marcel Dekker, Inc., New York, and Basel, 1987.

[50] Lauisell J., "New Examples of Quenching in Delay Differential Equations Having Time-Varying Delay", In Proc. 5th Eur: Control Conf., Karlshruhe, Germany, 1999.

[51] Lebedev L. P. and Cloud M. J., "The Calculus of Variations and Functional Analysis", World Scientific Functional Analysis, 2003.

[52] Leitmann G., "The Calculus of Variations and Optimal Control", Springer, New York and London, 1981.

[53] Lensnic D. "One-Dimensional Two-Point Boundary value problems", Inter. J. of Math. and Math. Sciences, Vol.23, pp.22-54, 1999.

[54] Luo J., "A Note on Exponential Stability in p-th Mean of Solutions of Stochastic Delay Differential Equation", J. of Comp. and App. Math., Vol.128, pp.143-148, 2007.

[55] Mackey M. C. and Glass L., "A Report on the Use of Delay Differential Equations in Numerical Modeling in the Bioscience", J. Science, Vol.197, pp.287-289, 1977.

[56] Magri F., "Variaional Formulation for Every Linear Equation", Int. J. Engng. Sci., Vol.12, pp.537-549, 1974.

[57] Mao X., "Exponential Stability of Nonlinear Differential Delay Equations", SIAM (Soc. Ind. Appl. Math.), J. Math. Anal., Vol.28, No.389, 1997.

[58] Marie N., "Variational Formulation of Delay Differential Equations", M.Sc. Thesis, Department of Mathematics, College of Education, Ibn Al-Haitham, University of Baghdad, Baghdad, Iraq, 2001.

[59] Mastinsek M., “Adjoints of Solutions Semigroups and Identifiably of Delay Differential Equations in Hilbert spaces”, J. Acta, Math. Univ. Comenianqe, Vol. LXIII, No.2, pp.193-206, 1994.

[60] Memarbashi R., "Variational Problem with Moving Boundaries Using Decomposition Method", Journal of Samnan, Samnan, Iran, Vol.2006, pp.6-16, 2006.

[61] Michiels W., Assche V. V. and Miculescu S. I., "Stabilization of Time-Delay Systems with a Controlled Time Varying Delay and Applications", IEEE Transactions on Automatic Control, Vol.50, No.4, 2005.

[62] Milton J.G., Longtin A., Beuter A., Mackey M.C. and Glass L., "Complex Dynamics and Bifurcations in Neurology", J. Theor. Bio., 1., Vol.138, pp.129-147, 1989.

[63] Mishkis A. D., “Linear Differential Equations with Retarded Argument”, 2nd Edition, Nauka, Moscow, 1972.

[64] Mori T. and Kokame H., "Stability of $d/dt[x(t)] = Ax(t) + Bx(t - \tau)$", IEEE Transactions on Automatic Control, Vol.34, No.4, pp.460-462, 1989.

[65] Mori T., Fukuma N. and Kuwahara M., "Simple Stability Criteria for Single and Composite Linear Systems with Time Delays", International Journal of Control, Vol.34, No.6, pp.1175-1184, 1981.

[66] Muroya Y., "A Global Stability Criterion in Nonautonomous Delay Differential Equations", J. Math. Anal. Appl., Vol.326, pp.209-227, 2007.

[67] Nibet R. M., Mckinstry and Gurney J.W.S., “A Strategic Model of Material Cycling in a Clased Ecosystem”, J. Mathematical Biosciences, Vol.64, pp.99-113, 1983.

[68] Niculescu S. I., "Delay Effects on Stability: A Robust Control Approach", Heiddberg, Germany: Springer-Verlag, Vol.269, 2001.

[69] Niculescu S., Duggard L. and Verriest E. I., "Stability and Robust Stability of Time-Delay Systems: A Guided Tour in Stability and Control of Time-Delay Systems", Springer, New York, pp.1-71, 1998.

[70] Pinny E., "Ordinary Difference Differential Equations", University of California, 1959.

[71] Rezounenko A. V., "Differential Equations with Discrete State-Dependent Delay, Uniqueness and Well-Posedness in the Space of Continuous Functions", J. Mathematics Subject Classifications, Vol.10, pp.1-14, 2008.

