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G-Spline Interpolation for Approximating the Solution of the Ordinary Differential Equations Using Linear Multistep Methods.

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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بسم الله الرحمن الرحيم

نَرفَعُ دَرَجَٰتٍ مَّن نَّشَاءُ وفَوقَ كُلِّ
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صدق الله العلي العظيم

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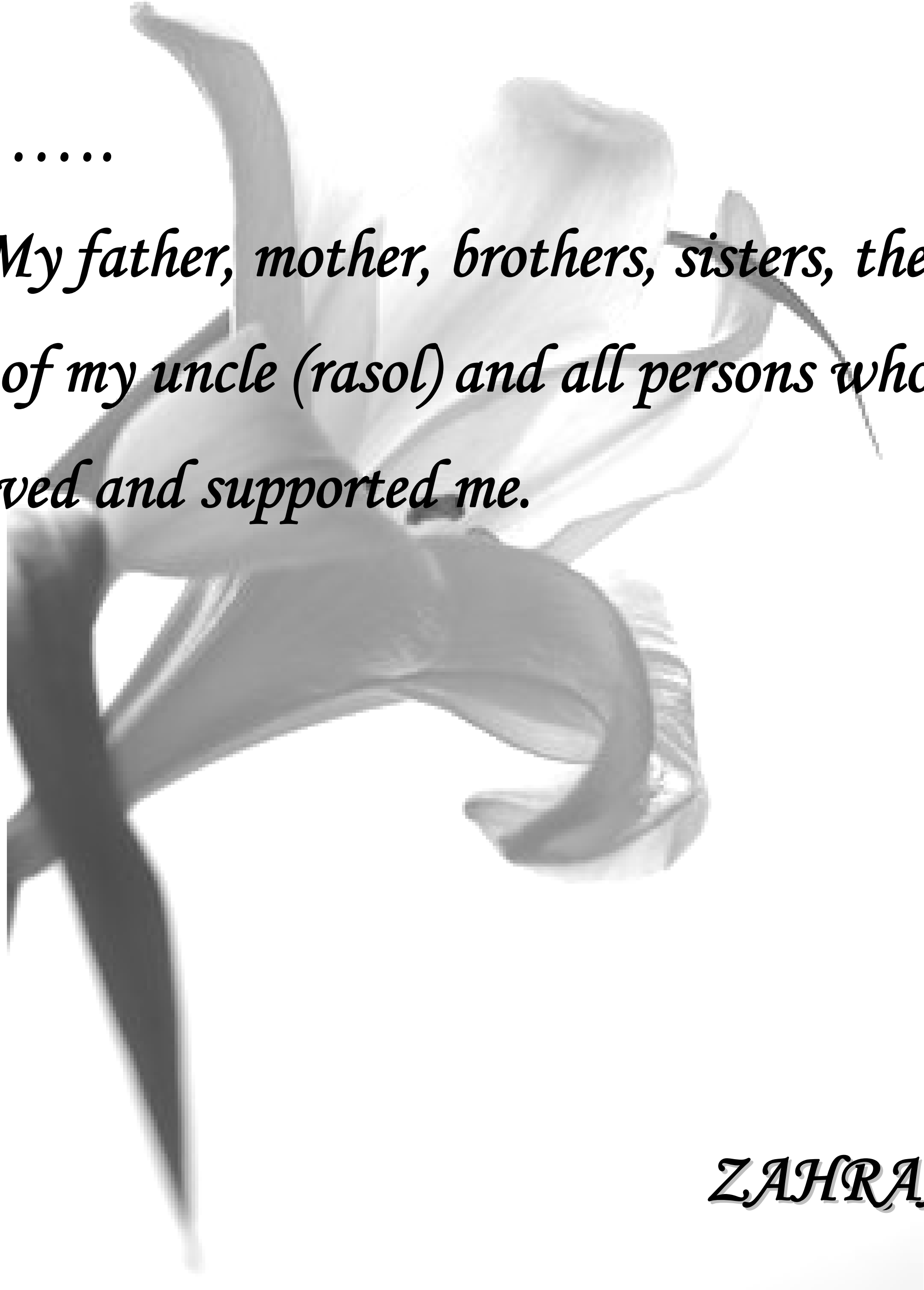
Dedication

To …….

My father, mother, brothers, sisters, the soul of my uncle (rasol) and all persons who motived and supported me.



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Zahraa Jawad

September, 2008

Supervisor Certification

I certify that this thesis was prepared under my supervision at Al-Nahrain University, College of Science, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Data: 28 / 9 / 2008

In view of the available recommendations, I forward this thesis for debate by the examining committee.

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Data: 28 /9 / 2008

Examining Committee Certification

We certify that we have read this thesis entitled "*G-Spline Interpolation for approximating the Solution of the Ordinary Differential Equations using Linear Multistep Methods*" and as an examining committee examined the student *(Zahraa Jawad Kadhom Al-Sodani)* in its contents and in what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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Data: / / 2009

Abstract

The main objectives of this thesis, is oriented toward function approximation using special type of spline functions, which is called the “G-spline “including the details of the subject.

The second objective consider the 1st order ordinary differential equations of the form:

$$y'(x) = F(x, y), \quad x \in [a, b].$$

$$y(a) = y_0.$$

Where the study concern the approximate solution of the above differential equation using linear multistep methods based on G-spline interpolation and then a generalization to this approach have been extended to solve Boundary value problems of the second order ordinary differential equations.

List of Symbols

LMM	**Linear Multistep Method.**
ODE	**Ordinary Differential Equation.**
C[a,b]	**The set of all continuous function on [a,b].**
C^n [a,b]	**Set of all continuously n-differentiable of [a,b].**
HB	**the Hermmite-Birkhoff problem.**
E	**Incidence matrix.**
Π_n	**Set of all polynomial of degree less than or equal to n.**
B-Spline	**Basis spline.**
L^2	**Set of all square function.**
S(x)	**the spline function of x.**

Contents

Introduction

Every one that has ever tried to apply simple linear interpolation method to find a value between pairs of data points will be only too aware such that attempts are extremely unlikely to provide reliable results if the data being used in any think other than broadly linear. In an attempt to deal within herent non-linearity, the next step usually involves some sort of polynomial interpolation. This general leads to more stable and robust interpolation and fitting, but is also potentially so difficult area as the end points, monotonicity, convexity and continuity of derivatives all make their influences felt in often contradictory ways.

One of the most popular ways of dealing with these issues is to use splines. In their most general form, splines can be considered as a mathematical model that associate a continuous representation of a curve or surface with a discrete set of points in a given space. Spline fitting is an extremely popular form of piecewise approximation using various forms of polynomials of degree n, or more general functions, on an interval in which they are fitted to the function at specified points, which are known as the control points or nodes or knots. The polynomial used can change, but the derivatives of the polynomials are required to match up to degree n-1 at each side of the knots, or to meet related interpolatory conditions. Boundary conditions are also imposed on the end points of the intervals. The heart of spline construction revolves around how the selection control points are effectively “blended” together using the polynomial function of choice, [Stephen, 2002].

It is more than 50 years since I.J.Schoenberg (1946) introduced “spline functions” to the mathematical literature. Since then, splines have proved to be enormously important in various branches of mathematics such as approximation theory, numerical analysis, numerical treatment of differential, integral and partial differential equations, and statistics, [Micula, 2003].

The construction of mathematical models to approximate real world problems has been one of the most important aspects of the theoretical development of each of the branches of science. It is often the case that these mathematical models involve an equation in which a function and its derivatives play important roles. Such equations are called differential equations. A derivative may be involved implicitly through the presence of differentials. The aim is to find methods for solving differential equations; that is, to find the unknown function or functions that satisfy the differential equation, [Rainville, 1989].

There is no general agreement on how the phrase “numerical analysis” should be interpreted. Some see “analysis” as the key word, and wish to embed the subject entirely in rigorous modern analysis, to others, "numerical" is the vital word, and the algorithm the only respectable product, [Lambert, 1973].

For purposes of interpolation, the use of spline function offers substational advantages such as by employing polynomials of relatively low degree, and then one can often avoid the marked undulatory behaviour that commonly arises from fitting a single polynomial exactly to a large number of empirical observations. [Osama, 2006].

This thesis consists of three chapters.

In chapter one, some basic concepts and definitions related to numerical solution ordinary differential equations and spline functions are studied. This chapter consists of four sections. In section 1.1, we discuss the general type of linear multistep method and some methods of derivation. In section 1.2, we discuss the spline functions, while in section 1.3, we discuss and list some types of splines, then in section 1.4, we discuss the approximation of linear functional and we give some theorems related to this subject.

In chapter two, an introduction to the what so called G-spline functions is given, as well as, its basic theory, including the proof of some fundamental results for completeness. This chapter consists of six sections. In section 2.1, we present the HB-problem, and then in section 2.2, we discuss what we meant by the normality of HB-problem. Section 2.3, deals with interpolation by G-spline functions and illustrating the uniqueness of the solution of HB-problem. Section

2.4, is about the spline formula. In section 2.5, the construction of G-spline was given. Finally, an illustrative example presented in section 2.6.

In chapter three, the construction of linear multistep of initial an boundary value problems by using G-spline functions is given. This chapter consists of four sections. In section 3.1, the constructions of linear multistep method with some propositions are given. In section 3.2, the compare between the explicit and implicit methods of linear multistep methods was given and the Predictor-Corrector method was introduced, in section 3.3, we give some examples for an initial value problem of ODE’s. Finally, in section 3.4, the solution of boundary value problem by using G-spline function with an example is discussed.

Chapter One
Basic Concepts

In this work, so many topics of applied mathematics are used, as well as, the connection between these concepts are presented. Therefore in this chapter, fundamental concepts are necessary to understand and recall these subjects are given for completeness and making this work of self contents as possible.

1.1 LINEAR MULTISTEP METHODS, [LAMBERT, 1973]:

Linear multistep methods are used in applied mathematics for evaluating the numerical solution of ordinary differential equations of the form:

$$y' = f(x, y)\,,\ y(a) = y_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.1)}$$

where $a \le x \le b$, $a, b \in R$

One step methods such as Euler's method refer only to one previous value of solution to determine the current value. Multistep methods refer to several previous function values in an effort to achieve greater accuracy. The general form of k-steps method is given by:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.2)}$$

Where h is the step size, α_j and β_j ($\forall j = 0,1,2,...,k$) are constants. For a linear k-step method, we require that $\alpha_k \neq 0$ or $\beta_k \neq 0$.

Furthermore, the coefficients in eq.(1.2) are not uniquely defined, and since multiplication throughout by a constant defines the same method, the coefficients may be normalized such that $\alpha_k = 1$ or $\sum_j \beta_j = 1$.

In addition, the method is said to be explicit if $\beta_k = 0$ and otherwise it is implicit.

1.1.1 The Root Condition:

The first and second characteristic polynomials of the linear multisteps method are given by:

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j \quad \ldots\ldots (1.3)$$

$$\delta(r) = \sum_{j=0}^{k} \beta_j r^j \quad \ldots\ldots (1.4)$$

In addition to the first order accuracy (also called consistency condition). A linear multistep method must satisfy the following condition:

Equation (1.2) is said to satisfy the root condition if all the roots of $\rho(r) = 0$, satisfy $|r_j| \leq 1$, for all j=0, 1, 2… k, then the method is said to be zero stable and if $|r_j| = 1$ then it must be a simple root (has no multiplicity).

1.1.2 Methods for Derivation Linear Multistep Methods:

In this subsection, we turn to the problem of determining the coefficients α_j, β_j appearing in eq.(1.2). Any specific linear multistep method may be derived in a number of different ways; we shall consider a selection of different approaches which cast some right on the nature of the approximation involved. These approaches may be summarized as follows:

***I Derivation through Taylor's expansion*:**

Consider the Taylor's expansion for $y(x_n+h)$ about x_n:

$$y(x_n+h)=y(x_n)+hy^{(1)}(x_n)+\frac{h^2}{2}y^{(2)}(x_n)+...$$

Where

$$y^{(q)}(x_n)=\left.\frac{d^q y}{dx^q}\right|_{x=x_n},\ \text{for all } q=1,2,\dots$$

If truncating this expansion after two terms and substitute for $y'(x)$ from the differential equation, one get:

$$y(x_n+h)=y(x_n)+hf(x_n,y(x_n)) \quad \dots\dots\dots\dots\dots\dots \ (1.5)$$

and the error term is given by:

$$\frac{h^2}{2!}y^{(2)}(x_n)+\frac{h^3}{3!}y^{(3)}(x_n)+... \quad \dots\dots\dots\dots \ (1.6)$$

Equation (1.5) expresses an approximate relation between exact values of the solution of eq.(1.1).We can also interpret this relation as an exact relation between approximate values of the solution of eq.(1.1) if we replace $y(x_n), y(x_n+h)$ by y_n, y_{n+1} respectively, yielding:

$$y_{n+1}=y_n+hf(x_n,y_n) \quad \dots\dots\dots\dots\dots\dots \ (1.7)$$

which is an explicit linear one-step method. It is, in fact, Euler's method, which is simplest of all linear multistep methods. The error associated with Euler's method is the expression of eq.(1.6) (multiplied by +1 or -1 according to the sense of the definition of error) and is called the local truncation error.

In addition, the local truncation error is of order h^2 (and termed as $O(h^2)$), and that it is identically zero if the solution of eq.(1.1) is a polynomial of degree not exceeding one.

II Derivation through numerical integration:

Consider the identity:

$$y(x_{n+2}) - y(x_n) \equiv \int_{x_n}^{x_{n+2}} y'(x)dx \qquad \text{...} (1.8)$$

Then using the differential equation (1.1) and replacing $y'(x)$ by $f(x, y)$ and, if our aim is to derive ,say, a linear two-step method , then the only available data for the approximate evaluation of the integral will be the values f_n, f_{n+1}, f_{n+2}. Let p(x) be the unique polynomial of degree two passing through the three points $(x_n, f_n), (x_{n+1}, f_{n+1})$ and (x_{n+2}, f_{n+2}). By using the Newton-Gregory forward interpolation formula ,

$$p(x) = p(x_n + rh) = f_n + r\Delta f_n + \frac{r(r-1)}{2!}\Delta^2 f_n \ .$$

where $r \in [0,2]$, $\Delta f_n = f_{n+1} - f_n$

Now, make the approximation:

$$\int_{x_n}^{x_{n+2}} y'(x)dx \equiv \int_{x_n}^{x_{n+2}} p(x)dx = \int_0^2 \left[f_n + r\Delta f_n + \frac{1}{2} r(r-1)\Delta^2 f_n \right] hdr$$

$$= h(2f_n + 2\Delta f_n + \frac{1}{3}\Delta^2 f_n).$$

Expanding Δf_n and $\Delta^2 f_n$ in terms of f_n, f_{n+1}, f_{n+2} and substituting in eq.(1.8) gives:

$$y_{n+2} = y_n + \frac{h}{3}(f_{n+2} + 4f_{n+1} + f_n) \qquad \text{......(1.9)}$$

which is Simpson's rule, the most accurate implicit linear two-step method.

This derivation is very close to the derivation of a Newton-Cotes quadrature formula for the numerical evaluation of $\int_a^b f(x)dx$. Indeed, eq.(1.9) is such a formula, and all Newton-Cotes a formulae can be regarded as a linear multistep methods.

III Derivation through interpolation:

This method is illustrated by deriving the implicit two-step method of eq.(1.9). Let y(x) be the solution of eq.(1.1), and approximated locally in the range $x_n \le x \le x_{n+2}$ by a polynomial $I(x)$, which should interpolate the points $(x_{n+j}, y_{n+j}), (j = 0,1,2),$ and moreover, that the derivative of $I(x)$ should coincide with the prescribed derivative f_{n+j} for j=0,1,2. This defines I(x) as an oscillatory or Hermit function which interpolates the conditions imposed on $I(x)$. Thus:

$$I(x_{n+j}) = y_{n+j},\ I'(x_{n+j}) = f_{n+j},\ j=0,1,2 \qquad \text{......(1.10)}$$

There are six conditions in all which produce six algebraic equations; let I have five free parameters, namely a, b, c, d and e that is, let I be a polynomial of degree four, namely:

$$I(x) = ax^4 + bx^3 + cx^2 + dx + e.$$

Eliminating the five undetermined coefficients a, b, c, d and e between the six equations resulting from eq.(1.10) yields the identity:

$$y_{n+2} - y_n = \frac{h}{3}(f_{n+2} + 4f_{n+1} + f_n).$$

Which is the linear multistep method of eq.(1.9). Derivation of eq.(1.9) by the method of Taylor expansions shows that that local truncation error is:

$$\pm \frac{1}{90} h^5 y^{(5)}(x_n) + \ldots$$

1.1.3 Family of Linear Multistep Methods:

Three families of linear multistep methods are commonly used, namely:

(I) Adam-Bashforth methods.

(II) Adam-Moulton method.

These families are illustrated as follows:

I Adams-Bashforth Methods:

The Adam-bashforth methods are explicit methods. Where the coefficients β_j of eq.(1.2) are chosen such that the methods has an order k (this determines the methods uniquely).

Among such Adams-Bashforth methods with k=1, 2, and 3 are:

- $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$.(Euler method);
- $y_n = y_{n-1} + h\left[\frac{3}{2} f(x_{n-1}, y_{n-1}) - \frac{1}{2} f(x_{n-2}, y_{n-2})\right]$; (Adam-Bashforth 2nd order method);
- $y_n = y_{n-1} + h\left[\frac{23}{12} f(x_{n-1}, y_{n-1}) - \frac{4}{3} f(x_{n-2}, y_{n-2}) + \frac{5}{12} f(x_{n-3}, y_{n-3})\right]$; (Adam-Bashforth third order method);

Adam-Bashforth methods or explicit linear multistep methods are used commonly in solving non-linear ordinary differential equations numerically and are used to find predictable values for implicit methods (which will be discussed later).

II Adam-Moulton Methods:

Adams-Moulton methods are implicit methods. Again the β_j of eq.(1.2) coefficients are chosen to obtain the highest possible order. However, the Adams-Moulton methods are implicit methods, since by removing the restriction that $\beta_0 = 0$, the k-step Adams-Moulton method can reach an order k+1, while a k-steps Adams-Bashforth method has only order k.

