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Some Approximated Methods for Solving Fractional Non-Local Problems

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By

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﴿۞وَمَن يَتَّقِ اللَّهَ يَجْعَل لَّهُ مَخْرَجًا۞ وَيَرْزُقْهُ مِنْ حَيْثُ لَا
يَحْتَسِب وَمَن يَتَوَكَّلْ عَلَى اللَّهِ فَهُوَ حَسْبُهُ إِنَّ اللَّهَ بَالِغُ أَمْرِهِ
قَدْ جَعَلَ اللَّهُ لِكُلِّ شَيْءٍ قَدْرًا۞۞﴾

صدق اللّٰه العظيم

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Content

Abstract

The aim of this work is to study some types of the fractional non-local problems. These types are the fractional non-local initial value problems that consist of the non-linear fractional Fredholm, Volterra, Fredholm-Volterra integro-differential equations together with the non-linear non-local initial conditions of the integral type.

This study includes the existence and the uniqueness of the solution for the non-linear fractional Fredholm-Volterra integro-differential equations together with the non-linear non-local initial conditions.

Also, Laplace transform method is used to solve special types of the linear fractional non-local problems with some illustrative examples.

Moreover, the generalized Taylor expansion method is presented to solve the non-local initial value problems that consist of the linear fractional Fredholm-Volterra integro-differential equations together with the linear non-local initial conditions with some illustrative examples.

Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics, [28].

Many authors and researchers concerned with the fractional problems such as Domenico and Luigi in 1996, [21] proved the existence and uniqueness theorems for some classes of nonlinear fractional differential equations by using Schauder fixed point theorem ,.Al-shather in 2003, [8] presented some approximated solutions for solving the fractional delay integro-differential equations, Abdul-Razzak in 2004, [2] gave some algorithms for solving fractional Fredholm integro- differential equations, Al Azawi in 2004, [6] gave results in fractional calculas, Al-Rahhal D.in 2005, [7] used some numerical methods for solving the fractional integro differential equations, Mohamed in 2007, [30] used the finite difference methods for solving fractional differential equations, Abdul Sattar in 2008, [3] gave some solutions of fractional boundary value problems, Mehdi and Majid in 2010, [29] presented some definitions of fractional derivatives and fractional Integrals and gave more explicit formulas of fractional derivative and integral of some special functions and presented some applications of the theory of fractional calculus, Azizollah, Dumetra and Ravi in 2013, [10] proved the existence and uniqueness of solutions for two classes of infinite delay nonlinear fractional order differential equations involving Riemann-Liouville fractional derivatives, Armand and Mohammadi in 2014, [9] discussed existence the and uniqueness of solutions of nonlinear differential equations of fractional order with fuzzy initial condition by using contraction mapping principle and the fixed point theorem.

Another subject that deals with this work is the non-local problems that is the problems with non-local conditions. Many researchers concerned with the nonlocal problem such as Kerefov in1979, [27] studied the nonlocal boundary value problems for the parabolic differential equations, Chabrowski in 1984, [17] studied the nonlocal initial value problem for the parabolic differential equations, Chabrowski in 1988, [18], the existence and uniqueness of solutions of the nonlocal problem for the linear elliptic equation with nonlocal condition, Byszewski in 1991, [15] studied theorems about the existence and uniqueness of solutions of a semi linear evolution nonlocal Cauchy problems, Pulkina in 1999, [37] used the Schauder fixed point theorem to prove the existence of the linear second order hyperbolic equation with the linear integral conditions, Abdelkader in 2003, [1] discussed the existence and uniqueness for the solutions of the nonlocal initial value problems for the non-linear ordinary differential equations, Saadatmandi and Dehgan in 2006, [39] used the shifted Legender technique for solving the one-dimentional wave equation with the one nonlocal linear integral boundary condition, Svajunas in 2010, [43] used the finite difference method to find the solution of the two-dimentional heat equation with the nonlocal linear integral condition, Kahtan in 2013, [25] used the finite difference method to solve special types of nonlocal problems for partial differential equations.

The fractional nonlocal Problems have been studied by several researchers such as, Symotyuk in 2001, [44] investigated Conditions for the existence and uniqueness of a classical solution of a nonlocal boundary-value problem for a differential equation with a regularized Riemann–Liouville fractional time derivative with variable coefficients, Mophou in 2009, [32] proved the existence of mild solutions to the Cauchy Problem for the fractional differential equation with nonlocal conditions, Xiwang in 2011, [47] studied the existence and uniqueness of solutions to the nonlocal problems for the fractional differential equation in Banach spaces, Ahmad in 2012, [4] studied the existence of solutions of the class of nonlinear Cupoto type fractional boundary value problems with nonlocal fractional integro-differential boundary conditions, Ahmad in 2013, [5] proved the existence of solutions of a nonlocal boundary value problem for nonlinear fractional order integro-differential equations.