[72] Russak I. B., "Calculus of Variations MA 4311 Lecture Notes", Naval Postgraduate School, Monterey, California, 2002.

[73] Salih S., "Approximated Solution of Linear Delay Differential Equations", M.Sc. Thesis, School of Applied Sciences, University of Technology, Baghdad, Iraq, 2004.

[74] Shampine L. F. and Thompson S., "Solving Delay Differential Equations with dde23", J. Comp. and Maths. with Appls., Vol.39, pp.43-54, 2000.

[75] Smith O.J., “Closer Control of Loops with Dead Time”, J. Chem. Eng. Prog. , Vol.53, pp.217-219, 1957.

[76] Stépán G., “Retarded Dynamical Systems”, Longman, Harlow, 1989.

[77] Su J. H., "Stability Analysis of Linear Systems with Time Delay", IEEE Transactions on Automatic Controls, Vol.39, No.6, pp.1341-1344, 1994.

[78] Su J. H., "The Asymptotic Stability of linear Autonomous Systems with Commensurate Time Delays", IEEE Transactions on Automatic Controls, Vol.40, No.6, pp.1114-1117, 1995.

[79] Taylor S. R., "Probabilistic Properties of Delay Differential Equations", Ph.D. Thesis, University of Waterloo, Waterloo, Canada, 2004.

[80] Techvenche J. M., "Asymptotic Stability of an Abstract Delay Functional-Differential Equations", J. Nonlinear Analysis: Modeling and Control, Vol.11, No.1, pp.79-93, 2006.

[81] Tiryaki A. and Aktas M. F., " Oscillation Criteria of a Certain Class of Third Order Nonlinear Delay Differential Equations with Damping", Department of Mathematics, Faculty of Arts and Science, J. Math. Anal. Appl., Vol.325, pp.54-68, 2007.

[82] Ulanowicz R. E., “Mass and Energy Flow in Closed Ecosystems”, J. theor. Biol., Vol.34, pp.239-253, 1972.

[83] Vandewall S. and Grander M., "Optimized Over Lapping Schwarz Methods for Parabolic PDE's with Time-Delay", Conference on Domain Decomposition Methods, Berlin, Germany, 2003.

[84] Verheyden K. and Lust K., "A Newton-Picard Collection Method for Periodic of Delay Differential Equations", Department of Computer Science, Report TW., pp.357-481, 2003.

[85] Wan F.Y.M., "Introduction to the Calculus of Variations and its Applications", CRC Press, 1995.

[86] Wang S. S., "Further Results on Stability of d/dt[x(t)] = Ax(t) + Bx(t – τ)", Systems and Control Letters, Vol.19, pp.165-168, 1992.

[87] Wangersky P.J. and Cunningham W. J., J. Ecology, Vol.38, pp.136, 1957.

[88] Whitaker S. Y., "A Biologically-Based Controlled Growth and Differentiation Model Using Delay Differential Equations: Development, Applications and Stability Analysis", Ph.D. Thesis, State University, North Carolina Graduate Faculty, 2000.

[89] Wiener J., "Generalized Solutions of Functional Differential Equations", World Scientific, 1993.

[90] Wright E. M., "The Linear Difference-Differential Equation with Asymptotically Constant Coefficients", Am. J. Math., Vol.70, No.2, pp.221-238, 1948.

[91] Wright E. M., "The Non-Linear Difference Differential Equation", Q.J. Math., Vol.17, pp.245-252, 1946.

Appendix A
Computer Programs

1- ODDE 1 Program:

$$x(t,a0,a1,a2) := \frac{2}{3}\cdot t + t\cdot(t-1)\cdot\left(a0 + a1\cdot t + a2\cdot t^2\right)$$

$$\frac{d^2}{dt^2}(x(t,a0,a1,a2)) \rightarrow 2\cdot a0 + 2\cdot a1\cdot t + 2\cdot a2\cdot t^2 + 2\cdot(t-1)\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(t-1)\cdot a2$$

$$xtt(t,a0,a1,a2) := 2\cdot a0 + 2\cdot a1\cdot t + 2\cdot a2\cdot t^2 + 2\cdot(t-1)\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(t-1)\cdot a2$$

$$Lx(t,a0,a1,a2) := xtt(t,a0,a1,a2) - (t-1)$$

$$j(a0,a1,a2) := \frac{1}{2}\cdot\int_0^1 (Lx(t,a0,a1,a2))^2\,dt$$

$$j(a0,a1,a2) \rightarrow a0 + \frac{12}{5}\cdot a2^2 + \frac{1}{6} + 4\cdot a2\cdot a1 + 2\cdot a2\cdot a0 + 2\cdot a1^2 + 2\cdot a0\cdot a1 + 2\cdot a0^2$$