Among Adams-Moulton methods with k=0, 1, 2, 3are:

- $y_n = y_{n-1} + hf(x_n, y_n)$, (The backward Euler method);
- $y_n = y_{n-1} + \frac{1}{2}h[f(x_n, y_n) + f(x_{n-1}, y_{n-1})]$, (The Trapezoidal rule);
- $y_n = y_{n-1} + h\left[\frac{5}{12}f(x_n, y_n) + \frac{2}{3}f(x_{n-1}, y_{n-1}) - \frac{1}{12}f(x_{n-2}, y_{n-2})\right]$;

  (Two-steps Adam-Moulton method),

- $$y_n = y_{n-1} + h\left[\frac{3}{8}f(x_n, y_n) + \frac{19}{24}f(x_{n-1}, y_{n-1}) - \frac{5}{24}f(x_{n-2}, y_{n-2}) + \frac{1}{24}f(x_{n-3}, y_{n-3})\right];$$

  (Three steps Adam's-Moulton method).

1.1.4 LMM for Solving a Special Case of Second Order ODE's:

It is remarkable that the linear multistep methods can be used to solve second order ordinary differential equation i.e., let us consider an initial value problem involving an ordinary differential equation of the second order, which can be written in the form:

$$y'' = f(x, y, y'), \; y(a) = y_0, \; y'(a) = y'_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.11)$$

In the form of a first order system $u' = v, v' = f(x, u, v)$ where $u = y, v = y'$. If, however, eq.(1.11) has the special form:

$$y'' = f(x, y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.12)$$

Then it is natural to ask whether there exist a direct method which does not require us to introduce the first derivative explicitly into an equation in which it does not already appear. We might ask the some sort of equation about special higher order equation of the form $y^{(m)} = f(x, y)$.We shall consider however, only special equation of the form of eq.(1.12), since these arise in a number of important applications, especially in mechanics [Lambert, 1973]. We take the standard initial value problem:

$$y'' = f(x, y) \;,\quad y(a) = y_0 \;,\quad y'(a) = y'_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.13)$$

We shall consider only linear k-step method of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.14)$$

where $\alpha_k = 1$, and α_0, β_0 does not both vanish. Since we can not approximate y'' with less than three discrete value of y, we intuitively expect that k must be at least two, and indeed this turns out to be the case.

The direct application of method of class of eq.(1.14) to problem (1.13), rather than the application of conventional linear multistep method to an equivalent first order system, is usually recommended.

1.2 SPLINE INTERPOLATION FUNCTIONS [GREVILLE, 1967]:

The name spline function was introduced by Schoenberg in 1946. The real explosion in the theory, and in practical applications, began in the early of 1960. Spline functions are used in many applications such partial differential equations.

When approximating functions for interpolation or for fitting measured data, it is necessary to have classes of functions which have enough flexibility to adapt to the given data, and which, at the same time, can be easily evaluated using computer. Traditionally polynomials have been used for this purpose, since these have some flexibility and can be computed easily.

However, for rapidly changing value of the function to be approximated the degree of the polynomial has to be increased, and the result is often a function exhibiting wild oscillations. The situation change as dramatically when the basic interval is divided into subintervals, and the approximating or fitting function is taken to be a piecewise polynomial over each subinterval. The polynomials are joined together at the interval endpoints (knots) in such away that certain degree of smoothness (differentiability) of the resulting function is guaranteed. If the degree of the polynomials is k, and the number of subintervals is (n+1), the resulting function is called a polynomial spline function of degree k (order k+1) with n-knots.

For practical problems, spline functions have the following useful properties:

- Smooth and flexible.
- Easy to store and manipulate on a computer.
- Easy to evaluate along with their derivatives and integrals.
- Easy to generalize to higher dimensions.

Now, suppose one want to interpolate n given data points $(x_i, y(x_i))$, for all (i=1,2,…,n) by means of a function $g(x) \in C^k[a,b], 1 \le k \le n$ and moreover, is the "smoothest" such function in the sense that :

$$\varphi = \int_a^b \left[g^{(k)}(x)\right]^2 dx \qquad \text{(1.15)}$$

is made as small as possible, where:

$a \le x_1 < x_2 < ... < x_n \le b$.

For $k > n$, the problem does not have a unique solution, as there are an infinite number of polynomials of degree k-1 that fit the data points exactly, and for any of these $\varphi = 0$.

For $k \le n$ there is a unique solution which is a piecewise function given in any interval $[x_i, x_{i+1}], \forall i = 1,2,...,n$ by a polynomial in each such interval.

The function g has a further property that in each of the intervals $(-\infty, x_1)$ and (x_n, ∞) it reduces to a polynomial of degree k-1. The function g just described above belongs to a class of functions known as "spline functions", [Greville, 1967].

Now, we are in a place to set the following definition of a general spline function:

Definition (1.2.1), [Greville, 1967]:

A spline function S(x) of degree m with knote points $x_1 < x_2 < ... < x_n$ is characterized by the two properties:

(a) S(x) is given in $[x_i, x_{i+1}]$, i=0,1,2,…,n , $x_0 = -\infty$, $x_{n+1} = \infty$, by some polynomial of degree at most m.

(b) $S(x) \in C^{m-1}$ in $(-\infty, \infty)$.

Definition (1.2.2), [Greville, 1967]:

A spline function of odd degree 2m-1 is said to be (natural spline function) if it satisfies the following conditions:

(a) $S(x) \in \prod_{2m-1}$ in (x_i, x_{i+1}), i=0, 1, 2,.., n-1.

(b) $S(x) \in C^{2m-2}$. $(-\infty, \infty)$.

(c) $S(x) \in \prod_{m-1}$ in $(-\infty, x_0)$ and (x_n, ∞).

where the symbol $\prod_{2m-1}$ is used to denote the set of all polynomials of degree $\leq$ 2m-1.

A function of general importance in defining and simplifying spline functions is given in the next definition:

Remark (1.2.3), (Truncated Power function):

A typical spline of order k is the truncated power function which is denoted by:

$$(x-a)_+^{k-1} = \begin{cases} (x-a)^{k-1}, & x > a \\ 0 \quad \text{if} & x \leq a \end{cases} \qquad \text{where } a \in R.$$

An important result that will be used in G-spline interpolation is the Peano's theorem, which has the following statement:

Theorem (1.2.4), (Peano's Theorem), [Greville, 1967]:

Let $\mathcal{L}$ be a linear operator of the class defined by:

$$\mathcal{L}(f) = \sum_{r=0}^{k-1} \int_a^b f^{(r)}(x) d\mu_r(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.16)}$$

where $\mu_r(x)$ are functions of bounded variation, $f \in C^{k-1}[a,b]$ and having the additional property that $\mathcal{L}(p)=0$ for every polynomial p of degree $\leq k-1$, then for all $f \in C^{k-1}[a,b]$:

$$\mathcal{L}(f) = \int_a^b K(t) f^{(k)}(t)dt \qquad \text{(1.17)}$$

where

$$K(t) = \frac{1}{(k-1)!} \mathcal{L}_x[(x-t)_+^{k-1}] \qquad \text{(1.18)}$$

and the notation $\mathcal{L}_x$ means that the operator $\mathcal{L}$ is applied to the function within brackets considered as a function of x.

1.3 TYPES OF SPLINE FUNCTIONS:

Several types of spline functions may be used in applications and among such types of spline functions are:

1.3.1 Linear Spline function, [deBoor, 1978]:

Linear spline interpolation function is the simplest form of spline interpolation. The data points are graphically connected by straight lines and then the resultant spline is just a polygon. Algebraically, each pice S_i is a linear function constructed as:

$$S_i(x) = y_i + \frac{(y_{i+1} - y_i)}{(x_{i+1} - x_i)}(x - x_i)$$

The spline function must be continous at each data point, that is:

$$S_i(x_i) = S_{i+1}(x_i), \ i=1, 2, \ldots, n-1$$

1.3.2 Quadratic Spline function, [deBoor, 1978]:

In these splines, a quadratic polynomial approximates the data between two consecutive data points; the analysis of interpolation is as follows:

Given $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$, and to find quadratic splines $f_1, f_2, ..., f_n$ through the data, the splines are given by:

$$f_1(x) = a_1x^2 + b_1x + c_1 \quad , \; x_0 \le x \le x_1$$

$$f_2(x) = a_2x^2 + b_2x + c_2 \quad , \; x_1 \le x \le x_2$$

$$\vdots$$

$$f_n(x) = a_nx^2 + b_nx + c_n \quad , \; x_{n-1} \le x \le x_n$$

Therefore, there are 3n coefficients, a_i, b_i and $c_i, \forall i = 1,2,...,n$; and to find the 3n unknowns, one needs to setup 3n equation and then simultaneously solve the resulting system of algebraic equations.

These 3n equations are found by the following approach:

1. Each quadratic spline goes through two consecutive data points and we can construct the following equations:

$$a_1x_0^2 + b_1x_0 + c_1 = f_1(x_0)$$
$$a_1x_1^2 + b_1x_1 + c_1 = f_1(x_1)$$
$$\vdots$$
$$a_nx_{n-1}^2 + b_nx_{n-1} + c_n = f_n(x_{n-1})$$
$$a_nx_n^2 + b_nx_n + c_n = f_n(x_n)$$

These conditions gives 2n equations as there are n quadratic splines going through two consecutive data points.

2. The first derivatives of the quadratic splines are continuous at the interior points. For example, the derivative of the first spline $a_1x^2 + b_1x + c_1$ is:

$$2a_1x + b_1$$

and the derivative of the second spline $a_2x^2 + b_2x + c_2$ is:

$$2a_2x + b_2$$

where the two derivatives are equal at $x = x_1$, gives:

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

Which implies that:

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$
$$\vdots$$
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

and in general,

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

Since there are (n-1) interior points, we have the (n-1) of such equations and so far the total number of equations is $(2n) + (n-1) = (3n-1)$ equations.

Finally, we still then need one more equation; it can be assume that the first spline is linear, that is $a_1 = 0$.

This gives us 3n equations of 3n unknowns; these can be solved by a number of techniques used to solve simulations linear equations.

1.3.3 Cubic Spline function, [deBoor, 1978]:

Suppose that $(x_k, y_k)_{k=0}^{n}$ are $n+1$ points, where $a = x_0 < x_1 < ... < x_n = b$, the function S(x) is called a cubic spline interpolation function if there exist n cubic polynomials $S_k(x)$ that satisfy the properties:

(I) $S(x_k) = y_k$, for k=0, 1, 2,…, n.

(II) $S_k(x_{k+1}) = S_{k+1}(x_{k+1})$, for k=0, 1, 2,…, n-2.

(III) $S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$, for k=0, 1, 2,.., n-2.

(IV) $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$, for k=0, 1, 2,…, n-2.

The general form for each cubic function joining x_i and x_{i+1} is:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \text{ i=0, 1, 2,…, n-1}$$

and hence to find each a_i, b_i, c_i and $d_i, \forall i$, one must apply the conditions (I)-(IV) which will give (4n-2) equations with (4n-4) unknowns, and hence in order to find a unique solution, additional boundary conditions are used to find additional two equations.

1.3.4 B-Spline functions [deBoor, 1978]:

Given m+1 knot points x_i with $x_0 \le x_1 \le \ldots \le x_m$. The B-spline of degree n is a parametric curve $S:[x_0, x_m] \to R^2$ composes of basis B-spline of degree n, as:

$$S(x) = \sum_{i=0}^{m} p_i b_{i,n}(x), \qquad x \in [x_0, x_m].$$

Where p_i are called control points or de boor points.

A polygon may be constructed by connecting the knote points with lines. Starting with p_0 and finishing with p_m, this polygon is called de Boor polygon. The m-n basis B-spline of degree n can be evaluated by using the (Cox-deBoor recursion formula):

$$b_{j,0}(x) = \begin{cases} 1 & \text{if} \quad x_j \le x < x_{j+1} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

and

$$b_{j,n}(x) = \frac{x - x_j}{x_{j+n} - x_j} b_{j,n-1}(x) + \frac{(x_{j+n+1} - x)}{x_{j+n+1} - x_{j+1}} b_{j+1,n-1}(x).$$

When the knots are equidistant then the B-spline is called uniform otherwise it is called non-uniform.

1.4 APPROXIMATION PROBLEM, [GREVILLE, 1967]:

Interest in the interpolating natural spline function stems primalty from its "best approximating" properties; let $\mathcal{L}$ denote a linear operator, defined for $f \in C^{k-1}[a,b]$, of the form:

$$\mathcal{L}(f) = \sum_{r=0}^{k-1} \int_a^b f^{(r)}(x) d\mu_r(x) \quad \text{……………………………...…………... (1.19)}$$

In particular, $\mathcal{L}(f)$ might be:

(a) The value of f for a particular argument ξ.

(b) The value of the r^{th} derivative of f where (r<k) for the argument ξ.

(c) The integral of f over (a,b) or over some subinterval .

Let s denote the natural spline function interpolation between the points $(x_i, f(x_i))$, $\forall (i = 1,2,...,n)$. Schoenberg has shown in (Schoenberg 1964) that for an arbitrary $\mathcal{L}$ of the class defined by eq.(1.19), $\mathcal{L}$(s) is a "best approximation" to $\mathcal{L}$(f) in the sense of Sard [Sard, 1963].

Let $\mathcal{F}$ be an arbitrary operator of the class defined by eq.(1.19) and consider the approximation of $\mathcal{F}$ by an operator of the form:

$$\mathcal{L}(f) = \sum_{j=1}^{n} \beta_j f(x_j) \quad \text{……………………………...…………………… (1.20)}$$

where the coefficients β_j are determined subject to the requirement that $\mathcal{L}$(p)=$\mathcal{F}$(p) for every polynomial p of degree k-1 or less.

In general, this requirement does not completely determine the β_j's.

Clearly one such approximation to $\mathcal{F}$ is obtained by taking:

$$\mathcal{L}(f)=\mathcal{F}(s) \qquad \text{(1.21)}$$

Where s is the interpolating natural spline function for f of degree 2k-1 with the knots x_i.

The operator R=$\mathcal{F}$-$\mathcal{L}$ satisfies the conditions of Peano's theorem and therefore:

$$\mathcal{F}(f)=\mathcal{L}(f)+\int_a^b K(t)f^{(k)}(t)dt \qquad \text{(1.22)}$$

where

$$K(t)=\frac{1}{(k-1)!}R_x[(x-t)_+^{k-1}] \qquad \text{(1.23)}$$

where R_x is an operator applied to the function within brackets considerd as a function of x.

It appears from eq.(1.22) that $\mathcal{L}$ is a good approximation to $\mathcal{F}$ for all function f if $|K(t)|$ is small for all $t\in[a,b]$. Therefore Sard in (1963) calls $\mathcal{L}$ a best approximation to $\mathcal{F}$ if the coefficients β_j in eq.(1.20) are determined (subject to the requirement of exactness for polynomials of degree$\leq k-1$), so that the quantity:

$$J=\int_a^b [K(t)]^2 dt \qquad \text{(1.24)}$$

is small enough as possible.

Schoenberg has shown in (Schoenberg 1964) that an arbitrary operator $\mathcal{F}$ of the class defined by eq.(1.19) the best approximation in this sense is given by eq.(1.21). This useful property is expressed by the following theorem:

Theorem (1.4.1), (Schoenberg's Theorem), [Greville, 1967]:

let $\mathcal{F}$ be an arbitrary operator of the class defined by eq.(1.19) and $a \le x_1 < x_2 < ... < x_n \le b$. Let $\mathcal{L}'$ denote any approximation to $\mathcal{F}$ in the form of eq.(1.20), exact for polynomials of degree $\le k-1$ and let $\mathcal{L}$ denote the approximation given by eq.(1.21); let J' and J be the corresponding values of the quantity defined by eq.(1.24),then $J \le J'$, with equality only if $\mathcal{L}' = \mathcal{L}$.

Proof:

Let β'_j and β_j be the coefficients in eq.(1.20) corresponding to $\mathcal{L}'$ and $\mathcal{L}$ respectively and if $J = \int_a^b [K(t)]^2 dt$ and $J' = \int_a^b [K_1(t)]^2 dt$ where $K(t)$ and $K_1(t)$ be the corresponding peano kernels. Hence, one can write J' in terms of J, since:

$$J' = \int_a^b [K_1(t)]^2 dt .$$

or in another form:

$$J' = J + \int_a^b [K_1(t)]^2 dt - 2\int_a^b K_1(t)K(t)dt + \int_a^b [K(t)]^2 dt + 2\int_a^b K_1(t)K(t)dt -$$

$$2\int_a^b [K(t)]^2 dt$$

$$= J + \int_a^b [K_1(t) - K(t)]^2 dt + 2\int_a^b K(t)[K_1(t) - K(t)]dt \qquad \text{……..……… (1.25)}$$

To prove that $J \leq J'$ it will suffice to show that the last integral in eq.(1.25) vanishes, and from the definition of R, R is linear operator and apply this operator in eq.(1.20), to get:

$$R_x(f) = \sum_{j=1}^{n} \beta_j f(x_j).$$

and hence for K, K from their definition given by(1.23), this implies:

$$R_x\left[(x-t)_+^{k-1}\right] = \sum_{j=1}^{n} \beta_j (x_j - t)_+^{k-1}.$$

Hence

$$K_1(t) - K(t) = \frac{1}{(k-1)!} R_x\left[(x-t)_+^{k-1}\right] - \frac{1}{(k-1)!} R_x\left[(x-t)_+^{k-1}\right]$$

$$= \frac{1}{(k-1)!}\sum_{j=1}^{n} \beta_j (x_j - t)_+^{k-1} - \frac{1}{(k-1)!}\sum_{j=1}^{n} \beta'_j (x_j - t)_+^{k-1}$$

$$= \sum_{j=1}^{n} c_j (x_j - t)_+^{k-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.26)$$

where

$$c_j = \frac{1}{(k-1)!}(\beta_j - \beta'_j),\ (j=1, 2,\ldots, n)$$

The right member of eq.(1.26), as a function of t, is a spline function of degree k-1.

Since $\mathcal{L}'$ and $\mathcal{L}$, as approximations to $\mathcal{F}$, are both exact for polynomials of degree $\leq k-1$, it follows from eq.(1.23) that $K_1(t)$ and $K(t)$ are vanish, and therefore $K_1(t) - K(t)$ vanishes, for t outside of (x_1, x_n).

Let G (t) is any function satisfying:

$$G^{(k)}(t) = K_1(t) - K(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.27)$$

then G (t) is a natural spline function of order 2k-1 with the knots x_i.

By eq.(1.22) and (1.27), the last integral in eq.(1.25) is the remainder when $\mathcal{L}$ is used to approximate $\mathcal{F}$, operating on the function G, but by virtue of the uniqueness of spline interpolating, the interpolating natural spline function for G with the knots x_i is G itself, this remainder therefore vanishes, $J' = J$ only if the first integral in eq.(1.25) also vanishes, this implies that $K_1(t) = K(t)$, and it follow from eq.(1.24) that $\mathcal{L}' = \mathcal{L}$.