The purpose of this work is to study some types of the fractional non-local initial value problems for the non-linear fractional Fredholm-Volterra integro-differential equations together with the non-linear non-local initial conditions of the integral type.

This thesis consists of three chapters.

In chapter one we use the Banach fixed point theorem to discuss the existence and the uniqueness of the solution for the non-linear fractional Fredholm-Volterra integro-differential equations together with the non-linear non-local initial conditions.

In chapter two, we use the Laplace transform method to solve special types of the non-local initial value problems for the linear fractional Volterra integro-differential equations of the difference kernel.

In chapter three, we devote the generalized Taylor expansion method for solving the linear non-local initial value problems for the linear fractional Fredholm-Volterra integro-differential equations together with the linear non-local initial conditions. All computations that appeared in this work are obtained by using the Mathcad software package.

Introduction:

The non-local condition is the condition which appears when values of the function on the boundary or on the initial are connected to values inside the domain. Every problem with non-local condition is termed as non-local problem, [13].

The nonlocal problems play an important role in many real life applications and they arise in various fields of mathematical physics (like string oscillation telegraph equation), [24], biology and biotechnology (like evolution of dominant genes and propagation nerve pluses), [33], and in other fields.

The aim of this chapter is study special types of the non-local problems, namely the non-local initial value problems. This study includes the non-local initial value problems that consist of the non-linear fractional Fredholm-Volterra integro-differential equations together with the non-linear non-local initial conditions of the integral type.

This chapter consists of two sections:

In section one, we give some fundamental concepts of fractional calculus.

In section two, we use the Banach fixed point theorem to prove the existence and uniqueness of the solutions for special types of non-local initial value problems of the non-linear fractional Fredholm-Volterra integro-differential equations.

1.1 Some Fundamental Concepts of Fractional Calculus:

In this section, we give some basic concepts and definitions related to the subject of fractional calculus.

We start this section by giving the following definitions:

Definition (1.1), [35]:

The Gamma function Γ of a positive real number x, is defined by:

$$\Gamma(x)=\int_0^\infty y^{x-1}e^{-y}dy,\quad x>0$$

Some of the most important properties of the Gamma function are listed below, [35]:

(i) $\Gamma(n+1)=n!,\ n\in N.$

(ii) $\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!\sqrt{\pi}}{4^n n!},\ n\in N.$

(iii) $\Gamma(x+1)=x\Gamma(x),\quad x>0.$

Definition (1.2), [35]:

The Beta function β with positive parameters p and q is defined by:

$$\beta(p,q)=\int_0^1 y^{p-1}(1-y)^{q-1}dy,\quad p>0,\ q>0$$

If either p or q is non-positive real number, the above integral diverges.

One of the most important properties of the Beta function is, [35]:

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\quad p>0,\ q>0.$$

Definitions (1.3), [22]:

The Mittag-Leffler function is defined by:

$$E_{\alpha}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(\alpha i+1)},\quad \alpha, z\in C,\ \mathrm{Re}(\alpha)\geq 0$$

and the generalized Mittag-Leffler function is defined by:

$$E_{\alpha,\beta}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(\alpha i+\beta)},\quad \alpha, \beta, z\in C,\ \mathrm{Re}(\alpha)\geq 0$$

Examples (1.4):

(1) $E_{0}(z)=E_{0,1}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(1)}=\sum_{i=0}^{\infty}z^{i}=\frac{1}{1-z},\quad |z|<1.$

(2) $E_{1,0}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(i)}=\sum_{i=0}^{\infty}\frac{z^{i}}{(i-1)!}=ze^{z},\ z\in C.$

(3) $E_{1}(z)=E_{1,1}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(i+1)}=\sum_{i=0}^{\infty}\frac{z^{i}}{i!}=e^{z},\quad z\in C.$

(4) $E_{1,2}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(i+2)}=\frac{e^{z}-1}{z},\ z\in C\backslash\{0\}.$

Remark (1.5):

Gamma, Beta, Mittag-Leffler and the generalized Mittag-Leffler functions are one of the most important notations in fractional calculus, since they play an important role in fractional differentiation and integration formulas.

Definitions (1.6), [35]:

Let u be an absolutely continuous function on [a,b], the left and the right hand Riemman-Liouville fractional derivatives of u of order $\alpha > 0$, can be defined as:

$$ {}_xD^{\alpha}_{a^+}u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{u(y)}{(x-y)^{\alpha-n+1}} dy, \quad a \le x \le b $$

and

$$ {}_xD^{\alpha}_{b^-}u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{u(y)}{(x-y)^{\alpha-n+1}} dy, \quad a \le x \le b $$

respectively, where $n-1 < \alpha \le n$, $n \in N$.