$a0 := 0 \qquad a1 := 1 \qquad a2 := 3$

$m := \text{Minimize}(j, a0, a1, a2)$

$$m = \begin{pmatrix} -0.333 \\ 0.167 \\ -2.716\times 10^{-9} \end{pmatrix}$$

2- ODDE 2 Program:

$$x(t,a0,a1,a2) := \frac{5}{3}\cdot t + t\cdot(t-1)\cdot\left(a0 + a1\cdot t + a2\cdot t^2\right)$$

$$\frac{d^2}{dt^2}(x(t,a0,a1,a2)) \rightarrow 2\cdot a0 + 2\cdot a1\cdot t + 2\cdot a2\cdot t^2 + 2\cdot(t-1)\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(t-1)\cdot a2$$

$$xtt(t,a0,a1,a2) := 2\cdot a0 + 2\cdot a1\cdot t + 2\cdot a2\cdot t^2 + 2\cdot(t-1)\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(a1 + 2\cdot a2\cdot t) + 2\cdot t\cdot(t-1)\cdot a2$$

$$\frac{d}{dt}x(t,a0,a1,a2) \rightarrow \frac{5}{3} + (t-1)\cdot\left(a0 + a1\cdot t + a2\cdot t^2\right) + t\cdot\left(a0 + a1\cdot t + a2\cdot t^2\right) + t\cdot(t-1)\cdot(a1 + 2\cdot a2\cdot t)$$

$$xt(t,a0,a1,a2) := \frac{5}{3} + (t-1)\cdot\left(a0 + a1\cdot t + a2\cdot t^2\right) + t\cdot\left(a0 + a1\cdot t + a2\cdot t^2\right) + t\cdot(t-1)\cdot(a1 + 2\cdot a2\cdot t)$$

$Lx(t, a0, a1, a2) := xtt(t, a0, a1, a2) - (t - 1) - 2$

$j(a0, a1, a2) := \frac{1}{2} \cdot \int_{0}^{1} (Lx(t, a0, a1, a2))^2 \, dt$

$j(a0, a1, a2) \rightarrow -3 \cdot a0 - 2 \cdot a1 + \frac{12}{5} \cdot a2^2 + \frac{7}{6} - 2 \cdot a2 + 4 \cdot a2 \cdot a1 + 2 \cdot a2 \cdot a0 + 2 \cdot a1^2 + 2 \cdot a0 \cdot a1 + 2 \cdot a0^2$

$a0 := 0 \qquad a1 := 1 \qquad a2 := 2$

$m := Minimize(j, a0, a1, a2)$

$m = \begin{pmatrix} 0.667 \\ 0.167 \\ 1.478 \times 10^{-10} \end{pmatrix}$

3- ODDE 3 Program:

$x(t, a0, a1) := 6.94 t - 2.997 + (t - .1) \cdot (t - 0.2) \cdot (a0 + a1 \cdot t)$

$u(t) := \left[-e^{\left(1 - \frac{1}{t}\right)} + t \right]$

$\frac{d}{dt}(x(t, a0, a1)) \rightarrow 6.94 + (t - .2) \cdot (a0 + a1 \cdot t) + (t - .1) \cdot (a0 + a1 \cdot t) + (t - .1) \cdot (t - .2) \cdot a1$

$xt(t, a0, a1) := 6.94 + (t - .2) \cdot (a0 + a1 \cdot t) + (t - .1) \cdot (a0 + a1 \cdot t) + (t - .1) \cdot (t - .2) \cdot a1$