#

Chapter Two
Interpolation By G-spline Functions

In previous literatures, G-spline interpolation functions had prove its efficiency in approximating functions, since it may by applied in general for any function f depending only on the interpolatory conditions rather than the function.

This chapter presents the basic theory of G-spline interpolation functions, as well as, with some illustrating examples.

2.1 THE HERMITE-BIRKHOFF PROBLEM [SCHOENBERG, 1968]:

In this section, it is convenient to discuss the Hermit-Birkhoff problem, consider the knots points:

$$x_1 < x_2 < ... < x_k$$

to be distinct and real and let e={(i, j)}be the set of all distinct order pairs (i, j) such that i assume takes each of the values 1,2,…,k once or several times, and $j \in \{0,1,2,...,\alpha\}$ where α is the maximum value of derivative to be specified at the knots. The value $j = \alpha$ being assumed for some pair (i, j).

Now, let us consider the problem of finding the function $f(x) \in C^{\alpha}[a,b]$, which satisfy the interpolatory conditions:

$$f^{(j)}(x_i) = y_i^{(j)} \quad \text{for} \quad (i, j) \in e \quad \ldots\ldots\ldots\ldots (2.1)$$

where $y_i^{(j)}$ are prescribed reals for each $(i,j) \in e$, and define an "incidence matrix" E by:

$$E = [a_{ij}]; i = 1,2,...,k; j = 0,1,2,...,\alpha \ .$$

Where a_{ij} take the value zero or one and defined by:

$$a_{ij} = \begin{cases} 1 & \text{if} \quad (i,j) \in e \\ 0 & \text{if} \quad (i,j) \notin e \end{cases}$$

and a_{ij} being in the ith row and jth column and it is required that each row of E, also its last column, should contain at least elements equals to 1. The matrix E will likewise describe the set of equations (2.1) if we define the set e by:

$$e = \{(i,j) | a_{ij} = 1\}.$$

Of importance is the integer $n = \sum_{i,j} a_{ij}$ which is the number of interpolatory conditions required to constitute the system (2.1).

Remark (2.1.1), [Schoenberg, 1968]:

The interpolation problem (2.1) was first studied by G.D.Birkhoff in 1906 and a special case is obtained if it is assumed that E has the additional property:

$$\text{If } 0 \le j' < j \quad \text{and} \quad a_{ij} = 1 \quad \text{then} \quad a_{ij'} = 1.$$

Then it can be seen that each knot x_i of the system (2.1) prescribe the value $f(x_i)$ and also a certain number of consecutive derivatives $f^{(j)}(x_i)$ for (i=1,2,…,k) and (j=1,2,…,α_{i-1}) then eq.(2.1) may be called as an Hermite interpolation problem [Schoenberg 1965] .

Therefore it is appropriate to refer to eq.(2.1) as a Hermite-Birkhoff interpolation problem which shall be abbreviated as HB-problem.

Definition (2.1.2), [Schoenberg, 1968]:

The HB-problem (2.1) is said to be normal provided that eq.(2.1) has a unique solution $f(x) \in \prod_{n-1}$.

A necessary condition for eq.(2.1) to be normal is satisfaction of the inequality:

$$n > \alpha \qquad \ldots\ldots (2.2)$$

While if $n \le \alpha$ then $n-1 < \alpha$, which implies that:

$$f^{(\alpha)}(x) \notin \prod_{n-1}$$

Which is a contradiction, since $f^{(\alpha)}(x) \in \prod_{n-1}$ and satisfy eq.(2.1) that involves $f^{(\alpha)}(x)$. For this result it can be concluded that every Hermite system is normal.

Assuming eq.(2.1) to be normal and let for each $(i,j) \in e$ there exist $L_{ij}(x)$ which is the unique element of $\prod_{n-1}$ which is called (the fundamental functions) such that:

$$L_{ij}^{(s)}(x_r) = \begin{cases} 1 & \text{if} \quad (r,s) = (i,j) \\ 0 & \text{if} \quad (r,s) \ne (i,j) \end{cases} \qquad \ldots\ldots (2.3)$$

By this term of the fundamental function it can be expresse the unique solution of eq.(2.1) in $\prod_{n-1}$ by:

$$f(x) = \sum_{(i,j)\in e} y_i^{(j)} L_{ij}(x) \qquad \ldots\ldots (2.4)$$

Moreover, if $f(x) \in C^{\alpha}$ then by the interpolatory condition:

$$f(x) = \sum_{(i,j)\in e} f^{(j)}(x_i) L_{ij}(x) + Rf \qquad \ldots\ldots (2.5)$$

Where Rf is the remainder that is equal to zero if $f(x) \in \prod_{n-1}$ which means that eq.(2.5) is exact. Equation (2.5) is said to be HB-interpolation formula.

2.2 ON NORMAL SYSTEM OF HB-PROBLEM AND RELATED CONSEPTS:

The condition that the HB-problem (2.1) to be normal may be equivalently expressed by the following:

If

$P(x) \in \prod_{n-1}$

$P^{(j)}(x_i) = 0 \quad \text{if} \quad (i, j) \in e$

Then

P(x) =0

Definition (2.2.1), [Schoenberg, 1968]:

Let m be a natural number, then the HB-problem is said to be m-poised provided that:

$$p(x) \in \prod_{m-1} \qquad \ldots\ldots (2.6)$$

$$p^{(j)}(x_i) = 0 \quad \text{if} \quad (i, j) \in e \qquad \ldots\ldots (2.7)$$

Then

$$P(x) = 0 \qquad \ldots\ldots (2.8)$$

The next lemma gives some important properties of HB-problem which is given in [Schoenberg 1968] without proof and for details of the proof see [Osama, 2006].

Lemma (2.2.2):

(i) The HB-problem (2.1) is normal if and only if it is n-poised.

(ii) If (2.1) is m-poised then the inequality $m \le n$ must be hold.

(iii) If (2.1) is m-poised and $1 \le m' < m$ then it is also m'-poised.

Proof (i):

If the HB-problem is normal, then to prove that it is n-poised. By the definition of normal (2.1.2), HB-problem has a unique solution such that $f(x) \in \prod_{n-1}$.

Hence $f(x)$ is a solution of the HB-problem then it is satisfying the interpolatary conditions:

$$f^{(j)}(x_i) = y_i^{(j)} \quad \text{for} \quad (i,j) \in e.$$

If $f^{(j)}(x_i) = 0$, then from the uniqueness of the solution of the HB-problem (2.1), it implies that f(x) =0, which means that the HB-problem is n-poised.

Conversely, if the HB-problem is n-poised, then $f(x) \in \prod_{n-1}$ and $f^{(j)}(x_i) = 0$, and hence f(x) =0.

To prove that the HB-problem (2.1) is normal, we must prove that the solution is unique.

Suppose that there exist two solutions for the HB-problem which is $f(x) \in \prod_{n-1}$ and $g(x) \in \prod_{n-1}$, that satisfy the interpolatary conditions.

Define a function H(x) =f(x)-g(x) then:

$$H(x) \in \prod_{n-1} \quad \text{and} \quad H^{(j)}(x_i) = 0$$

Since the HB-problem is n-poised, then H(x) =0. Therefore, f(x) =g(x).

Therefore, the HB-problem (2.1) is normal.

#

Proof (ii):

Suppose that the HB-problem (2.1) is m-poised and for contrary let m>n.

Since the problem is m-poised, hence:

If $f(x) \in \prod_{m-1}$ and $f^{(j)}(x_i) = 0, (i,j) \in e$ then f(x) =0.

Since $f(x) \in \prod_{m-1}$ then f(x) depends on m parameters but we have only n equations, since m>n.

Then the number of unknown parameters is grater than the number of equations. Therefore, we have an infinite number of solutions, which is a contradiction.

Therefore, $m \le n$ must hold.

#

Proof (iii):

Let $f(x) \in \prod_{m'-1}$, and since $1 < m' < m$, then from the fact that $\prod_{m'-1} \subset \prod_{m-1}$. Hence $f(x) \in \prod_{m-1}$.

Since the problem (2.1) is m- poised, then:

$f(x) \in \prod_{m-1}$ and $f^{(j)}(x_i) = 0$ and hence f(x) =0; i.e., $f(x) \in \prod_{m'-1}$ and $f^{(j)}(x_i) = 0$ then f(x) =0.

Therefore, the HB-problem is m'-poised.

#

Remark (2.2.3):

A non-normal system (2.1) may be m-poised for some value m<n.

As an example, consider the HB-problem:

$$f(x_1) = y_1, f'(x_2) = y'_2, f(x_3) = y_3 \quad , x_2 = \frac{1}{2}(x_1 + x_3) \quad \text{……..… (2.9)}$$

with

e= {(1, 0), (2, 1), (3, 0)}.

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

eq.(2.9) is not 3-poised, consider:

$$p(x) = (x - x_1)(x - x_3) \ne 0$$

1. $p(x) \in \prod_2 \subseteq \prod_3$
2. $p^{(j)}(x_i) = 0, \forall (i, j) \in e$, then

$$p^{(0)}(x_1) = (x_1 - x_1)(x_1 - x_2) = 0$$

Also:

$$p'(x) = (x - x_3) + (x - x_1)$$
$$= 2x - x_1 - x_3$$

Therefore:

$$p'(x_2) = 2\frac{(x_1 + x_3)}{2} - x_1 - x_3 = 0$$

and

$$p^{(0)}(x_3) = 0$$

Hence, from the definition of m-poised problems (2.2.1) the HB-problem (2.9) is 2-poised.

Consequently, the system is not n-poised, i.e., not normal.

Remark (2.2.4):

The condition (2.1) may be m-poised can be expressed as follows:

Let

$$p(x) = \sum_{v=0}^{m-1} a_v \frac{x^v}{v!}.$$

Then eq.(2.7) becomes:

$$p^{(j)}(x_i) = \sum_{v-j=0}^{m-1} a_v \frac{x_i^{v-j}}{(v-j)!} = 0 \quad \text{for} \quad (i,j) \in e$$

such that

$$\frac{x^{v-j}}{(v-j)!} = \begin{cases} 1 & \text{if} \quad v-j=0 \\ 0 & \text{if} \quad v-j<0 \end{cases}$$

Therefore, (2.1) is m-poised if and only if the matrix with entries:

$$\frac{x_i^{v-j}}{(v-j)!} \text{ has rank m} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.10)$$

where v-j=0, 1, 2… m-1 is the column of the matrix and $(i, j) \in e$ corresponds a row of the matrix.

2.3 THE G-SPLINE INTERPOLATION:

In this section, we shall assume that the natural number m satisfies the condition that; the HB-problem (2.1) is m-poised in definition (2.2.1).

Now, by the definition of the matrix E, where:

$E = [a_{ij}], (i, j) \in e$

Let us add $(m - \alpha - 1)$ columns of zero elements to the matrix E where α is the highest derivative that appears in the interpolatory problem, and let a new incidence matrix be:

Let $E^* = [a_{ij}^*]$, $(i = 1,2,..., k)$ and $(j = 0,1,.., m - 1)$

where

$$a_{ij}^* = \begin{cases} a_{ij} & \text{if} \quad j \le \alpha \\ 0 & \text{if} \quad j = \alpha + 1, \alpha + 2, ..., m - 1 \end{cases}$$

If m=α+1, then $E^* = E$.

<u>Definition (2.3.1), [Schoenberg, 1968]:</u>

A function S(x) is called a natural G-spline for the knots $x_1, x_2, ..., x_k$ and the matrix E^* with order m provided that S(x) satisfy the following conditions:

(I) $S(x) \in \prod_{2m-1}$ in $(x_i, x_{i+1}), (i = 1,2,..., k - 1)$.

(II) $S(x) \in \prod_{m-1}$ in $(-\infty, x_1)$ and in $(x_k, +\infty)$.

(III) $S(x) \in C^{m-1}(-\infty, \infty)$.

(IV) If $a^*_{ij} = 0$ then $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$, i.e,

$$S^{(2m-j-1)}(x_i - 0) = S^{(2m-j-1)}(x_i + 0).$$

Now, the set of all natural G-spline interpolation functions is denoted by:

$$\zeta_m = \zeta(E^*; x_1, x_2, ..., x_k) \qquad \text{.......................................} (2.11)$$

where ζ_m is a non empty set shown by the inclusion relation:

$$\prod_{m-1} \subset \zeta_m.$$

and if $S(x) \in \prod_{m-1}$, then S(x) satisfies all conditions from (I) to (IV).

2.3.1 Special Case examples [Schoenberg, 1968]:

1. The Lagrange problem:

Assume that $\alpha = 0$ and n=k and e={(i,0), i=0,1,2,…,k}, the corresponding problem (2.1) being:

$$f(x_1) = y_1, f(x_2) = y_2, ..., f(x_n) = y_n.$$

Let us inspect the corresponding class ζ_m of G-spline by:

$$\alpha < m \le n \text{ such that } 0 < m \le n.$$

By definition of the matrix E and the matrix E^* it can be seen that $a^*_{ij} = 0$ for each (j=1, 2,…, m-1) and (i=1,2,…,n=k) and by condition (IV) we can conclude that $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$ for (j=1, 2, …, m-1) or equivalently that $S^{(v)}(x)$ is continuous at $x = x_i$ for v=m, m+1, …,2m-2.

This conclusion and condition (III) shows that:

$$S(x) \in C^{2m-2}(-\infty, \infty).$$

2. The Hermite problem:

Assume that (2.1) is a HB- problem:

$$f(x_i) = y_i, f'(x_i) = y'(x_i), ..., f^{(\alpha_i - 1)}(x_i) = y_i^{(\alpha_i - 1)}, \text{ (i=1, 2,…, k)}$$

Here $\alpha = \max_i \{\alpha_i - 1\}$ and choose any m such that $\max_i \alpha_i \le m \le n$, since

$$a^*_{ij} = 0 \quad \text{if} \quad j = \alpha_i, \alpha_{i+1}, ..., m-1.$$

From condition (IV)it is clear that $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$ if (j=$\alpha_i, ..., m-1$) or equivalently, that. $S^{(v)}(x)$ is continuous at $x = x_i$ if $v = m, ..., 2m - \alpha_i - 1$.

Combining this with condition (III), we conclude that:

$$S(x) \in C^{2m-\alpha_i-1} \text{ near } x = x_i \text{ (i=1, 2,.., k)} \qquad \text{……….…………...(2.12)}$$

Recognize in condition (I),(II) of definition (2.3.1) and (2.12) the characteristic properties of the natural spline functions of degree 2m-1(m is identical with the natural spline function of degree 2m-1) having x_i (i=1,2,..,k) as a multiple knot of multiplicity α_i $(\alpha_i \le m)$.

In the next theorem, we establish the uniqueness of the natural spline function.

Theorem (2.3.2), [Schoenberg, 1968]:

Let the HB-problem (2.1) be m-poised, m satisfies the inequality $\alpha < m \le n$. Then (2.1) with the prescribed $y_i^{(j)}$ has a unique solution:

$$S(x) \in \zeta_m(E^*; x_1, x_2, ..., x_k) \qquad \text{……………………………...….. (2.13)}$$

Proof:

Problem (2.1) is equivalently a linear problem. Setting up all polynomial components of S(x) with indeterminate coefficient. From the condition (I) of definition (2.3.1) one can be obtain 2m(k-1) of unknowns, and from the condition (II) of definition (2.3.1) when $x \in (-\infty, x_1)$ we have m unknowns also when $x \in (x_k, \infty)$ we have m unknowns then the total of m+2m(k-1) +m=2mk unknowns are obtained.

Now, to count the number of equations:

From condition (III) of definition (2.3.1) one can obtain mk equations. The number of equations resulting from condition (IV) is equal to the number of vanishing elements of E^*, since their total number is mk, while $n = \sum a_{ij}$ is the number of non vanishing elements then (mk-n) equations can be obtained.

Finally, (2.1) furnishes n-equations and obtain a total of equations is:

mk+ (mk-n) +n=2mk.

The numbers of equations are equal to the number of unknowns; it suffices to establish the following:

If

$$S(x) \in \zeta_m \quad \text{and} \quad S^{(j)}(x_i) = 0 \text{ for } (i,j) \in e \qquad \text{.....................} \quad (2.14)$$

Then

$$S(x) = 0 \text{ for all real } x \qquad \text{.....................} \quad (2.15)$$

Select a finite interval I= $[x_0, x_{k+1}]$ such that $x_0 < x_1$, $x_k < x_{k+1}$

To show that eq.(2.14) implies that:

$$J = \int_I (S^{(m)}(x))^2 \, dx = 0 \qquad \text{.....................} \quad (2.16)$$

First, suppose that:

$$J = \int_{x_0}^{x_1} (S^{(m)}(x))^2 dx + \int_{x_1}^{x_2} (S^{(m)}(x))^2 dx + \dots + \int_{x_k}^{x_{k+1}} (S^{(m)}(x))^2 dx$$

and integrating by parts repeatedly for each of these integrals yields:

$$\int_{x_i}^{x_{i+1}} S^{(m)}(x) S^{(m)}(x) dx = S^{(m)}(x) S^{(m-1)}(x) \Big|_{x_i}^{x_{i+1}} - S^{(m+1)}(x) S^{(m-2)}(x) \Big|_{x_i}^{x_{i+1}} + \dots \pm$$

$$S^{(2m-1)}(x) S^{(0)}(x) \Big|_{x_i}^{x_{i+1}} \mp \int_{x_i}^{x_{i+1}} S^{(2m)}(x) S^{(0)}(x) dx$$

Now, observe the following:

1. Each of the very last integrals on the right hand side vanishes, because $S(x) \in \prod_{2m-1}$ in each interval by conditions (I) and (II).

2. From the "finite parts" obtain that at each x_i (i=1,2,…,k) the following sum of terms:

   $$\sum_{j=0}^{m-1} \pm \Delta_i^{(j)} .$$

   where

   $$\Delta_i^{(j)} = \underset{\text{at } x=x_i}{\text{jump of}}\, S^{(2m-j-1)}(x)S^{(j)}(x), \text{ (j=0, 1, 2,…, m-1).}$$

   Since $S(x) \in C^{m-1}$ by condition (III), hence:

   $$\Delta_i^{(j)} = S^{(2m-j-1)}(x_i+0)S^{(j)}(x_i+0) - S^{(2m-j-1)}(x_i-0)S^{(j)}(x_i-0)$$

   $$= S^{(j)}(x_i)[S^{(2m-j-1)}(x_i+0) - S^{(2m-j-1)}(x_i-0)]$$

   This expression is vanishes for the following reasons:

   (i) If $a_{ij}^* = 1$ then $S^{(j)}(x_i) = 0$ by (2.14) because $(i, j) \in e$.

   (ii) If $a_{ij}^* = 0$ then by condition (IV) of definition (2.3.1) $S^{(2m-j-1)}(x)$ is continous at $x = x_i$ then the limit in right is equal to the limit in left, this implies to $\Delta_i^{(j)} = 0$.

3. The finite parts at x_0 and x_{k+1} also vanish in view of condition (II). Thus eq.(2.16) is established.

   Hence, eq.(2.16) shows that:

   $S^{(m)}(x) = 0$ for all real x.

   Therefore,

   $$S(x) \in \prod_{m-1} \quad \text{……………………………………………..….. (2.17)}$$

However, by assumption, problem (2.1) is m-poised. Now eq.(2.16) and the set of equations (2.14) directly imply the desired conclusion (2.15). The existence and uniqueness of G-spline solution $S(x) \in \zeta_m$ of m-poised HB-problem (2.1) is established.

#

Corollary (2.3.3), [Schoenberg, 1968]:

If the HB-problem (2.1) is normal or n-poised we may choose m=n, when

$$\zeta_m(E^*; x_1, x_2, \ldots, x_k) = \prod_{n-1}.$$

where n, m are integer.

2.3.2 The G-Spline Interpolation Formula [Schoenberg, 1968]:

It is convenient to summarize our results as follows:

Under the assumptions of theorem (2.3.2) define now within ζ_m the G-spline $L_{ij}(x)$ satisfying the relation:

$$L_{ij}^{(s)}(x_r) = \begin{cases} 0 & \text{if } (r,s) \neq (i,j) \\ 1 & \text{if } (r,s) = (i,j) \end{cases}$$

where $(i,j) \in e$.

If $f(x) \in C^{\alpha}$, then f(x) may be written as:

$$f(x) = \sum_{(i,j) \in e} f^{(j)}(x_i) L_{ij}(x) + Rf \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.18)}$$

Where the right hand sum represents the G-spline interpolating of f(x) at the data of HB-problem (2.1) and Rf is the remainder. Equation (2.1) refers to the G-spline interpolation formula and this formula is exact for all elements of ζ_m and in particular for the elements of $\prod_{m-1}$.

The following theorem illustrate the optimal property of G-spline interpolation functions which may be called the minimum norm property:

Theorem (2.3.4), [Schoenberg, 1968]:

Let $I=[x_0,x_{k+1}]$ such that $x_0<x_1<...<x_{k+1}$ and let $f(x)\in C^m(I)$ with $f^{(m-1)}(x)$ is absolutely continuous and $f^{(m)}(x)\in L^2(I)$. If the HB-problem (2.1) is m-poised, and $\alpha<m\le n$, and let S(x) is the unique G-spline function satisfying the equations:

$$S^{(j)}(x_i)=f^{(j)}(x_i) \quad ,(i,j)\in e$$

Then

$$\int_I (f^{(m)}(x))^2 dx > \int_I (S^{(m)}(x))^2 dx$$

Proof:

Since $f(x)\in C^m(I)$ with $f^{(m-1)}(x)$ is absolutely continuous and $f^{(m)}\in L^2(I)$ then:

$$\int_I (f^{(m)}(x)-S^{(m)}(x))^2 dx = \int_I (f^{(m)}(x))^2 dx - 2\int_I f^{(m)}(x)S^{(m)}(x)dx + \int_I (S^{(m)}(x))^2 dx$$

$$= \int_I (f^{(m)}(x))^2 dx - 2\int_I (f^{(m)}(x)-S^{(m)}(x))S^{(m)}(x)dx - \int_I (S^{(m)}(x))^2 dx \quad \text{……………. …..…. (2.19)}$$

To prove that:

$$J=\int_I (f^{(m)}(x)-S^{(m)}(x))S^{(m)}(x)dx = 0 \quad \text{…………………..…. (2.20)}$$

First, let:

$$J=\int_{x_0}^{x_1}(f^{(m)}(x)-S^{(m)}(x))S^{(m)}(x)dx+\int_{x_1}^{x_2}(f^{(m)}(x)-S^{(m)}(x))S^{(m)}(x)dx$$

$$+\ldots+\int_{x_k}^{x_{k+1}}(f^{(m)}(x)-S^{(m)}(x))S^{(m)}(x)dx\,.$$

and integrate by parts repeatedly each of these integrals according to the following scheme for each (i= 0, 1,…, k):

$$\int_{x_i}^{x_{i+1}} S^{(m)}(x)(f^{(m)}(x)-S^{(m)}(x))dx = S^{(m)}(x)(f^{(m-1)}(x)-S^{(m-1)}(x))\Big|_{x_i}^{x_{i+1}} -$$

$$S^{(m+1)}(x)(f^{(m-2)}(x)-S^{(m-2)}(x))\Big|_{x_i}^{x_{i+1}} +\ldots\pm S^{(2m-1)}(x)(f(x)-S(x))\Big|_{x_i}^{x_{i+1}} \pm$$

$$\int_{x_i}^{x_{i+1}} S^{(2m)}(x)(f(x)-S(x))dx\,.$$

The last integrals on the right hand side of the last formula are vanishes since $S(x)\in\prod_{2m-1}$ in each interval by conditions (I) and (II) of definition (2.3.1).

From the "finite parts" obtain at each x_i, (i=1, 2,…, k), a sum of terms:

$$\sum_{j=0}^{m-1}\pm\Delta_i^{(j)}\,.$$

By the same prove of (2) in theorem (2.3.2) implies that:

$$\Delta_i^{(j)}=0$$

Hence, the equation (2.20) is established and therefore eq.(2.19) becomes:

$$\int_I (f^{(m)}(x)-S^{(m)}(x))^2dx=\int_I (f^{(m)}(x))^2dx-\int_I (S^{(m)}(x))^2dx\,.$$

Then

$$\int_I (f^{(m)}(x))^2 dx = \int_I (S^{(m)}(x))^2 dx + \int_I (f^{(m)}(x) - S^{(m)}(x))^2 dx .$$

and since $\int_I (f^{(m)}(x) - S^{(m)}(x))^2 dx > 0$, therefore:

$$\int_I (f^{(m)}(x))^2 dx > \int_I (S^{(m)}(x))^2 dx .$$

#

2.4 THE BEST APPROXIMATION PROBLEM OF LINEAR FUNCTIONALS WITH THE SENSE OF G-SPLINE INTERPOLATION FORMULA, [SCHOENBERG, 1968]:

Let I=[a,b] be a finite interval containing the real knot points $x_1 < x_2 < ... < x_k$ and let us consider the linear functional, $\mathcal{L}f : C^{\alpha}[a,b] \to R$ of the form:

$$\mathcal{L}f = \sum_{j=0}^{\alpha} \int_a^b a_j(x) f^{(j)}(x) dx + \sum_{j=0}^{\alpha} \sum_{i=1}^{n_j} b_{ji} f^{(j)}(x_{ji}) \quad \text{.................... (2.21)}$$

Where $a_j(x)$ $\forall j$ are piecewise continuous functions in I, $x_{ji} \in I$ and b_{ji} are constants.

We can approximate the functional (2.21) using the formula :

$$\mathcal{L}f = \sum_{(i,j)\in e} \beta_{ij} f^{(j)}(x_i) + Rf \quad \text{.. (2.22)}$$

Therefore, in order to find the approximation $\mathcal{L}f$ which is best in some sense, we propose to determine the reals β_{ij}.

Now, two procedures may be used to find β_{ij} which are associated with the name of Newton and Cotes procedure and Sard procedure, which are:

1. Newton-cotes:

There are $n = \sum a_{ij}$ (i.e., there is n parameters β_{ij}), therefore require that the formula (2.22) should be exact (i.e. Rf=0) if $f \in \prod_{n-1}$. If β_{ij} are the constants determined by this condition, then eq.(2.22) represents the best approximation of $\mathcal{L}$f.

In order to derive this approximation, substitute:

$$f(x) = \frac{x^v}{v!} \text{ , (v=0, 1,…, n-1)} \qquad \text{.....(2.23)}$$

into eq.(2.22) (with Rf=0) and obtain for the determination of the β_{ij}'s, the following system of n-equations in n unknowns:

$$\mathcal{L}\frac{x^v}{v!} = \sum_{(i,j)\in e} \beta_{ij} \frac{x_i^{v-j}}{(v-j)!}, \text{ (v=0, 1,…, n-1)} \qquad \text{.....(2.24)}$$

The linear system related to equation (2.24) is non-singular if and only if the HB-problem (2.1) is normal or n-poised.

Recall that the HB-interpolation formula is given by:

$$f(x) = \sum_{(i,j)\in e} f^{(j)}(x_i)L_{ij}(x) \quad \text{if } f \in \prod_{n-1}.$$

and take $\mathcal{L}$ to the both sides of the last equation, yields:

$$\mathcal{L}f = \sum_{(i,j)\in e} f^{(j)}(x_i)\, \mathcal{L}L_{ij}(x) \quad \text{if } f \in \prod_{n-1}.$$

Then comparing it with eq.(2.22), one have:

$$\beta_{ij} = \mathcal{L}L_{ij}(x), \forall (i,j) \in e \qquad \text{..... (2.25)}$$

2. *Sard's procedure:*

Whether the HB-problem (2.1) is n-poised or not, assume that m satisfying:

$$\alpha < m < n \qquad \text{..... (2.26)}$$

such that:

The HB-problem (2.1) is m-poised ………………………...... (2.27)

Now, require the approximation (2.22) to be exact if $f(x) \in \prod_{m-1}$, this requirement is equivalent to the system of equations:

$$\mathcal{L}\left(\frac{x^{(v)}}{v!}\right) = \sum_{(i,j)\in e} \beta_{ij} \frac{x_i^{v-j}}{(v-j)!} \text{ , (v=0, 1, 2,…, m-1)} \qquad \text{……………...(2.28)}$$

and eq.(2.10) shows that the matrix related to this system has rank m.

Assuming eq.(2.28) to hold, we still have n-m free parameters among the β_{ij}.

However, Rf defined by eq.(2.22) is also a linear functional of the form (2.21) with the property that:

Rf=0 if $f \in \prod_{m-1}$.

assuming that $f(x) \in C^m[a,b]$ then by Peano's theorem, write

$$Rf = \int_I K(x) f^{(m)}(x) dx \, .$$

where K(x) is a kernel which depends on n-m free parameters among the β_{ij} but not on f(x). These n-m parameters are now determined by the requirement that:

$$\min \int_I (K(x))^2 dx \qquad \text{……………………………….... (2.29)}$$

The β_{ij}, thereby completely determined, are substituted into eq.(2.22) thereby producing the best approximation to $\mathcal{L}$f of order m in the sense of Sard (1963).

The main result of part (2) is in the following theorem:

Theorem (2.4.1), [Schoenberg, 1968]:

If the assumptions (2.26) and (2.27) hold, then Sard's best approximation (2.22) to $\mathcal{L}$f of order m is obtained by operating with $\mathcal{L}$ on both sides of the G-spline interpolation formula (2.18) of order m. In other words, the coefficients β_{ij} obtained as solution of the minimum problem (2.29) with m side conditions in eq.(2.28) are:

$$\beta_{ij} = \mathcal{L} L_{ij}(x) \qquad \text{.....} (2.30)$$

where the $L_{ij}(x)$ are the fundamental function of eq.(2.18).

Proof:

We wish to compare the functional:

$$Rf = \mathcal{L} f - \sum_{e} \beta_{ij} f^{(j)}(x_i) \qquad \text{.....} (2.31)$$

where β_{ij} are defined by eq. (2.30) with the functional:

$$\tilde{R}f = \mathcal{L} f - \sum_{e} \tilde{\beta}_{ij} f^{(j)}(x_i) \qquad \text{.....} (2.32)$$

whose coefficients $\tilde{\beta}_{ij}$ are required only to satisfy the m equations (2.28). Evidently:

$$Rf = 0 \quad \text{and} \quad \tilde{R}f = 0 \quad \text{if } f \in \prod_{m-1} \qquad \text{.....} (2.33)$$

By Peano's theorem, we have the representations:

$$Rf = \int_I K(x) f^{(m)}(x)dx \quad , \quad \tilde{R}f = \int_I \tilde{K}(x) f^{(m)}(x)dx \qquad \text{.....} (2.34)$$

where the formulas:

$$K(x) = R_t \frac{(t-x)_+^{m-1}}{(m-1)!} \quad , \quad \tilde{K}(x) = \tilde{R}_t \frac{(t-x)_+^{m-1}}{(m-1)!} \qquad \text{.....} (2.35)$$

Define these kernels for all real x.

Let us consider their deference:

$$\vartheta(x) = \tilde{K}(x) - K(x) .$$

Using eqs.(2.35), (2.31) and (2.32) then $\vartheta(x)$ may be written as :

$$\vartheta(x) = (R_t - \tilde{R}_t)\frac{(t-x)_+^{m-1}}{(m-1)!} = \sum_e c_{ij}\frac{\partial^j}{\partial t^j}\frac{(t-x)_+^{m-1}}{(m-1)!}\Bigg|_{t=x_i} \quad \text{……... (2.36)}$$

where

$$c_{ij} = \beta_{ij} - \tilde{\beta}_{ij} \quad \text{…………………………………………….... (2.37)}$$

Now, by eq.(2.35), and in view of (2.33) K(x) and $\tilde{K}(x)$ are vanish outside of I=[a,b], for x>b because there $(t-x)_+^{m-1} = 0,\ (t \in I)$. By the definition of the truncated power functions, and for x<a because then $(t-x)_+^{m-1}$is in $a \le t \le b$ a polynomial in t of degree m-1, K(x) and $\tilde{K}(x)$ need not to be vanish in $[a, x_1]$ or in $[x_k, b]$.

However, their difference $\vartheta(x)$ vanishe also in these two intervals. Evidently, if $x \in [x_k, b]$ from eq.(2.36) for $\vartheta(x)$and becomes equally clear for $x \in [a, x_1]$.

Instead of eq.(2.35) if the equivalent expressions are used:

$$K(x) = (-1)^m R_t \frac{(x-t)_+^{m-1}}{(m-1)!} \quad , \tilde{K}(x) = (-1)^m \tilde{R}_t \frac{(x-t)_+^{m-1}}{(m-1)!}$$

Thus

$$\vartheta(x) = 0 \text{ every where out side } [x_1, x_k] \quad \text{…………………….(2.38)}$$

Now, consider a function S(x) satisfying:

$$S^{(m)}(x) = \tilde{K}(x) - K(x) \quad \text{……………………………………..… (2.39)}$$

It is clear from eq.(2.36) and (2.38) that S(x) satisfies the conditions (I) and (II) in definition (2.3.1) of the class $\zeta_m$of G-spline.

Evidently, condition (III), that is $S(x) \in C^{m-1}(-\infty, \infty)$, is also satisfied.

Finally, let us verify condition (IV) and thus conclude that:

$S(x) \in \zeta_m$.

For this purpose, look at those terms of the sum of eq.(2.36) which correspond to the same node x_i(i fixed), they are:

$$\sum_{(i,j)\in e} c_{ij'} \frac{(x_i - x)_+^{m-j'-1}}{(m-j'-1)!} \qquad \text{.. (2.40)}$$

It is only discontinous derivatives are D^{m-j-1}, where j is such that $(i, j) \in e$.

However,

$$S^{(2m-j-1)}(x) = \vartheta^{(m-j-1)}(x).$$

and therefore it is continuous at x_i if $(i, j) \notin e$ or $a_{ij}^* = 0$.

This is precisely condition (IV) and eq.(2.39) is established.

On the other hand, we know that the G-spline interpolation formula (2.18) is exact for all $f(x) \in \zeta_m$, but also the approximation formula:

$$\mathcal{L}f = \sum_e \beta_{ij} f^{(j)}(x_i) + Rf.$$

obtained from eq.(2.18) by operating on both sides with $\mathcal{L}$, must also be exact if $f(x) \in \zeta_m$.

Therefore, if substituting our G-spline f(x) =S(x) into the identity:

$$f(x) = \sum_{(i,j)\in e} \beta_{ij} f^{(j)}(x_i) + \int_I K(x) f^{(m)}(x)dx.$$

The remainder term must vanish because $f^{(m)}(x) = S^{(m)}(x)$ and by eq.(2.39) therefore we conclude that:

$$\int_I K(x)(\tilde{K}(x) - K(x))dx = 0.$$

and a direct consequence of this equation is the following equation:

$$\int_I (\tilde{K}(x))^2 dx = \int_I (K(x))^2 dx + \int_I (\tilde{K}(x) - K(x))^2 dx \qquad \text{................ (2.41)}$$

and from this it can be obtain the desired inequality:

$$\int_I (\tilde{K}(x))^2 dx > \int_I (K(x))^2 dx \qquad \text{.. (2.42)}$$

Unless, by eq.(2.41), $\vartheta(x)$ vanishes identically, when $c_{ij} = 0$ and by eq.(2.37) that the two approximations having the coefficients $\tilde{\beta}_{ij}$ and β_{ij} respectively, are identical.

The method of the proof just given was first used by Greville in establishing theorem (2.4.1) for the simplest case of Lagrange data [Greville 1967].

As in the case of Lagrange data [Schoenberg 1964] we obtain now for the interpolation functional $\mathcal{L}f = f(x)$ the following corollary where the details of the proof is given in [Osama 2006]:

Corollary (2.4.2), [Schoenberg, 1968]:

The g-spline interpolation formula (2.18) is also the best interpolation of order m in the sense of Sard of the HB-problem (2.1).

2.5 THE CONSTRUCTION OF THE G-SPLINE INTERPOLATION FUNCTIONS, [SCHOENBERG, 1968]:

The most difficulty in the study of G-spline function is the constructing of G-spline itself, because the literatures give the results directly, therefore in this section, we will illustrate in details the method of construction:

The construction in more efficient way leading to a system of only m+n equations (instead of 2mk) is as follows:

From conditions (I), (II) and (III) of definition (2.3.1) it is clear that a G-spline function S(x) must take the form:

$$S(x) = P_{m-1}(x) + \sum_{i=1}^{k}\sum_{j=0}^{m-1} c_{ij} \frac{(x - x_i)_+^{2m-j-1}}{(2m-j-1)!} \qquad \ldots\ldots\ldots\ldots\ldots\ldots.... (2.43)$$

where $P_{m-1} \in \prod_{m-1}$ and c_{ij} are constants.