$Lx(t, a0, a1) := ((xt(t, a0, a1))) + \ln(t - u(t))$

$j(a0, a1) := \frac{1}{2} \cdot \int_{0.1}^{0.2} (Lx(t, a0, a1))^2 - 2 \cdot (Lx(t, a0, a1)) \, dt$

$a0 := 0 \qquad a1 := 1$

$m := Minimize(j, a0, a1)$

$m = \begin{pmatrix} -40.202 \\ 109.133 \end{pmatrix}$

4- ODDE 4 Program:

$x(t, a0, a1) := 0.892526 t + t \cdot (t - 1) \cdot (a0 + a1 \cdot t)$

$\frac{d}{dt}(x(t, a0, a1)) \rightarrow .892526 + (t - 1) \cdot (a0 + a1 \cdot t) + t \cdot (a0 + a1 \cdot t) + t \cdot (t - 1) \cdot a1$

$xt(t, a0, a1) := \frac{d}{dt}(x(t, a0, a1))$

$u(t, a0, a1) := 0.5 \cdot t \cdot x(t, a0, a1)$

$LS(t,a0,a1) := [xt(t,a0,a1) - x(t,a0,a1) - (t - u(t,a0,a1)) - t]^2$

$Lx(t,a0,a1) := xt(t,a0,a1) - x(t,a0,a1) - (t - u(t,a0,a1))$

$$j(a0,a1) := \frac{1}{2}\cdot\int_0^1 (Lx(t,a0,a1))^2 - 2\cdot(t\cdot Lx(t,a0,a1))\,dt$$

$a0 := 0 \quad a1 := 1$

$z := \text{Minimize}(j,a0,a1)$

$$z = \begin{pmatrix} 1.267 \\ 0.148 \end{pmatrix}$$

$t := 0, 0.1 .. 1$

$t =$	$LS(t,z_0,z_1) =$
0	0.14
0.1	0.103
0.2	0.08
0.3	0.066
0.4	0.057
0.5	0.052
0.6	0.048
0.7	0.044
0.8	0.038
0.9	0.03
1	0.019

5- PDDE 1 Program:

$$u(x,t,a0,a1,a2,a3,a4,a5) := \left(x^2 + t^2\right) + \left[x\cdot(x-1)\cdot t\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right)\right]$$

$$\frac{d^2}{dx^2}u(x,t,a0,a1,a2,a3,a4,a5) \rightarrow 2 + 2\cdot t\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) + 2\cdot(x-1)\cdot t\cdot(a1 + 2\cdot a3\cdot x +$$

$$a4\cdot t) + 2\cdot x\cdot t\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot a3$$

$$uxx(x,t,a0,a1,a2,a3,a4,a5) := 2 + 2\cdot t\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) + 2\cdot(x-1)\cdot t\cdot(a1 +$$

$$2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot t\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot a3$$

$$\frac{d}{dt}u(x,t,a0,a1,a2,a3,a4,a5) \rightarrow 2\cdot t + x\cdot(x-1)\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) + x\cdot(x-1)\cdot t\cdot(a2 + a4\cdot x + 2\cdot a5\cdot t)$$

$$ut(x,t,a0,a1,a2,a3,a4,a5) := 2\cdot t + x\cdot(x-1)\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) + x\cdot(x-1)\cdot t\cdot(a2 + a4\cdot x + 2\cdot a5\cdot t)$$

$$Du(x,t) := (x-1)^3\cdot t + (x-1)^2$$

$$f(x,t) := 2\cdot(1-t) - (x-1)^3\cdot t - (x-1)^2$$

$Lu(x,t,a0,a1,a2,a3,a4,a5) := uxx(x,t,a0,a1,a2,a3,a4,a5) - ut(x,t,a0,a1,a2,a3,a4,a5) - Du(x,t)$

$Lu(x,t,a0,a1,a2,a3,a4,a5) \rightarrow 2 + 2\cdot t\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) + (2\cdot x - 2)\cdot$

$t\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot t\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot a3 - 2\cdot t - x\cdot(x-1)\cdot\left(a0 + a1\cdot x + \right.$