Conversely, any function in eq.(2.43) satisfies these conditions with the exception of the requirement that:

$$S(x) \in \prod_{m-1} \quad \text{if } x_k < x \qquad \text{.............................. .... (2.44)}$$

From eq.(2.43) it can see that $S^{(2m-j-1)}(x)$ is continuous at $x = x_1$ if and only if $c_{ij} = 0$, while condition IV in definition (2.3.1) requires that this is the case if and only if $a^*_{ij} = 0$. Leaving out all such terms and obtains:

$$S(x) = P_{m-1}(x) + \sum_{(i,j)\in e} c_{ij} \frac{(x - x_i)_+^{2m-j-1}}{(2m - j - 1)!} \qquad \text{.......................... (2.45)}$$

as the appropriate expression. To insure also eq.(2.44) expand all binomials and equating to zero the coefficients of $x^m, x^{m+1}, ..., x^{2m-1}$, we obtain the equations:

$$\sum_{\substack{(i,j)\in e \\ j\le v}} \frac{c_{ij}}{(2m - j - 1)!} \binom{2m - j - 1}{2m - v - 1} (-x_i)^{v-j} = 0, \text{ (v=0,1,2,…,m-1)}$$

$$\text{.................. (2.46)}$$

If we consider also the n equations of the m-poised HB-problem:

$$S^{(j)}(x_i) = y_i^{(j)}, \; (i, j) \in e \qquad \text{.. (2.47)}$$

Therefore, from eqs.(2.46) and (2.47) we get m+n equations which if solved will produce the unique interpolating G-spline function S(x). Writing the solution so as to exhibit the $y_i^{(j)}$, to get:

$$S(x) = \sum_{(i,j)\in e} y_i^{(j)} L_{ij}(x), \; (i, j) \in e$$

2.6 ILLUSTRATIVE EXAMPLE:

As an illustration to the discussion above, consider the following HB-problem:

$$f(0)=\theta_0 \,,\, f'(0)=\theta'_0 \,,\, f(1)=\theta_1 \,,\, f'(1)=\theta'_1 \,,\, f(2)=\theta_2 ,\, f'(2)=\theta'_2 \quad ..(2.48)$$

and to find the G-spline function which interpolate the problem (2.48). In this problem we have $\alpha=1$, $n=6$ and it is clear that it is 4-poised.

The incidence matrix is given by:

$$E=\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}$$

And the HB-set e will take the form:

$$e=\{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1)\}.$$

The G-spline interpolation function will be:

$$S_3(x)=P_3(x)+\sum_{(i,j)\in e} c_{ij}\frac{(x-x_i)_+^{2m-j-1}}{(2m-j-1)!}.$$

$$=a_0+a_1x+a_2x^2+a_3x^3+\frac{1}{7!}c_{00}x_+^7+\frac{1}{6!}c_{01}x_+^6+\frac{1}{7!}c_{10}(x-1)_+^7$$

$$+\frac{1}{6!}c_{11}(x-1)_+^6+\frac{1}{7!}c_{20}(x-2)_+^7+\frac{1}{6!}c_{21}(s-2)_+^6.$$

Now, to find the fundamental G-spline functions $L_{00}(x)$, $L_{01}(x)$, $L_{10}(x)$, $L_{11}(x)$, $L_{20}(x)$ and $L_{21}(x)$, we must solve the following linear system of algebraic equations obtained from eqs.(2.46) and (2.47):

$$\frac{c_{00}}{7!}+\frac{c_{10}}{7!}+\frac{c_{20}}{7!}=0.$$

$$\frac{c_{01}}{6!}-\frac{c_{10}}{6!}+\frac{c_{11}}{6!}-\frac{2}{6!}c_{20}+\frac{c_{21}}{6!}=0.$$

$$\frac{c_{10}}{5!2!}-\frac{c_{11}}{5!}+\frac{2}{5!}c_{20}-\frac{2}{5!}c_{21}=0.$$

$$\frac{-c_{10}}{4!3!}+\frac{c_{11}}{4!2!}-\frac{8}{4!3!}c_{20}+\frac{c_{21}}{3!2!}=0.$$

$$a_0 = \theta_0.$$
$$a_1 = \theta'_0.$$
$$a_0 + a_1 + a_2 + a_3 + \frac{c_{00}}{7!} + \frac{c_{01}}{6!} = \theta_1.$$
$$a_1 + 2a_2 + 3a_3 + \frac{c_{00}}{6!} + \frac{c_{01}}{5!} = \theta'_1.$$
$$a_0 + 2a_1 + 4a_2 + 8a_3 + \frac{128}{7!}c_{00} + \frac{64}{6!}c_{01} + \frac{c_{10}}{7!} + \frac{c_{11}}{6!} = \theta_2.$$
$$a_0 + 4a_2 + 12a_3 + \frac{2^6}{6!}c_{00} + \frac{2^5}{5!}c_{01} + \frac{1}{6!}c_{10} + \frac{1}{5!}c_{11} = \theta'_2.$$

Hence, we get:

$$a_0 = \theta_0.$$
$$a_1 = \theta'_0.$$
$$a_2 = -\frac{33}{8}\theta_0 - \frac{11}{4}\theta'_0 + \frac{7}{2}\theta_1 - \frac{5}{2}\theta'_1 + \frac{7}{8}\theta_2 - \frac{1}{4}\theta'_2.$$

$$a_3 = \frac{281}{78}\theta_0 + \frac{25}{12}\theta'_0 - \frac{35}{13}\theta_1 + \frac{19}{6}\theta'_1 - \frac{133}{104}\theta_2 + \frac{115}{312}\theta'_2.$$

$$c_{00} = \frac{20160}{13}\theta_0 + 1260\theta'_0 - \frac{5040}{13}\theta_1 + 2520\theta'_1 - \frac{22050}{13}\theta_2 + \frac{6930}{13}\theta'_2.$$

$$c_{01} = -\frac{7350}{13}\theta_0 - 420\theta'_0 + \frac{2520}{13}\theta_1 - 840\theta'_1 + \frac{6930}{13}\theta_2 - \frac{2100}{13}\theta'_2.$$

$$c_{10} = -\frac{7560}{13}\theta_0 + \frac{10080}{13}\theta_1 - \frac{5040}{13}\theta_2 + \frac{2520}{13}\theta'_2.$$

$$c_{11} = -1680\theta_0 - 1680\theta'_0 - 3360\theta'_1 + 2520\theta_2 - 840\theta'_2.$$

$$c_{20} = -\frac{3570}{13}\theta_0 - 420\theta'_0 - \frac{2520}{13}\theta_1 - 840\theta'_1 + \frac{9450}{13}\theta_2 - \frac{3360}{13}\theta'_2.$$

Therefore, upon substituting $c_{00}, c_{01}, c_{10}, c_{11}, c_{20}, c_{21}, a_0, a_1, a_2$ and a_3 into $S_4(x)$ and writing $S(x)$ in terms of $\theta_0, \theta'_0, \theta_1, \theta'_1, \theta_2$, θ'_2 gives:

$$S(x) = \theta_0 L_{00}(x) + \theta'_0 L_{01}(x) + \theta_1 L_{10}(x) + \theta'_1 L_{11}(x) + \theta_2 L_{20}(x) + \theta'_2 L_{21}(x)$$

where

$$L_{00}(x) = 1 - \frac{33}{8}x^2 + \frac{281}{78}x^3 + \frac{20160}{6.552\times10^4}x_+^7 - \frac{7350}{9.36\times10^3}x_+^6 - \frac{7560}{6.552\times10^4}(x-1)_+^7 - \frac{1680}{6!}(x-1)_+^6 - \frac{12600}{6.552\times10^4}(x-2)_+^7 - \frac{3570}{9.36\times10^3}(x-2)_+^6.$$

$$L_{01}(x) = x - \frac{11}{4}x^2 + \frac{25}{12}x^3 + \frac{1260}{7!}x_+^7 - \frac{420}{6!}x_+^6 - \frac{1680}{6!}(x-1)_+^6 - \frac{1260}{7!}(x-2)_+^7 - \frac{420}{6!}(x-2)_+^6.$$

$$L_{10}(x) = \frac{7}{2}x^2 - \frac{35}{13}x^3 - \frac{5040}{6.552\times10^4}x_+^7 + \frac{2520}{9.36\times10^3}x_+^6 + \frac{10080}{6.552\times10^4}(x-1)_+^7 - \frac{5040}{6.552\times10^4}(x-2)_+^7 - \frac{2520}{9.36\times10^3}(x-2)_+^6.$$

$$L_{11}(x) = -\frac{5}{2}x^2 + \frac{19}{6}x^3 + \frac{2520}{7!}x_+^7 - \frac{840}{6!}x_+^6 - \frac{3360}{6!}(s-1)_+^6 - \frac{2520}{7!}(x-2)_+^7 - \frac{840}{6!}(s-2)_+^6.$$

$$L_{20}(x) = \frac{7}{8}x^2 - \frac{133}{104}x^3 - \frac{22050}{6.552\times10^4}x_+^7 + \frac{6930}{9.36\times10^3}x_+^6 - \frac{5040}{6.552\times10^4}(x-1)_+^7 + \frac{2520}{6!}(x-1)_+^6 + \frac{27090}{6.552\times10^4}(x-2)_+^7 + \frac{9450}{9.36\times10^3}(x-2)_+^6.$$

$$L_{21}(x) = -\frac{1}{4}x^2 + \frac{115}{312}x^3 + \frac{6930}{6.552\times10^4}x_+^7 - \frac{2100}{9.36\times10^3}x_+^6 + \frac{2520}{6.552\times10^4}(x-1)_+^7 -$$

$$\frac{840}{6!}(x-1)_+^6 - \frac{9450}{6.552\times10^4}(x-2)_+^7 - \frac{3360}{9.36\times10^3}(x-2)_+^6.$$

The approximate G-spline function for the function $f(x) = e^x + 3x^2$, with knot points $x_0 = 0, x_1 = 1, x_2 = 2$; is presented in figure (2.1).

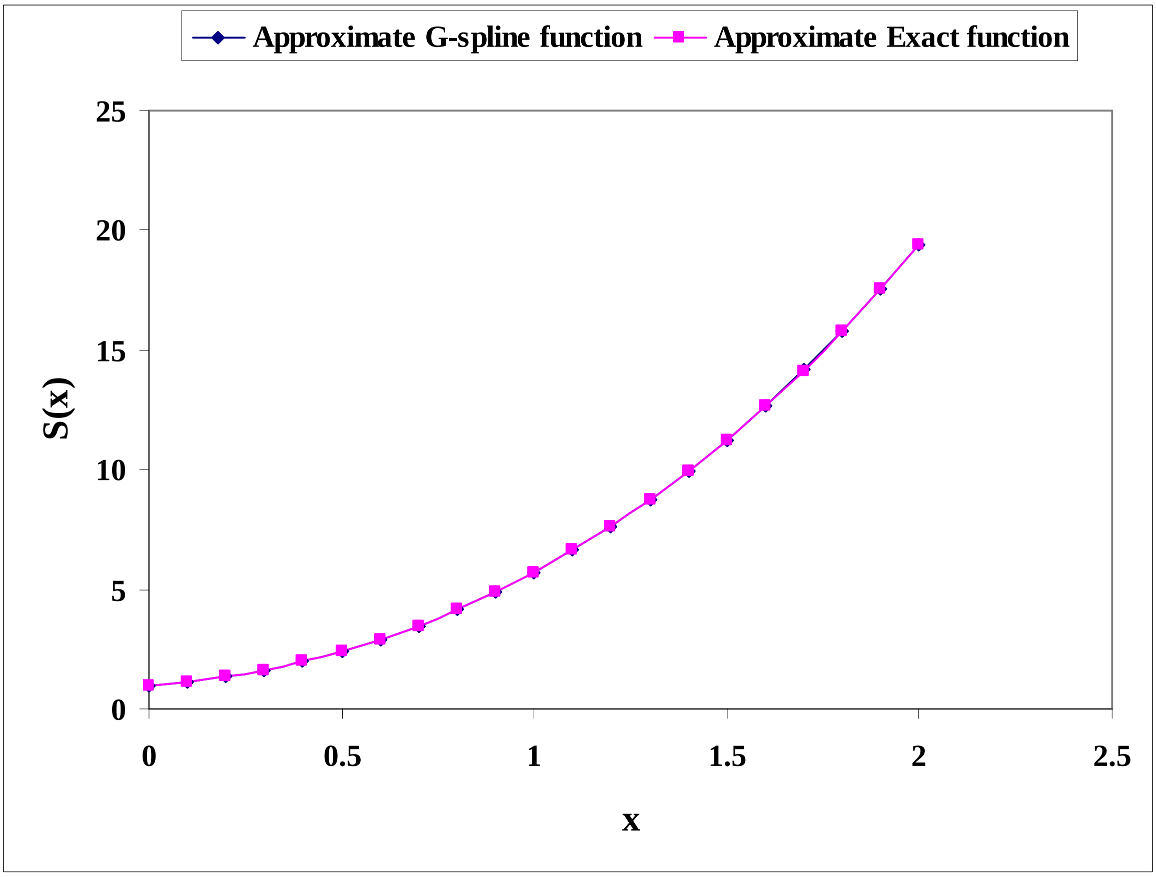


Figure (2.1) Approximate G-spline function to the f(x) $= e^x + 3x^2$.

Chapter Three

Solution of Ordinary Differential Equation Using Linear Multistep Method Based on G-Spline Interpolation

An ordinary differential equation of order n relates an unknown function y(x) to its derivatives $y', y'', ..., y^{(n)}$. The importance of differential equations in sciences stems from the fact that it is often relatively easy to reason a bout how an unknown function charges relative to its current value, therefore the central difference of this chapter is to approximate the solution of ordinary differential equations based on G-spline interpolation functions.

3.1 CONSTRUCTION OF LINEAR MULTISTEP METHOD USING G-SPLINE FUNCTION [BYRNE, 1972]:

Consider the construction of an mth order linear multistep formula of the general type:

$$y_k = y_q + \sum_{(i,j)\in e} \beta_{ij} h^{j+1} f^{(j)}(x_i, y_i) \qquad \text{.................................... (3.1)}$$

Where y_i is an approximation to $y(x_i)$, $0 \le q < k$ and $[x_n, x_{n+k}] \subset [a,b]$.

To this objective, we pick k and p along with the m-poised HB-problem corresponding to the n values that is described in chapter two.

The next result is given in [Byren, 1972] without proof, here the details of the proof is given for completeness.

Proposition (3.1.1):

Consider the initial value problem:

$$y' = f(x,y), \quad x \in [a,b] \qquad \ldots\ldots (3.2)$$
$$y(x_0) = y_0.$$

Then the m-th order linear multistep formula of eq.(3.2) is given by eq.(3.1).

Proof:

The technique that will be used involves of writing $y(x_k)$, which is the exact solution of eq.(3.2) evaluated at $x_k = a + kh$, as:

$$y(x_k) - y(x_q) = \int_{x_q}^{x_k} f(x, y(x))dx \quad ,0 \le q < k \qquad \ldots\ldots (3.3)$$

and then replacing f by its G-spline interpolation:

$$y(x_k) - y(x_q) = \int_{x_q}^{x_k} \sum_{(i,j)\in e} L_{ij}(x) f^{(j)}(x_i, y_i)\, dx.$$

$$= \sum_{(i,j)\in e} \int_{x_q}^{x_k} L_{ij}(x) f^{(j)}(x_i, y_i)\, dx\ .$$

and then make change of variables, yields:

$$y_k - y_q = \sum_{(i,j)\in e} \int_q^k hL_{ij}(s) f^{(j)}(x_i, y_i) ds \qquad \ldots\ldots (3.4)$$

Where $x = x_n + sh$, $0 \le s \le k$, h is the step size, n=0,1,…

Now, from theorem (2.4.1) one can write eq.(3.4) by:

$$y_k - y_q = \sum_{(i,j)\in e} h\beta_{ij} f^{(j)}(x_i, y_i) \qquad \ldots\ldots (3.5)$$

where $\beta_{ij} = \int_q^k L_{ij}(s)ds$.

and after differentiating j-times the function f by the chain rule, then:

$$y_k - y_q = \sum_{(i,j)\in e} h^{j+1}\beta_{ij} f^{(j)}(x_i, y_i).$$

#

Now, let L_{m-1} denote the class of linear functionals F of the form:

$$Ff = \sum_{i=0}^{m}\int_a^b f^{(j)}(x, y(x))d\mu_i(x) \qquad \text{.......................... (3.6)}$$

where μ_i is of bounded variation on I=[a,b].

If:

$$Rf = \left[\int_{x_{n+q}}^{x_{n+k}} f(x, y(x))dx - \sum_{(i,j)\in e} h^{j+1}\beta_{ij} f^{(j)}(x_{n+i}, y(x_{n+i}))\right] \qquad \text{.......... (3.7)}$$

then $R\in L_{m-1}$. By Peao's kernel theorem construction, Rf=0 for $f \in \prod_{m-1}$, and one can get:

$$Rf = \int_{x_n}^{x_{n+k}} f^{(m)}(x, y(x))K(x)dx \qquad \text{...................... (3.8)}$$

with

$$K(x) = \left[\int_{x_{n+q}}^{x_{n+k}} \frac{(z-x)_+^{m-1}}{(m-1)!}dz - \sum_{(i,j)\in e} h^{j+1}\beta_{ij}\frac{(x_{n+i}-x)_+^{m-j-1}}{(m-j-1)!}\right]$$

making change of variables, then K will be:

$$K(x) = h^m\left[\int_q^k \frac{(z-s)_+^{m-1}}{(m-1)!}dz - \sum_{(i,j)\in e} \beta_{ij}\frac{(i-s)_+^{m-j-1}}{(m-j-1)!}\right]$$

and

$$K(x) = h^m k(s) \qquad \ldots\ldots (3.9)$$

where $s \in [0, k]$, x=a+(n+s)h and K is the Peano kernel and,

$$k(s) = \int_q^k \frac{(z-s)_+^{m-1}}{(m-1)!} dz - \sum_{(i,j)\in e} \beta_{ij} \frac{(i-s)_+^{m-j-1}}{(m-j-1)!}.$$

The next proposition gives the error bounds of the LMM with G-spline functions, which appears in [Byrne, 1972] without proof.

<u>Proposition (3.1.2):</u>

The local truncation error by using the LMM in eq.(3.1) is bounded and:

$$\|Rf\| \le h^{m+\frac{1}{2}} \|k\| \left\|f^{(m)}\right\|.$$

Proof:

From eq.(3.8):

$$\|Rf\| = \left\| \int_{x_n}^{x_{n+1}} f^{(m)}(x, y(x)) K(x) dx \right\|.$$

$$\le \int_{x_n}^{x_{n+1}} \left\|f^{(m)}(x, y(x)) K(x)\right\| dx.$$

$$\le \int_{x_n}^{x_{n+1}} \left\|f^{(m)}(x, y(x))\right\| \|K(x)\| dx. \qquad \ldots\ldots (3.10)$$

Now, putting eq.(3.9) in eq.(3.10),yields:

$$\|Rf\| = \left\|f^{(m)}\right\| \int_{x_n}^{x_{n+1}} \left\|h^m k(s)\right\| dx.$$

$$= \left\|f^{(m)}\right\| \left\|h^m\right\| \int_{x_n}^{x_{n+1}} \|k(s)\| dx.$$

$$= h^m \left\| f^{(m)} \right\| \; \left\| k(s) \right\| \int_{x_n}^{x_{n+1}} dx .$$

$$= h^m \left\| f^{(m)} \right\| \; \left\| k(s) \right\| \; x \Big|_{x_n}^{x_{n+1}}$$

$$= h^{m+1} \left\| f^{(m)} \right\| \; \left\| k(s) \right\| .$$

$$\leq h^{m+\frac{1}{2}} \left\| f^{(m)} \right\| \quad \left\| k(s) \right\| . \qquad \text{……...……………………..……... (3.11)}$$

Since if 0<h<1 then $h^n < h^m$ if n>m, and hence:

$$\left\| Rf \right\| \leq h^{m+\frac{1}{2}} \left\| f^{(m)} \right\| \left\| k(s) \right\| .$$

#

Remark (3.1.3):

For simplicity and computational purpose, eq.(3.1) may be shifted n times to give the following more general formula:

$$y_{n+k} - y_{n+q} = \sum_{(i,j)\in e} h^{j+1} \beta_{ij} f^{(j)}(x_{n+i}, y_{n+i}) . \qquad \text{…………….... (3.13)}$$

or in more reliable form as:

$$y_{n+k} - y_{n+q} = \sum_{j=0}^{p} \sum_{i=0}^{k} h^{j+1} \beta_{ij} f^{(j)}(x_{n+i}, y_{n+i}) .$$

3.2 EXPLICIT AND IMPLICIT FORMULAS, [LAMBERT, 1973]:

In this section, it was common in practice to write the right-hand side of a certain linear multistep method in terms of a power series in a difference operator. A typical example is:

$$y_{n+1} - y_n = h(1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \mathbf{L})f_{n+1} \quad \ldots\ldots\ldots\ldots\ldots (3.14)$$

where $\nabla f_{n+1} = f_{n+1} - f_n$ and truncating the series after two terms, gives:

$$y_{n+1} - y_n = \frac{1}{2}h(f_{n+1} + f_n).$$

which is the Trapezoidal rule.

While, truncated after three terms, gives:

$$y_{n+1} - y_n = \frac{1}{12}h(5f_{n+1} + 8f_n - f_{n-1}).$$

which is a method that is equivalent to Adams-Moulton method.

One reason for expressing a numerical method in a form such as eq.(3.14) lay in the technique, common in desk computation, of including higher difference of f if they became significantly large at some stage of the calculation. This is equivalent to exchanging the linear multistep method for one step with higher step number.

The existence of formula like eq.(3.14) has resulted in "family" names being given to class of linear multistep methods, of different step number, which share a common form for the first characteristic polynomial $\rho(\zeta)$.

Thus methods for which $\rho(\zeta) = \zeta^k - \zeta^{k-1}$ are called Adams method. They have the property that all the spurious roots of ρ are located at the origin; such methods are thus zero-stable.

Adams methods which are explicit are called Adams-Bashforth methods, while these which are implicit are called Adam-Moulton methods. Explicit method for which $\rho(\zeta) = \zeta^k - \zeta^{k-2}$ are called Nystrom methods, and implicit methods with the same form for ρ are called the generalized Miline-Simpson methods; both of these families are clearly zero-stable, since they have one spurious root at -1 and the rest at the origin.

Clearly there exist many linear multistep methods which do not belong to any of the families named above. We now specify a selection of linear multistep methods by quoting coefficients α_j, β_j where j=0,1,2,…, k for k=1,2,3,4 giving explicit and implicit methods for each step number.

In each case, we retain just enough parameters a,b,c,… (an arbitrary real number) to enable us to have complete control over the values taken by the spurious roots of $\rho(\zeta)$, these parameters must be chosen so that the spurious roots all lie in zero-stable configuration, and it is remarkable that all of the methods quoted are consistent, so that the principal root of $\rho(\zeta)$ is always 1.

The methods have the highest order that can be attained whilst retaining the given number of free parameters; the order p and the error constant C_{p+1} are quoted in each case, and next we will consider examples for each case:

Explicit methods:

for k=1, we have:

$$\beta_0 = 1, \quad \alpha_1 = 1, \quad \alpha_0 = -1;$$

$$p = 1, \quad C_{p+1} = \frac{1}{2}.$$

for k=2, we have:

$$\beta_1 = \frac{1}{2}(3 - a), \quad \beta_0 = -\frac{1}{2}(1 + a),$$

$$\alpha_2 = 1, \quad \alpha_1 = -1 - a, \quad \alpha_0 = a,$$

$$p = 2; \quad C_{p+1} = \frac{1}{2}(5 + a).$$

and there exist no value for a which will causes the order to exceed 2 and the method to be zero-stable.

k=3, then:

$$\beta_2 = \frac{1}{12}(23 - 5a - b), \quad \beta_1 = \frac{1}{3}(-4 - 2a + 2b), \quad \beta_0 = \frac{1}{12}(5 + a + 5b)$$

$$\alpha_3 = 1, \quad \alpha_2 = -1 - a, \quad \alpha_1 = a + b, \quad \alpha_0 = -b$$

$$p = 3; \quad C_{p+1} = \frac{1}{24}(9 + a + b)$$

and there is no value for a and b which cause the order to exceed and the method is zero-stable.

for k=4, then:

$$\beta_3 = \frac{1}{24}(55 - 9a - b - c), \quad \beta_2 = \frac{1}{24}(-59 - 19a + 13b + 5c),$$

$$\beta_1 = \frac{1}{24}(37 + 5a + 13b - 19c), \quad \beta_0 = \frac{1}{24}(-9 - a - b - 9c),$$

$$\alpha_4 = 1, \alpha_3 = (-1 - a), \alpha_2 = a + b, \alpha_1 = -b - c, \alpha_0 = c,$$

$$p = 4; \quad C_{p+1} = \frac{1}{720}(251 + 19a + 11b + 19c).$$

and there is no value for a, b and c which cases the order to exceed and the method to be zero-stable.

Implicit methods:

for k=1, then:

$$\beta_1 = \frac{1}{2}, \quad \beta_0 = \frac{1}{2},$$

$$\alpha_1 = 1, \quad \alpha_0 = -1,$$

$$p = 2; \quad C_{p+1} = -\frac{1}{2}.$$

for k=2, then:

$$\beta_2 = \frac{1}{12}(5+a), \quad \beta_1 = \frac{2}{3}(1-a), \quad \beta_0 = \frac{1}{12}(-1-5a)$$
$$\alpha_2 = 1, \quad \alpha_1 = -1-a, \quad \alpha_0 = a$$
$$\text{if } a \neq -1, \text{ then } p = 3; \quad C_{p+1} = -\frac{1}{24}(1+a).$$
$$\text{if } a = -1, \text{ then } p = 4; \quad C_{p+1} = -\frac{1}{90}.$$

if k=3, then we have:

$$\beta_3 = \frac{1}{24}(9+a+b), \quad \beta_2 = \frac{1}{24}(19-13a-5b),$$
$$\beta_1 = \frac{1}{24}(-5-13a+19b), \quad \beta_0 = \frac{1}{24}(1+a+9b)$$
$$\alpha_3 = 1, \quad \alpha_2 = -1-a, \quad \alpha_1 = a+b, \quad \alpha_0 = -b$$
$$p = 4; \quad C_{p+1} = -\frac{1}{720}(19+11a+19b).$$

and there is no value for a and b which case the order to exceed 3 and the method to be zero-stable.

for k=4, then:

$$\beta_4 = \frac{1}{720}(251+19a+11b+19c), \quad \beta_3 = \frac{1}{360}(323-173a-37b-53c),$$
$$\beta_2 = \frac{1}{30}(-11-19a+19b+11c), \quad \beta_1 = \frac{1}{360}(53+37a+173b-323c),$$
$$\beta_0 = \frac{1}{720}(-19-11a-19b-251c).$$
$$\alpha_4 = 1, \quad \alpha_3 = -1-a, \quad \alpha_2 = a+b, \quad \alpha_1 = -b-c, \quad \alpha_0 = c.$$
$$\text{if } 27+11a+11b+27c \neq 0 \text{ then,}$$

$$p = 5; \quad C_{p+1} = -\frac{1}{1440}(27+11a+11b+27c).$$
$$\text{if } 27+11a+11b+27c = 0 \text{ then,}$$
$$p = 6; \quad C_{p+1} = -\frac{1}{15120}(74+10a-10b-74c).$$

and there exist no values for a, b and c which case the order to exceed 6 and the method to be zero=stable.

Definition (3.2.1), [Lambert, 1973]:

The linear multistep method given by eq.(1.1) is said to be absolutely stable if all the roots of:

$\pi(r,\hbar)=\rho(r)-\hbar\delta(r)=0$

Where $\pi(r,\hbar)$ is the stability polynomial of the method defined by ρ and δ, $\hbar=h\lambda$,

satisfy $|r_s|<1$, s=1,2,..,k.

3.2.1 Comparison Between Explicit and Implicit LMM's:

It is clear for the methods quoted that, for a given step number k, the highest attainable order for zero-stable method is less in the case of an explicit method than in the case of implicit one; but we have seen that there is no serious difficultly in obtaining additional starting values and thus there is no apparent reason why we should not prefer an explicit method of higher step number. Implicit method, however, possess advantages over explicit methods other than the higher order for a given step number.

As an illustration, compare between explicit Adams-Bashforth methods and implicit Adams-Moulton methods for step number k=1, 2, 3, 4; in each case, the method considerd has the highest attainable order, the coefficients can be obtained from the general methods quoted by setting each of the parameters a, b and c to zero, the result of this comparison are shown in the next table, where p is the order, C_{p+1} is the error constant.

Table (3.1)

Adams-Bash forth(explicit)				
k	1	2	3	4
P	1	2	3	4
C_{P+1}	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{8}$	$\frac{251}{720}$
α	-2	-1	$-\frac{6}{11}$	$-\frac{3}{10}$

Table (3.2)

Adams-Moulton (implicit)				
k	1	2	3	4
P	2	3	4	5
C_{P+1}	$-\frac{1}{2}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{3}{160}$
α	$-\infty$	-6	-3	$-\frac{90}{49}$

Where (α,0) is the interval of absolutely stability. If we compare explicit and implicit methods of the same order, it is clear that the latter, besides having error constant of smaller absolute value, possess a considerable advantage in the size of the interval of absolute stability. Thus, for example, if we wish to use a fourth-order Adams method, we have the choice between the explicit four-step method and the implicit three-step; the latter has an interval of absolute stability which is ten times greater than that of the former and, moreover, its error constant is smaller by a factor of approximately $\frac{1}{13}$.

These considerations, which are typical of more general comparisons between explicit and implicit methods, so for our implicit methods that explicit LMM are seldom used on their own; they do, however, play an important ancillary role in predictor-corrector pairs.

3.2.2 Predictor-Corrector Methods:

Suppose that we intend to use an implicit linear k-step method to solve certain initial value problem. At each step one must solve for y_{n+k} the difference equation:

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, y_{n+k}) + h\sum_{j=0}^{k-1} \beta_j f_{n+j} \qquad \text{……... (3.15)}$$

where y_{n+j}, f_{n+j}, j=0, 1, 2, … , k-1 are known. In general, this equation is non-linear and a unique solution y_{n+k} exists and can be approached arbitrary closely by the iteration:

$$y_{n+k}^{[s+1]} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, y_{n+k}^{[s]}) + h\sum_{j=0}^{k-1} \beta_j f_{n+j} \qquad \text{…….... (3.16)}$$

where s=0,1,2,… and $y_{n+k}^{[0]}$ is arbitrary.

Each step of the iteration (3.16) clearly involves an evaluation of $f(x_{n+k}, y_{n+k}^{[s]})$, and we are concerned to keep to a minimum the number of times the iteration (3.16) is applied particularly so when the evaluation of f at given values of its arguments is time consuming, (such an evaluation may call several subroutines). We would therefore like to make the initial guess $y_{n+k}^{[0]}$ as accurate as possible and this is done by using a separate explicit method to estimate y_{n+k} and taking this predicted value to be the initial guess $y_{n+k}^{[0]}$. The explicit method is called the predictor, and the implicit method in eq.(3.15) is the corrector.

Now, proceeding in one of the two different ways, the first consists of continuing the iteration (3.16) until the iterates have converges (in practice, until some criterion is satisfied such as:

$$\left|y_{n+k}^{[s+1]} - y_{n+k}^{[s]}\right| < \varepsilon,$$

where ε is a pre-assigned tolerance of order of the local round-off error, say, is satisfied). Then regarding the value $y_{n+k}^{[s+1]}$ so obtained as an acceptable approximation to the exact solution y_{n+k} of eq.(3.15). Since each iteration corresponds to one application of the corrector, this mode of operation of the predictor-corrector method is called correcting to convergence.

In this mode, one can not tell in advance how many iterations will be necessary, that is, how many function evaluations will be required at each step. On the other hand, the accepted value $y_{n+k}^{[s+1]}$ being independent of the initial guess $y_{n+k}^{[0]}$, the local truncation error and the weak stability characteristics of the over all method are precisely those of corrector alone, the properties of the predictor are of no importance.

In particular, h must be chosen so that h lies within the interval of absolute or relative stability of the corrector; no harm will be done if this value of h does not lie within a stability interval of the predictor (or even if the predictor is not zero-stable).

Let P indicate an application of the predictor, C a single application of the corrector, and E an evaluation of f in terms of known value of its arguments.

Suppose that one can compute $y_{n+k}^{[0]}$ from the predictor, and evaluate $f_{n+k}^{[0]} = f(x_{n+k}, y_{n+k}^{[0]})$, then apply the corrector once to get $y_{n+k}^{[1]}$; the calculation so far is denoted by PEC.

A further evaluation of $f_{n+k}^{[1]} = f(x_{n+k}, y_{n+k}^{[1]})$ followed by a second application of the corrector yields $y_{n+k}^{[2]}$, and the calculation is now denoted by PECEC, or $P(EC)^2$.

Applying the corrector m-times is similarly denoted by $P(EC)^m$. Since m is fixed, $y_{n+k}^{[m]}$ is accepted as the numerical solution at x_{n+k}. At this stage, the last computed value we have for f_{n+k} is $f_{n+k}^{[m-1]} = f(x_{n+k}, y_{n+k}^{[m-1]})$, and a further discussion is made to make, namely, whether or not to evaluate $f_{n+k}^{[m]} = f(x_{n+k}, y_{n+k}^{[m]})$.

If this final evaluation is made then one can denote the mode by $P(EC)^m E$, and if not by $P(EC)^m$. This choice clearly affects the next step of the calculation, since both predicted and corrected values for y_{n+k+1} will depend on whether f_{n+k} is taken to be $f_{n+k}^{[m]}$ or $f_{n+k}^{[m-1]}$.

Note that, for a given m, both $P(EC)^m E$ and $P(EC)^m$ modes apply the corrector the same number of times; but the former calls for one more function evaluation per-step than the latter.

Now, define the above mode precisely. It will turn out to be advantageous if the predictor and the corrector are separately of the same order, and this requirement may well make it necessary for the step number of the predictor to be grater than that of the corrector (see, for example, table (3.1) and table (3.2)).

Let the linear multistep method used as predictor by the characteristic polynomials:

$$\begin{aligned} \rho^*(\xi) &= \sum_{j=0}^{k} \alpha_j^* \xi^j , \quad \alpha_k^* = 1 \\ \delta^*(\xi) &= \sum_{j=0}^{k-1} \beta_j^* \xi^j \end{aligned} \qquad \ldots\ldots (3.17)$$

and that the used corrector by:

$$\rho(\xi)=\sum_{j=0}^{k}\alpha_j\xi^j\ ,\ \alpha_k=1$$
$$\delta(\xi)=\sum_{j=0}^{k}\beta_j\xi^j \qquad \text{……………………………..…. (3.18)}$$

Then the modes $P(EC)^m E$ and $P(EC)^m$ described above are formally defined as follows for m=1,2,… :

$P(EC)^m E$:

$$y_{n+k}^{[0]}+\sum_{j=0}^{k-1}\alpha_j^* y_{n+j}^{[m]}=h\sum_{j=0}^{k-1}\beta_j^* f_{n+j}^{[m]}\ ,$$

$$f_{n+k}^{[s]}=f(x_{n+k},y_{n+k}^{[s]})\ .$$

……………. (3.19)

$$y_{n+k}^{[s+1]}+\sum_{j=0}^{k-1}\alpha_j y_{n+j}^{[m]}=h\beta_k f_{n+k}^{[s]}+h\sum_{j=0}^{k-1}\beta_j f_{n+j}^{[m]}\ ,$$

$$f_{n+k}^{[m]}=f(x_{n+k},y_{n+k}^{[m]})\ .$$

where s=0,1,2,…,m-1.

$P(EC)^m$:

$$y_{n+k}^{[0]}+\sum_{j=0}^{k-1}\alpha_j^* y_{n+j}^{[m]}=h\sum_{j=0}^{k-1}\beta_j^* f_{n+j}^{[m-1]}\ ,$$

……………..... (3.20)

$$y_{n+k}^{[s+1]}+\sum_{j=0}^{k-1}\alpha_j y_{n+j}^{[m]}=h\beta_k f_{n+k}^{[s]}+h\sum_{j=0}^{k-1}\beta_j f_{n+j}^{[m-1]}\ .$$

where s=0,1,2,…,m-1.

Remark (3.2.3), [Lambert, 1973]:

Note that as $m \to \infty$, the results of computing with either of the above modes will tend to those given by the mode of correcting to convergence. In practice, it is usual to use a mode for which m is grater than 2.