$LS(x,t,a0,a1,a2,a3,a4,a5) := (Lu(x,t,a0,a1,a2,a3,a4,a5) - f(x,t))^2$

$$J(a0,a1,a2,a3,a4,a5) := \frac{1}{2}\cdot\int_0^1\int_0^1 \left[(Lu(x,t,a0,a1,a2,a3,a4,a5))^2 - 2\cdot[f(x,t)\cdot(Lu(x,t,a0,a1,a2,a3,a4,a5))]\right] dx\,dt$$

$a0 := 0 \quad a1 := 1 \quad a2 := 2 \quad a3 := 3 \quad a4 := 4 \quad a5 := 5$

$m := Minimize(J, a0, a1, a2, a3, a4, a5)$

$$m = \begin{pmatrix} 4.131\times 10^{-6} \\ -4.828\times 10^{-7} \\ 2.421\times 10^{-7} \\ 2.144\times 10^{-6} \\ -3.36\times 10^{-6} \\ -4.293\times 10^{-6} \end{pmatrix}$$

6- PDDE 2 Program:

$u(x,t,a0,a1,a2,a3,a4,a5) := x^3\cdot t + x^2 + x\cdot(x-1)\cdot t\cdot(t-1)\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right)$

$\frac{d^2}{dx^2}u(x,t,a0,a1,a2,a3,a4,a5) \rightarrow 6\cdot x\cdot t + 2 + 2\cdot t\cdot(t-1)\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) +$

$2\cdot(x-1)\cdot t\cdot(t-1)\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot t\cdot(t-1)\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot(t-1)\cdot a3$

$uxx(x,t,a0,a1,a2,a3,a4,a5) := 6\cdot x\cdot t + 2 + 2\cdot t\cdot(t-1)\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) +$

$2\cdot(x-1)\cdot t\cdot(t-1)\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot t\cdot(t-1)\cdot(a1 + 2\cdot a3\cdot x + a4\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot(t-1)\cdot a3$

$\frac{d^2}{dt^2}u(x,t,a0,a1,a2,a3,a4,a5) \rightarrow 2\cdot x\cdot(x-1)\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) + 2\cdot x\cdot(x-1)\cdot(t-1)\cdot(a2 +$

$a4\cdot x + 2\cdot a5\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot(a2 + a4\cdot x + 2\cdot a5\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot(t-1)\cdot a5$

$utt(x,t,a0,a1,a2,a3,a4,a5) := 2\cdot x\cdot(x-1)\cdot\left(a0 + a1\cdot x + a2\cdot t + a3\cdot x^2 + a4\cdot x\cdot t + a5\cdot t^2\right) + 2\cdot x\cdot(x-1)\cdot(t-1)\cdot$

$(a2 + a4\cdot x + 2\cdot a5\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot(a2 + a4\cdot x + 2\cdot a5\cdot t) + 2\cdot x\cdot(x-1)\cdot t\cdot(t-1)\cdot a5$

$f(x,t) := 6\cdot x\cdot t + 2 + 2\cdot 3\cdot [x - (x\cdot t)]^2$

$Duxt(x,t) := 3\cdot [x - (x\cdot t)]^2$

$LS(x,t,a0,a1,a2,a3,a4,a5) := (uxx(x,t,a0,a1,a2,a3,a4,a5) + utt(x,t,a0,a1,a2,a3,a4,a5) + 2\cdot Duxt(x,t) - f(x,t))^2$

$Lu(x,t,a0,a1,a2,a3,a4,a5) := uxx(x,t,a0,a1,a2,a3,a4,a5) + utt(x,t,a0,a1,a2,a3,a4,a5) + 2\cdot Duxt(x,t)$

$$J(a0,a1,a2,a3,a4,a5) := \frac{1}{2}\cdot\int_0^1\int_0^1 \left[(Lu(x,t,a0,a1,a2,a3,a4,a5))^2 - 2\cdot[f(x,t)\cdot(Lu(x,t,a0,a1,a2,a3,a4,a5))]\right] dx\,dt$$

$a0 := 0 \quad a1 := 1 \quad a2 := 2 \quad a3 := 3 \quad a4 := 4 \quad a5 := 5$

$z := Minimize(J,a0,a1,a2,a3,a4,a5)$

$$z = \begin{pmatrix} 4.516\times 10^{-7} \\ 1.151\times 10^{-6} \\ -3.294\times 10^{-7} \\ 1.98\times 10^{-6} \\ 1.853\times 10^{-7} \\ -1.861\times 10^{-6} \end{pmatrix}$$

7- PDDE 3 Program:

$d(t,p,a0,a1,a2,a3,a4,a5) := t\cdot p\cdot (p-1)\cdot\left(a0 + a1\cdot t + a2\cdot p + a3\cdot t^2 + a4\cdot t\cdot p + a5\cdot p^2\right)$

$\frac{d^2}{dp^2}d(t,p,a0,a1,a2,a3,a4,a5) \rightarrow 2\cdot t\cdot\left(a0 + a1\cdot t + a2\cdot p + a3\cdot t^2 + a4\cdot t\cdot p + a5\cdot p^2\right) + 2\cdot t\cdot (p-1)\cdot (a2 +$

$a4\cdot t + 2\cdot a5\cdot p) + 2\cdot t\cdot p\cdot (a2 + a4\cdot t + 2\cdot a5\cdot p) + 2\cdot t\cdot p\cdot (p-1)\cdot a5$

$dpp(t,p,a0,a1,a2,a3,a4,a5) := 2\cdot t\cdot\left(a0 + a1\cdot t + a2\cdot p + a3\cdot t^2 + a4\cdot t\cdot p + a5\cdot p^2\right) + 2\cdot t\cdot (p-1)\cdot (a2 +$

$a4\cdot t + 2\cdot a5\cdot p) + 2\cdot t\cdot p\cdot (a2 + a4\cdot t + 2\cdot a5\cdot p) + 2\cdot t\cdot p\cdot (p-1)\cdot a5$

$\frac{d}{dt}d(t,p,a0,a1,a2,a3,a4,a5) \rightarrow p\cdot (p-1)\cdot\left(a0 + a1\cdot t + a2\cdot p + a3\cdot t^2 + a4\cdot t\cdot p + a5\cdot p^2\right) + t\cdot p\cdot (p-1)\cdot (a1 +$

$2\cdot a3\cdot t + a4\cdot p)$

$dt(t,p,a0,a1,a2,a3,a4,a5) := p\cdot (p-1)\cdot\left(a0 + a1\cdot t + a2\cdot p + a3\cdot t^2 + a4\cdot t\cdot p + a5\cdot p^2\right) + t\cdot p\cdot (p-1)\cdot (a1 + 2\cdot a3\cdot t + a4\cdot p)$

$Ld(t,p,a0,a1,a2,a3,a4,a5) := dt(t,p,a0,a1,a2,a3,a4,a5) - dpp(t,p,a0,a1,a2,a3,a4,a5) + \left[\frac{22}{7}\cdot p\cdot (p-1)\right]$

$LS(t,p,a0,a1,a2,a3,a4,a5) := (Ld(t,p,a0,a1,a2,a3,a4,a5))^2$

$$J(a0,a1,a2,a3,a4,a5) := \frac{1}{2}\left[\int_0^1\int_0^1 \left(Ld(t,p,a0,a1,a2,a3,a4,a5)^2\right) dt\, dp\right]$$

$$\frac{17}{20}\cdot a0^2 + \frac{2129}{2520}\cdot a5^2$$

$$\frac{d}{da0}J(a0,a1,a2,a3,a4,a5) \rightarrow \frac{22}{35} + \frac{17}{20}\cdot a2 + \frac{41}{30}\cdot a1 + \frac{17}{10}\cdot a0 + \frac{7}{6}\cdot a3 + \frac{41}{60}\cdot a4 + \frac{163}{210}\cdot a5$$

$$\frac{d}{da1}J(a0,a1,a2,a3,a4,a5) \rightarrow \frac{143}{315} + \frac{41}{60}\cdot a2 + \frac{53}{45}\cdot a1 + \frac{41}{30}\cdot a0 + \frac{21}{20}\cdot a3 + \frac{53}{90}\cdot a4 + \frac{64}{105}\cdot a5$$