3.3 ILLUSTRATIVE EXAMPLES:

In this section, some illustrative examples are considerd to show the applicability of the discussed numerical methods and to compare between these methods.

Example (3.3.1):

Consider the linear ordinary differential equation:

$$y'(x) = xy \,,\; y(0) = 1,\; x \in [0,2]$$

where the exact solution is given by:

$$y(x) = e^{\frac{x^2}{2}} .$$

Suppose that a two-step method must be constructed in such away that it is exact for $y \in \Pi_4$.

To construct such a method via G-spline, we shall use the example in chapter two section (2.6) such that:

$\Delta = \{0, 1, 2\}$.

be the knot points and let:

e={(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)}.

to seek for $S_4(s) \in S_4(E^*, \Delta)$ with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Now, using eq.(3.13) over [1,2], yields:

$$y_{n+2} = y_{n+1} + h\{\beta_{00} f(x_n, y_n) + \beta_{10} f(x_{n+1}, y_{n+1}) + \beta_{20} f(x_{n+2}, y_{n+2})\}$$
$$+ h^2\{\beta_{01} f'(x_n, y_n) + \beta_{11} f'(x_{n+1}, y_{n+1}) + \beta_{21} f'(x_{n+2}, y_{n+2})\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.21)$$

where

$$\beta_{00} = \int_1^2 L_{00}(s)ds = 0.098.$$

$$\beta_{01} = \int_1^2 L_{01}(s)ds = -0.052.$$

$$\beta_{10} = \int_1^2 L_{10}(s)ds = 0.522.$$

$$\beta_{11} = \int_1^2 L_{11}(s)ds = 0.146.$$

$$\beta_{20} = \int_1^2 L_{20}(s)ds = 0.442.$$

$$\beta_{21} = \int_1^2 L_{21}(s)ds = -0.062.$$

and since f(x,y)=xy , then $f'(x, y) = y(x^2 + 1)$.

If $L_{ij}(s)$ where $(i, j) \in e$ are given as in example of section (2.6).

Table (3.3) illustrate the numerical results obtained by applying eq.(3.21) and its comparison with the exact solution, we have:

Table (3.3)

The numerical results of example (3.3.1)

x_i	y_i	$y(x_i)$	Absolute error
0	1	1	0
0.1	1.005	1.005	0
0.2	1.0195	1.02	0.0005
0.3	1.044	1.046	0.002
0.4	1.081	1.083	0.002
0.5	1.132	1.133	0.001
0.6	1.197	1.197	0
0.7	1.280	1.278	0.002
0.8	1.383	1.377	0.006
0.9	1.510	1.499	0.011
1	1.666	1.649	0.017
1.1	1.857	1.831	0.026
1.2	2.092	2.054	0.038
1.3	2.381	2.328	0.053
1.4	2.738	2.664	0.074
1.5	3.181	3.08	0.101
1.6	3.734	3.597	0.137
1.7	4.428	4.242	0.186
1.8	5.305	5.053	0.252
1.9	6.421	6.08	0.341
2	7.852	7.389	0.463

Example (3.3.2):

Consider the linear ordinary differential equation:

$$y' = y + x\ ,\ y(0) = 1, \quad x \in [0,3].$$

where the exact solution is given by:

$$y(x) = 2e^{x} - x - 1.$$

Consider it is required that a three step method must be constructed in such away that it is exact for $y \in \Pi_4$.

To construct such a method via G-spline functions, an HB-problem must be first chosen. The choice is for the knot points are:

$\Delta = \{0, 1, 2, 3\}$.

and let

$e=\{(0,0),(1,0),(1,1),(2,0),(3,0)\}$.

To seek for $S_4(s) \in S_4(E^*, \Delta)$ with:

$$E=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

and for which:

$$S_4^{(j)}(i) = \varphi^{(j)}(i) \quad , (i, j) \in e.$$

and to find the G-spline function which interpolate the following HB-problems:

$$f(0)=\varphi_0\ ,\ f(1) = \varphi_1\ ,\ f'(1) = \varphi_1'\ ,\ f(2) = \varphi_2\ ,\ f(3) = \varphi_3.$$

In this problem the maximum order of derivatives is $\alpha = 1$ and n=5 and it is clear that it is 4-poised.

The G-spline interpolation function will be of the form:

$$S_4(s) = P_3(s) + \sum_{(i,j)\in e} c_{ij} \frac{(s - s_i)_+^{(2m-j-1)}}{(2m - j - 1)!}.$$

$$= a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \frac{1}{7!} c_{00} s_+^7 + \frac{1}{7!} c_{10} (s-1)_+^7 + \frac{1}{7!} c_{11} (s-1)_+^6 +$$

$$\frac{1}{7!} c_{20} (s-2)_+^7 + \frac{1}{7!} c_{30} (s-3)_+^7 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3.22)}$$

and to find the fundamental G-spline functions $L_{00}(s), L_{10}(s), L_{11}(s), L_{20}(s)$, and $L_{30}(s)$, one must solve the following linear system of algebraic equations which is obtained from eqs.(2.46) and (2.47):

$$a_0 = \varphi_0.$$

$$a_0 + a_1 + a_2 + a_3 + \frac{1}{7!} c_{00} = \varphi_1.$$

$$a_1 + 2a_2 + 3a_3 + \frac{1}{6!} c_{00} = \varphi_1'.$$

$$a_0 + 2a_1 + 4a_2 + 8a_3 + \frac{2^7}{7!} c_{00} + \frac{1}{7!} c_{10} + \frac{1}{6!} c_{11} = \varphi_2.$$

$$a_0 + 3a_1 + 9a_2 + 27a_3 + \frac{3^7}{7!} c_{00} + \frac{2^7}{7!} c_{10} + \frac{2^6}{6!} c_{11} + \frac{1}{7!} c_{20} = \varphi_3.$$

$$\frac{1}{7!} c_{00} + \frac{1}{7!} c_{10} + \frac{1}{7!} c_{20} + \frac{1}{7!} c_{30} = 0.$$

$$-\frac{1}{6!} c_{10} + \frac{1}{6!} c_{11} - \frac{2}{6!} c_{20} - \frac{3}{6!} c_{30} = 0.$$

$$\frac{1}{5!2!} c_{10} - \frac{1}{5!} c_{11} + \frac{2}{5!} c_{20} + \frac{9}{5!2!} c_{30} = 0.$$

$$-\frac{1}{4!3!} c_{10} + \frac{1}{4!2!} c_{11} - \frac{8}{4!3!} c_{20} - \frac{27}{4!3!} c_{30} = 0.$$

Solving this system numerically yields:

$$a_0 = \varphi_0.$$

$$a_1 = -\frac{2299}{852} \varphi_0 + \frac{967}{568} \varphi_1 - \frac{737}{284} \varphi_1' + \frac{311}{284} \varphi_2 - \frac{169}{1704} \varphi_3.$$

$$a_2 = \frac{343}{142} \varphi_0 - \frac{107}{284} \varphi_1 + \frac{603}{142} \varphi_1' - \frac{319}{142} \varphi_2 + \frac{59}{184} \varphi_3.$$

$$a_3 = -\frac{205}{284}\varphi_0 - \frac{189}{568}\varphi_1 - \frac{473}{284}\varphi_1' + \frac{331}{284}\varphi_2 - \frac{63}{568}\varphi_3.$$

$$c_{00} = \frac{1680}{71}\varphi_0 + \frac{2520}{71}\varphi_1 + \frac{5040}{71}\varphi_1' - \frac{5040}{71}\varphi_2 + \frac{840}{71}\varphi_3.$$

$$c_{10} = \frac{2520}{71}\varphi_0 + \frac{3780}{71}\varphi_1 + \frac{7560}{71}\varphi_1' - \frac{7560}{71}\varphi_2 + \frac{1260}{71}\varphi_3.$$

$$c_{11} = -\frac{5040}{71}\varphi_0 - \frac{7560}{71}\varphi_1 - \frac{15120}{71}\varphi_1' + \frac{15120}{71}\varphi_2 - \frac{2520}{71}\varphi_3.$$

$$c_{20} = -\frac{5040}{71}\varphi_0 - \frac{7560}{71}\varphi_1 - \frac{15120}{71}\varphi_1' + \frac{15120}{71}\varphi_2 - \frac{2520}{71}\varphi_3.$$

$$c_{30} = \frac{840}{71}\varphi_0 + \frac{1260}{71}\varphi_1 + \frac{2520}{71}\varphi_1' - \frac{2520}{71}\varphi_2 + \frac{420}{71}\varphi_3.$$

Therefore, upon substituting $c_{00}, c_{10}, c_{11}, c_{20}, c_{30}, a_0, a_1, a_2$ and a_3 into $S_4(s)$ and writing S(s) in terms of $\varphi_0, \varphi_1, \varphi_1', \varphi_2$ and φ_3 gives:

$$S(s) = \varphi_0 L_{00}(s) + \varphi_1 L_{10}(s) + \varphi_1' L_{11}(s) + \varphi_2 L_{20}(s) + \varphi_3 L_{30}(s).$$

where

$$L_{00} = 1 - \frac{2299}{852}s + \frac{343}{142}s^2 - \frac{205}{284}s^3 + \frac{1}{213}s_+^7 + \frac{1}{142}(s-1)_+^7 -$$

$$\frac{7}{71}(s-1)_+^6 - \frac{1}{71}(s-2)_+^7 + \frac{1}{426}(s-3)_+^7.$$

$$L_{10} = \frac{967}{568}s - \frac{107}{284}s^2 - \frac{189}{568}s^3 + \frac{1}{142}s_+^7 + \frac{3}{284}(s-1)_+^7 - \frac{21}{142}(s-1)_+^6 -$$

$$\frac{3}{142}(s-2)_+^7+\frac{1}{284}(s-3)_+^7.$$

$$L_{11}=-\frac{737}{284}s+\frac{603}{142}s^2-\frac{473}{284}s^3+\frac{1}{71}s_+^7+\frac{3}{142}(s-1)_+^7-\frac{21}{71}(s-1)_+^6-$$
$$\frac{3}{71}(s-2)_+^7+\frac{1}{142}(s-3)_+^7.$$

$$L_{20}=\frac{311}{284}s-\frac{319}{142}s^2+\frac{331}{284}s^3-\frac{1}{71}s_+^7-\frac{3}{142}(s-1)_+^7+\frac{21}{71}(s-1)_+^6+$$
$$\frac{3}{71}(s-2)_+^7-\frac{1}{142}(s-3)_+^7.$$

$$L_{30}=-\frac{169}{1704}s+\frac{59}{284}s^2-\frac{63}{568}s^3+\frac{1}{426}s_+^7+\frac{1}{284}(s-1)_+^7-\frac{7}{142}(s-1)_+^6-$$
$$\frac{1}{142}(s-2)_+^7+\frac{1}{852}(s-3)_+^7.$$

Now, using eq.(3.13) over [2,3] yields the closed formula:

$$y_{n+3}=y_{n+2}+h[\beta_{00}f(x_n,y_n)+\beta_{10}f(x_{n+1},y_{n+1})+\beta_{20}f(x_{n+2},y_{n+2})+$$
$$\beta_{30}f(x_{n+3},y_{n+3})]+h^2[\beta_{11}f'(x_{n+1},y_{n+1})].$$

Where

$$\beta_{00}=\int_2^3 L_{00}(s)ds=-0.043.$$

$$\beta_{10}=\int_2^3 L_{10}(s)ds=-0.336.$$

$$\beta_{11}=\int_2^3 L_{11}(s)ds=-0.255.$$

$$\beta_{20}=\int_2^3 L_{20}(s)ds=1.047.$$

$$\beta_{30} = \int_{2}^{3} L_{30}(s)ds = 0.332.$$

And hence the following table of results is obtained with a comparison with the exact solution.

Table (3.4)

The numerical results of example (3.3.2).

x_i	$y_i(x)$	$y(x)$	Absolute error
0	1	1	0
0.2	1.243	1.243	0
0.4	1.582	1.584	0.002
0.6	2.041	2.044	0.003
0.8	2.646	2.651	0.005
1	3.429	3.437	0.008
1.2	4.429	4.440	0.011
1.4	5.696	5.71	0.014
1.6	7.287	7.306	0.019
1.8	9.274	9.299	0.025
2	11.746	11.778	0.032
2.2	14.809	14.85	0.041
2.4	18.594	18.646	0.052
2.6	23.261	23.327	0.066
2.8	29.006	29.089	0.083
3	36.067	36.171	0.104

Example (3.3.3):

In this example we can use the same G-spline functions for the 2-step method as in example (3.3.1) to solve non-linear ordinary differential equation function, such that:

$$y' = e^{-y} \ , y(0) = 1.$$

where the exact solution is given by:

$$y(x) = \ln(x + c), \text{ where } c = e^1.$$

Now, by using Euler's method to find y_1, we have:

$$y_{n+1} = y_n + hf(x_n, y_n).$$
$$y_1 = y_0 + 0.1f(x_0, y_0).$$
$$=1.037.$$

Then we have the table:

Table (3.5)

The numerical result of example (3.3.3)

x_i	$y_i(x)$	$y(x)$	Absolute error
0	1	1	0
0.2	1.074	1.071	0.003
0.4	1.143	1.137	0.006
0.6	1.208	1.199	0.009
0.8	1.269	1.258	0.011
1	1.326	1.313	0.013
1.2	1.380	1.366	0.014
1.4	1.431	1.415	0.016
1.6	1.480	1.4463	0.017
1.8	1.526	1.508	0.018
2	1.570	1.551	0.019

Example (3.3.4):

Consider the linear ordinary differential equation:

$$y'(x) = \sin^2(x) + xe^x - y, \quad y(0) = 1$$

where the exact solution is given by:

$$y(x) = \frac{13}{20}e^{-x} + \frac{1}{10}\cos(2x) + \frac{1}{5}\sin(2x) + (\frac{x}{2} - \frac{1}{4})e^x + \frac{1}{2}.$$

Suppose that a two-step method must be constructed in such away that it is exact for $y \in \prod_4$

To construct such a method via G-spline, we shall use the example in chapter two section (2.6) such that:

$\Delta = \{0, 1, 2\}$.

be the knot points and let:

e={(0,0), (0,1), (1,0), (1,1), (2,0),(2,1)}.

to seek for $S_4(s) \in S_4(E^*, \Delta)$ with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Now, using the eq.(3.13) over [1,2] and formulated it to predictor and corrector equations , yields:

$$y^p_{n+2} = y^c_{n+1} + h\{\beta_{00}f(x_n, y^c_n) + \beta_{10}f(x_{n+1}, y^c_{n+1}) + \beta_{20}f(x_{n+2}, y^p_{n+2})\}$$
$$+ h^2\{\beta_{01}f'(x_n, y^c_n) + \beta_{11}f'(x_{n+1}, y^c_{n+1}) + \beta_{21}f'(x_{n+2}, y^p_{n+2})\}.$$

……………..…….……... (3.23)

$$y^c_{n+2} = y^c_{n+1} + h\{\beta_{00}f(x_n, y^c_n) + \beta_{10}f(x_{n+1}, y^c_{n+1}) + \beta_{20}f(x_{n+2}, y^p_{n+2})\}$$
$$+ h^2\{\beta_{01}f'(x_n, y^c_n) + \beta_{11}f'(x_{n+1}, y^c_{n+1}) + \beta_{21}f'(x_{n+2}, y^p_{n+2})\}.$$

..............………………..… (3.24)

where

$$\beta_{00} = \int_1^2 L_{00}(s)ds = 0.098.$$

$$\beta_{01} = \int_1^2 L_{01}(s)ds = -0.052.$$

$$\beta_{10} = \int_1^2 L_{10}(s)ds = 0.522.$$

$$\beta_{11} = \int_1^2 L_{11}(s)ds = 0.146.$$

$$\beta_{20} = \int_1^2 L_{20}(s)ds = 0.442.$$

$$\beta_{21} = \int_1^2 L_{21}(s)ds = -0.062.$$

And since f(x,y)= $\sin^2(x) + xe^x - y$ then:

$$f'(x,y) = \frac{-4}{5}\sin(2x) + (\frac{x}{2} + \frac{3}{4})e^x - \frac{2}{5}\cos(2x) + \frac{13}{20}e^{-x}.$$

If $L_{ij}(s)$ where $(i,j) \in e$ are given as in example of section (2.6).

Table (3.6) illustrate the numerical results obtained by applying eq.(3.23) and (3.24) and its comparison with the exact solution .

Table (3.6)

The numerical result of example (3.3.4).

x_i	y_i^p	y_i^c	y(x)	Absolute error
0	1	1	1	0
0.2	1.0139	1.014	1.019	0.0003
0.4	1.0712	1.071	1.074	0.0002
0.6	1.1706	1.170	1.17	0.0002
0.8	1.3272	1.327	1.323	0.0006
1	1.5703	1.57	1.559	0.0002
1.2	1.9399	1.94	1.919	0.0003
1.4	2.4932	2.493	2.458	0.0003
1.6	3.2990	3.299	3.244	0
1.8	4.4462	4.447	4.362	0.0003
2	6.0394	6.039	5.913	0.0002

3.4 BOUNDARY VALUE PROBLEMS:

One of the most important mathematical models in mathematical physics is the boundary value problems (in short BVP's), in which the governing equation is either ordinary or partial differential equation, where boundary condition are given at least at two points.

An important form of BVP's is the Sturm-Liouville BVP, given by:

$$\frac{d}{dx}[P(x)y'(x)] + [Q(x) + \lambda R(x)]y(x) = 0 \, , \, x \in [a,b] \qquad \text{……... (3.25)}$$

with homogenous boundary conditions:

$$\begin{aligned} a_{11}y(a) + a_{12}y'(a) = 0 \\ a_{21}y(b) + a_{22}y'(b) = 0 \end{aligned} \qquad \text{…………………………………..... (3.26)}$$

where $a_{ij}, \forall i, j = 1,2$ are prescribed constants, and P, P′ ,Q′ and R are continuous function on the interval $a \le x \le b$, furthermore P(x)>0 and R(x)>0 for all $x \in [a, b]$, and λ is a constant.

The BVP given by eqs.(3.25) and (3.26) is the prototype of a large class of problems, which is called the Sturm-Liouville problems, which plays an important rule in mathematics.

A parameter λ for which eq.(3.25) has an trivial solution is called eigen value, and the corresponding solution is an eigen function of Sturm-Liouville problem. The problem of evaluation such values of λ and y(x) is called eigen value problem, [Sagan, 1961], [Hildebrand, 1976], [Zhidkov, 2000].

3.4.1 The Shooting Method for Solving Linear BVP's:

The shooting method for solving linear ordinary differential equation is based on the replacement of the boundary value problem by its two related initial value problems, as in usual case for solving boundary value problems.

Now, consider the linear second order boundary value problem:

$$y'' = p(x)y' + q(x)y + r(x)\,,\ a < x < b \qquad \ldots\ldots (3.27)$$

$$y(a) = \alpha\,,\ y(b) = \beta \qquad \ldots\ldots (3.28)$$

Satisfying:

(i) p(x), q(x) and r(x) are continuous on [a,b].

(ii) q(x)>0 on [a,b].

hence, the related two initial value problems are given by:

$$u'' = p(x)u' + q(x)u\,,\ a \le x \le b\,,\ u(a) = 0\,,\ u'(a) = 1 \qquad \ldots\ldots (3.29)$$

and

$$v'' = p(x)v' + q(x)v + r(x)\,,\ a \le x \le b\,,\ v(a) = \alpha\,,\ v'(a) = 0 \qquad \ldots\ldots (3.30)$$

the solution of the BVP can be obtained to be:

$$y(x) = v(x) + \mu u(x) \qquad \ldots\ldots (3.31)$$

where:

$$\mu=\frac{\beta - v(b)}{u(b)} \ , \ u(b) \neq 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (3.32)$$

If u(b)=0 then one can use the finite different method to solve the system of boundary value problem.

In order to solve the BVP using G-spline functions one can incorporate the method of LMM using G-spline functions discussed in section (3.1) to solve the related two initial value problems (3.29) and (3.30), and one can show that in the next example:

Example (3.4.1):

Consider the non-homogeneous BVP:

$$y'' = 2y' - y + (x^2 - 4x + 2) \, , \, 0 \le x \le 2$$

y(0)=1, y(2)=2.

where the exact solution is given by:

$$y(x) = e^x - 0.635xe^x + x^2$$

Now, to solve the homogeneous problem:

$$u'' = 2u' - u \, , \quad u(0) = 0 \, , \, u'(0) = 1 \, .$$

Let $u_1 = u$, then $u' = u_2 = f_1(x,u_1,u_2)$ and so $u_2' = 2u_2 - u_1 = f_2(x,u_1,u_2)$.

$$u_1(0) = 0, u_2(0) = 1 \, .$$

Therefore in matrix form:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Also, for the non-homogeneous problem:

$$v'' = 2v' - v + (x^2 - 4x + 2) \, , \, v(0) = 1 \, , \, v'(0) = 0$$

Let $v_1 = v$, then $v' = v_2 = g_1(x, v_1, v_2)$ and so $v'_2 = 2v_2 - v_1 + (x^2 - 4x + 2)$ such that $v'_2 = g_2(x, v_1, v_2)$.

$$v_1(0) = 1, v_2(0) = 0.$$

Therefore in matrix form:

$$\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x^2 - 4x + 2 \end{bmatrix}$$

Now:

$$\mu = \frac{2 - v(2)}{u(2)}$$

$$= 0.365.$$

Now, we can use eq. (3.13) of linear multistep method and formulated it to solve the BVP's, we have:

$$u^1_{n+k} = u^1_{n+q} + \sum_{(i,j)\in e} h^{j+1} \beta_{ij} f_1^{(j)}(x_{n+i}, u^1_{n+i}, u^2_{n+i}).$$

and

$$u^2_{n+k} = u^2_{n+q} + \sum_{(i,j)\in e} h^{j+1} \beta_{ij} f_2^{(j)}(x_{n+i}, u^1_{n+i}, u^2_{n+i}).$$

and the same work for:

$$v^1_{n+k} = v^1_{n+q} + \sum_{(i,j)\in e} h^{j+1} \beta_{ij} g_1^{(j)}(x_{n+i}, v^1_{n+i}, v^2_{n+i}).$$

and

$$v^2_{n+k} = v^2_{n+q} + \sum_{(i,j)\in e} h^{j+1} \beta_{ij} g_2^{(j)}(x_{n+i}, v^1_{n+i}, v^2_{n+i}).$$

Now, by using the same information of example (3.3.1), we have the following table:

Table (3.7)

The results of BVP by using G-spline functions.

x_i	u_1	u_2	v_1	v_2	y_1	y_2
0	0	1	1	0	1	0.365
0.1	0.1	1.2	1	0.1	1.036	0.538
0.2	0.237	1.459	1.013	0.178	1.1	0.711
0.3	0.404	1.760	1.034	0.231	1.181	0.873
0.4	0.604	2.110	1.059	0.254	1.279	1.024
0.5	0.844	2.516	1.085	0.242	1.393	1.16
0.6	1.129	2.986	1.109	0.189	1.521	1.279
0.7	1.467	3.529	1.125	0.091	1.66	1.379
0.8	1.866	4.156	1.128	-0.063	1.809	1.454
0.9	2.335	4.878	1.113	-0.280	1.965	1.5
1	2.885	5.708	1.072	-0.570	2.125	1.513
1.1	3.528	6.663	0.997	-0.943	2.285	1.489
1.2	4.277	7.759	0.878	-1.413	2.439	1.419
1.3	5.150	9.015	0.706	-1.995	2.586	1.295
1.4	6.162	10.455	0.466	-2.703	2.715	1.113
1.5	7.336	12.102	0.146	-3.559	2.824	0.858
1.6	8.693	13.986	-0.270	-4.582	2.903	0.523
1.7	10.261	16.138	-0.804	-5.796	2.941	0.094
1.8	12.069	18.594	-1.474	-7.231	2.931	-0.444
1.9	14.151	21.397	-2.307	-8.917	2.858	-1.107
2	16.545	24.592	-3.330	-10.889	2.709	-1.913

Table (3.8)

The results of BVP using Euler method.

x_i	u_1	u_2	v_1	v_2	y_1	y_2
0	0	1	1	0	1	0.365
0.1	0.1	1.2	1	0.1	1.036	0.538
0.2	0.22	1.43	1.01	0.181	1.09	0.703
0.3	0.363	1.694	1.028	0.24	1.16	0.858
0.4	0.532	1.996	1.052	0.274	1.246	0.976
0.5	0.732	2.343	1.080	0.28	1.347	1.135
0.6	0.966	2.738	1.108	0.253	1.461	1.252
0.7	1.24	3.189	1.133	0.189	1.586	1.353
0.8	1.559	3.703	1.152	0.083	1.721	1.435
0.9	1.929	4.287	1.16	-0.072	1.864	1.493
1	2.358	4.952	1.153	-0.282	2.014	1.525
1.1	2.853	5.706	1.125	-0.553	2.166	1.53
1.2	3.424	6.562	1.069	-0.895	2.319	1.5
1.3	4.08	7.532	0.98	-1.317	2.469	1.432
1.4	4.833	8.631	0.848	-1.83	2.612	1.32
1.5	5.696	9.873	0.665	-2.444	2.744	1.16
1.6	6.684	11.279	0.421	-3.175	2.861	0.942
1.7	7.811	12.866	0.103	-4.036	2.954	0.66
1.8	9.098	14.658	-0.3	-5.044	3.021	0.306
1.9	10.564	16.68	-0.805	-6.219	3.051	-0.131
2	12.232	18.959	-1.427	-7.581	3.038	-0.661

A comparison between the obtained results using G-spline interpolation method and using the Euler's method are illustrated in figs.(3.1) and (3.2).

Figer (3.1)

The comparison of y_1.

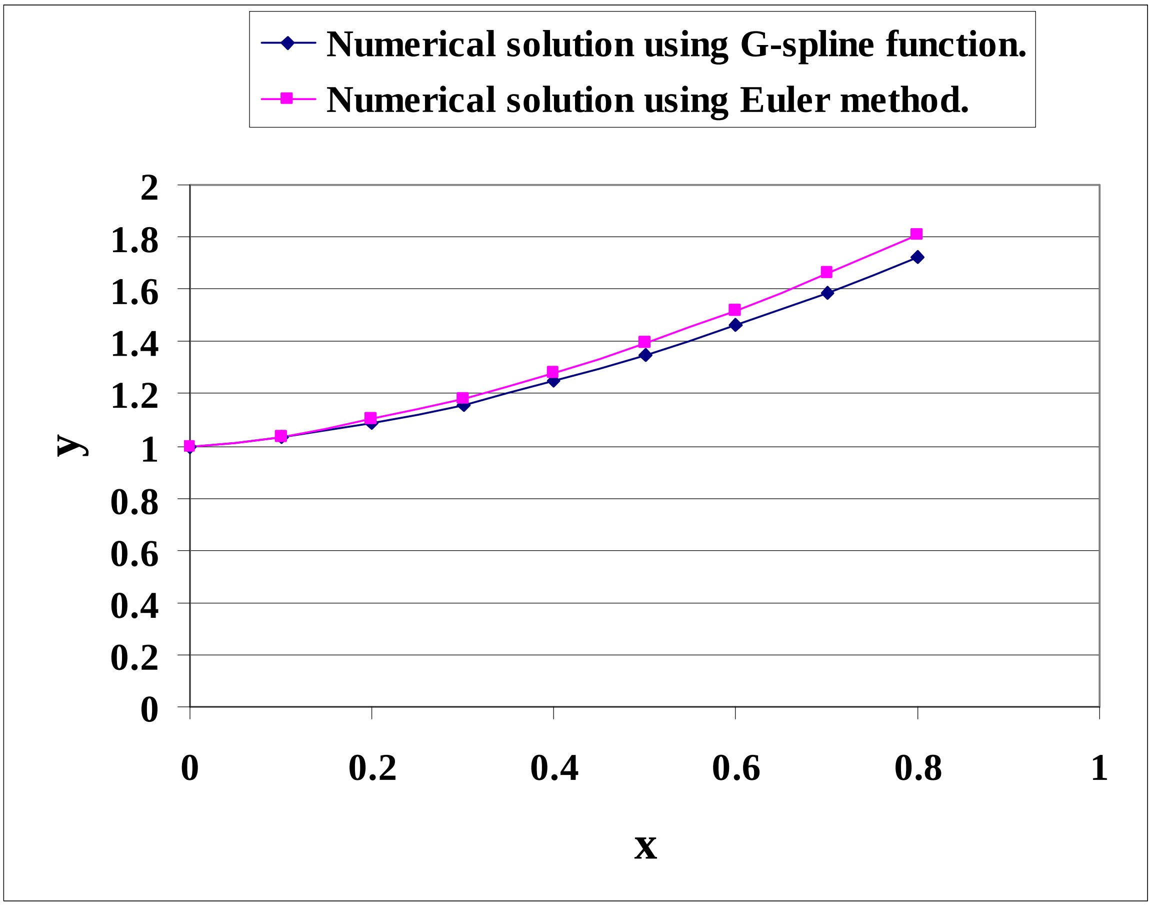
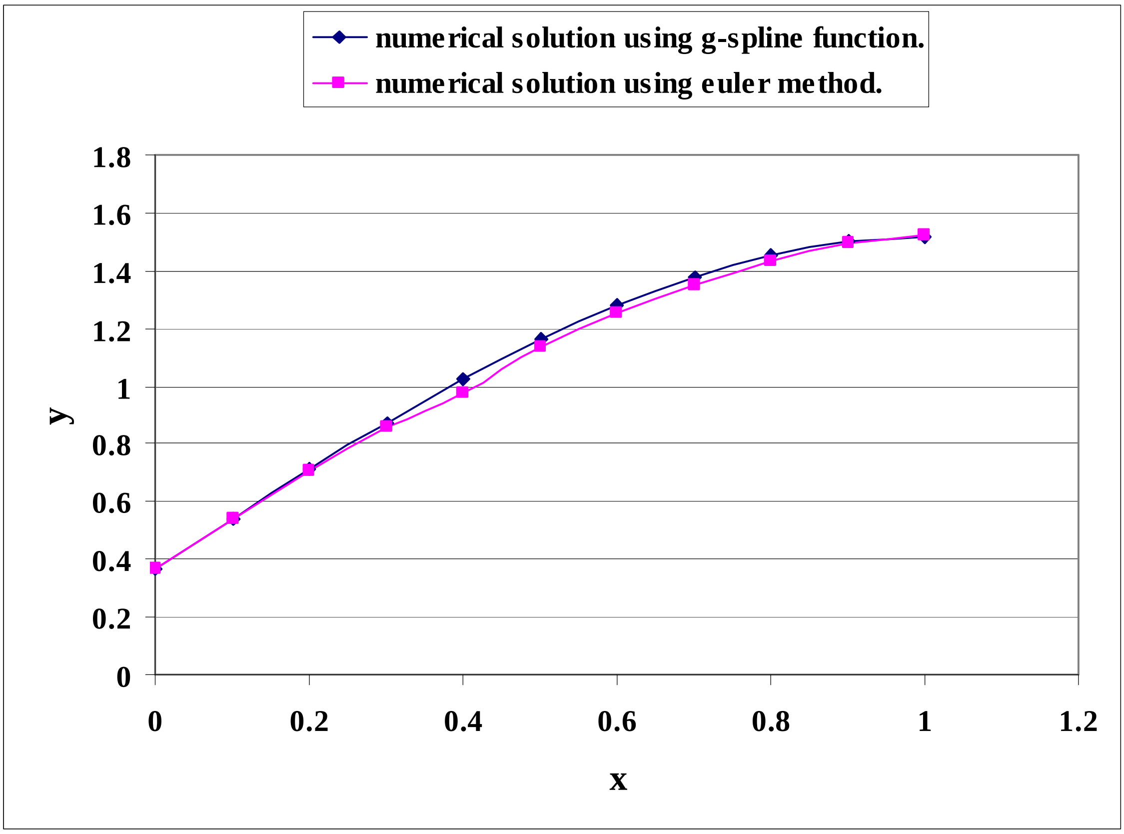


Figure (3.2)

***The comparison of** y_2.*



Conclusion and Recommendations

From the present study we may conclude the following:

1. The numerical results obtained using implicit 3-step methods are more accurated than those obtained by using implicit 2-step method.
2. The predictor-corrector method “with two correctors” give more accurate result than those obtained without corrector method “as its expected”.

Also, we may recommend the following problems for future work as open problems:

1. Using Rung-kutta methods to solve ordinary differential equations based on G-Spline interpolation methods.
2. Using G-spline interpolation functions to solve partial differential equations.
3. Using other spline interpolation methods to solve ordinary differential equations numerically by using linear multistep methods.

References

1. Ahlberge J. H., Nilson, E. N. and Walsh, J. L., “The theory of splines and their applications”, Academic press, New York, 1967.
2. Birkhoff G. D., “General Mean Value and Remainder Theorems with Applications to Mechanical Differentiation and Integration”, Trans. Amer. Math. Soc. 7 (1906), 107-136.
3. Byren G. D. and Chi D. H., “Linear Multistep Formulas Based on G-Splines”, Siam J. Numer. Anal. , Vol.9, No. 2, June 1972.
4. deBoor, C., “A practical guide to splines”, Springer-Verlag, New York, Inc., 1978.
5. Greville T. N., “Spline functions, interpolation and numerical quadrature”, A chapter in: Mathematical methods for digital computers, Vol.2, (Ralston A., and Wilf, H. S., eds.), Wiley, New York, 1967.
6. Hildebrond F. B., “Advanced Calculus for Approximations”, second Edition, Prentice-Hall, Inc., 1976.
7. Lambert J. D., “Computational Methods in Ordinary Differential Equations”, John Wiley &Sons, New York, U.S.A, 1973.
8. Micula G., “A variational Approach to Spline Functions Theory”, Rend. Sem. Mat. Univ. Pol. Torino, Vol. 61, 3(2003), Splines and Radial Functions.

.

9. Osama H. M., “Functions Approximation Using G-Spline and Its Generalization to two Dimensional Spaces”,PHD. Thesis, College of Science, Al-Nahrain University, 2006.

10. Osama H. M., “G-Spline Interpolation for Approximating the Solution of Fractional Differential Equations Using Linear Multistep Methods”, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2007.

11. Rainvill E. D., “Elementary Differential Equations”, Seven Edition, Macmillan Publishing Company, New York, U.S.A, Collier Macmillan Publishers, London, U.K, 1989.

12. Sagan H., “Boundary and Eigenvalue Problems in Mathematical Physics”, John Wiley and Sons, Inc., New York, 1961.

13. Sard A., “Linear approximation”, American Mathematical Society, Providence, 1963.

14. Schoenberg I. J., “Contribution to the problem of approximation of equi-distant data by analytic functions”, Parts A and B, Quart Appl. Math., 4, (1946), pp.45-99.

15. Schoenberg I. J., “On best approximation of linear operators”, Nedral. Akad. Wetensch. Indag. Math, 26 (1964), pp.155-163.

16. Schoenberg I. J., “On interpolation by spline functions and its minimal properties”, On approximation theory (proceedings of the conference

heald in the mathematical research institute at Oberwolfach, Black Forest, Augest, p.4-10, 1963), Birkhäuser Verlag, Basel, 1964.

17. Schoenberg I. J., "On the Ahlberg-Nilson extension of spline interpolation: the G-spline and their optimal properties, J. Math. Anal. Appl., 21 (1968), pp.207-231.
18. Stephen W., "An introduction to the mathematics and construction of splines", Addix software consultancy limited, Version 1.6, September, 2002.
19. Zhidkov P. E., "Basic Properties of Eigenfunctions of Non Linear Sturm-Liouville Problems", Electronic Journal of Differential Equations, Vol. 2000, PP. 1-13, edge. Math. Swt.edu, 2000.

المستخلص

الهدف الرئيسي من هذه الاطروحة هو دراسة تقريب الدوال بأستخدام نوع خاص من دوال السبلاين "Spline Functions"، والتي تدعى بدوال السبلاين – G ، حيث تضمنت الدراسة تفاصيل موضوع التقريب.

الهدف الثاني للأطروحة هو لدراسة الحلول التقريبية للمعادلات التفاضلية من الرتبة الاولى وبالصيغة:

$$y'(x) = F(x, y) \,,\; x \in [a, b]$$

$$y(a) = y_0$$

حيث كانت الدراسة، ايجاد الحلول العددية باستخدام طرائق متعددة الخطوات والمعتمدة على دوال التقريب من النوع سبلاين – G ومن ثم اعمام الاسلوب المتبع لدراسة حلول المعادلات التفاضلية الحدودية من الرتب الثانية.

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دوال السبلاين-G لتقريب حلول المعادلات التفاضلية الأعتيادية باستخدام طرائق متعددة الخطوات.

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