$$\frac{d}{da2}J(a0,a1,a2,a3,a4,a5) \rightarrow \frac{31}{21}\cdot a2 + \frac{11}{35} + \frac{41}{60}\cdot a1 + \frac{17}{20}\cdot a0 + \frac{7}{12}\cdot a3 + \frac{8}{7}\cdot a4 + \frac{403}{280}\cdot a5$$

$$\frac{d}{da3}J(a0,a1,a2,a3,a4,a5) \rightarrow \frac{11}{30} + \frac{1013}{1050}\cdot a3 + \frac{7}{6}\cdot a0 + \frac{21}{20}\cdot a1 + \frac{107}{210}\cdot a5 + \frac{7}{12}\cdot a2 + \frac{21}{40}\cdot a4$$

$$\frac{d}{da4}J(a0,a1,a2,a3,a4,a5) \rightarrow \frac{143}{630} + \frac{298}{315}\cdot a4 + \frac{53}{90}\cdot a1 + \frac{41}{60}\cdot a0 + \frac{929}{840}\cdot a5 + \frac{8}{7}\cdot a2 + \frac{21}{40}\cdot a3$$

$$\frac{d}{da5}J(a0,a1,a2,a3,a4,a5) \rightarrow \frac{55}{294} + \frac{2129}{1260}\cdot a5 + \frac{107}{210}\cdot a3 + \frac{64}{105}\cdot a1 + \frac{929}{840}\cdot a4 + \frac{163}{210}\cdot a0 + \frac{403}{280}\cdot a2$$

$$M := \begin{pmatrix} \frac{17}{10} & \frac{41}{30} & \frac{17}{20} & \frac{7}{6} & \frac{41}{60} & \frac{163}{210} \\ \frac{41}{30} & \frac{53}{45} & \frac{41}{60} & \frac{21}{20} & \frac{53}{90} & \frac{64}{105} \\ \frac{17}{20} & \frac{41}{60} & \frac{31}{21} & \frac{7}{12} & \frac{8}{7} & \frac{403}{280} \\ \frac{7}{6} & \frac{21}{20} & \frac{7}{12} & \frac{1013}{1050} & \frac{21}{40} & \frac{107}{210} \\ \frac{41}{60} & \frac{53}{90} & \frac{8}{7} & \frac{21}{40} & \frac{298}{315} & \frac{929}{840} \\ \frac{163}{210} & \frac{64}{105} & \frac{403}{280} & \frac{107}{210} & \frac{929}{840} & \frac{2129}{1260} \end{pmatrix} \qquad v := \begin{pmatrix} \frac{-22}{35} \\ \frac{-143}{315} \\ \frac{-11}{35} \\ \frac{-11}{30} \\ \frac{-143}{630} \\ \frac{-55}{294} \end{pmatrix}$$

$$M^{-1}\cdot v = \begin{pmatrix} -1.544 \\ 2.637 \\ -0.376 \\ -1.354 \\ -1.075\times 10^{-15} \\ 0.376 \end{pmatrix}$$

المستخلص

هذه الاطروحة لها الاهداف التالية:

الهدف الاول هو لدراسة الشكل العام وبعض المفاهيم الاساسية للمعادلات التفاضلية التباطؤية الاعتيادية والجزئية ذات التباطؤ المتغير (Variable Delays).

الهدف الثاني هو لايجاد الصياغة التغايرية (The Variational Formulation) للمعادلات التفاضلية التباطؤية ولكلا النوعين، أعتيادية وجزئية وايجاد الشرط الضروري للنهايات الصغرى للدالي تحت الدراسة في موضوع حسبان التغاير.

وأخيراً فان الهدف الثالث هو لايجاد النهايات الصغرى للصياغة التغايرية عدديا باستخدام طريقة رتز المباشرة (The direct-Ritz method) وذلك لايجاد الحل التقريبي للمعادلة التفاضلية التباطؤية ذات التباطؤ المتغير.

جمهـورية العـراق
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كليــة العلوم
قسم الريـــاضيـــات

الصياغة التغايرية لبعض الأنظمة التفاضلية ذات التباطؤ المتغير

رسالة
مقدمة إلى كلية العلوم - جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل

سارة علاء الدين عبد القادر

(بكالوريوس علوم، جامعة النهرين، 2005)

إشراف

أ.م.د. فاضــل صبحي فاضل

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