To illustrate the definitions of Riemman-Liouville fractional derivatives, consider the following example:

Example (1.7):

Let $u(x) = 2x$, $\alpha = \frac{1}{2}$ and $0 \le x \le 1$. The left and the right hand Riemman-Liouville fractional derivatives of u of order $\frac{1}{2}$ are:

$$ {}_xD^{\frac{1}{2}}_{0^+}u(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx} \int_0^x \frac{2y}{(x-y)^{\frac{1}{2}}} dy = 4\sqrt{\frac{x}{\pi}} $$

and

$$ {}_xD^{\frac{1}{2}}_{1^-}u(x) = \frac{-1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx} \int_x^1 \frac{2y}{(x-y)^{\frac{1}{2}}} dy = \frac{2(2x+1)}{3\sqrt{\pi}\sqrt{x-1}} + \frac{8\sqrt{x-1}}{3\sqrt{\pi}} $$

respectively.

Next, we give the left hand Riemman-Liouville fractional derivative of the power function. This proposition is very important to find the left hand Riemman- Liouville fractional derivative of any analytic function.

Proposition (1.8), [36]:

$$ {}_{x}D_{0^+}^{\alpha} x^{p} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, \quad x>0,\ p>-1. $$

Proof:

It is known that

$$ {}_{x}D_{0^+}^{\alpha} x^{p} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x y^p (x-y)^{n-\alpha-1} dy $$

Let $y=\lambda x,$ then the above equation reduces to:

$$ \begin{aligned} {}_{x}D_{0^+}^{\alpha} x^{p} &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[\int_0^1 (\lambda x)^p ((1-\lambda)x)^{n-\alpha-1} x d\lambda \right] \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[x^{p+n-\alpha} \int_0^1 \lambda^p (1-\lambda)^{n-\alpha-1} d\lambda \right] \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[x^{p+n-\alpha} \beta(p+1, n-\alpha) \right] \\ &= \frac{\Gamma(p+1)\Gamma(n-\alpha)}{\Gamma(n-\alpha)\Gamma(n-\alpha+p+1)} \frac{d^n}{dx^n} \left[x^{p+n-\alpha} \right] \end{aligned} $$

Thus

$$ {}^{c}D_{0^+}^{\alpha} x^p = \frac{\Gamma(p+1)\Gamma(p+n-\alpha+1)}{\Gamma(n-\alpha+p+1)\Gamma(p+n-\alpha-n+1)} x^{p+n-\alpha-n} $$

$$ = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, \ x>0. $$

Now, the following example gave the left hand Riemman-Liouville fractional derivative of the exponential function via proposition (1.8)

Example (1.9):

Let $u(x) = e^{\lambda x}$, $\lambda \in C$, $x>0$. The left hand Riemman-Liouville fractional derivative of u of order α is:

$$ {}_{x}D_{0^+}^{\alpha}\left(e^{\lambda x}\right) = {}_{x}D_{0^+}^{\alpha}\left(\sum_{i=0}^{\infty} \frac{(\lambda x)^i}{i!}\right) = \sum_{i=0}^{\infty} \frac{(\lambda)^i}{i!} \ {}_{x}D_{0^+}^{\alpha} x^i. $$

By using proposition (1.8), the above equation becomes

$$ {}_{x}D_{0^+}^{\alpha}\left(e^{\lambda x}\right) = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}\left(\frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)} x^{i-\alpha}\right) = x^{-\alpha} \sum_{i=0}^{\infty} \frac{(\lambda x)^i}{\Gamma(i-\alpha+1)} = x^{-\alpha} E_{1,1-\alpha}(\lambda x). $$

Remark (1.10):

If α=m, where m is a positive integer, then we choose n=m+1, so

$$ {}_{x}D_{a^+}^{m} u(x) = \frac{1}{\Gamma(1)} \frac{d^{m+1}}{dx^{m+1}} \int_a^x \frac{u(y)}{(x-y)^{m-(m+1)+1}} dy $$

$$ = \frac{d^{m+1}}{dx^{m+1}} \int_a^x u(y) dy $$

Thus

$${}_xD^m_{a^+}u(x) = \frac{d^m}{dx^m}u(x)$$

and

$${}_xD^m_{b^-}u(x) = \frac{(-1)^{m+1}}{\Gamma(m+1-m)}\frac{d^{m+1}}{dx^{m+1}}\int_x^b \frac{u(y)}{(x-y)^{m-(m+1)+1}}dy$$

$$=(-1)^{m+1}\frac{d^{m+1}}{dx^{m+1}}\int_x^b u(y)dy$$

$$=(-1)^m \frac{d^m}{dx^m}u(x).$$

Next, another definition of the fractional derivative is given by the Italian mathematician Caputo in1967.

Definitions (1.11) , [16]:

Let u be an absolutely continuous function on [a,b], the left and the right hand Caputo fractional derivatives of u of order $\alpha > 0$, can be defined as:

$${}^C D^\alpha_{a^+}u(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^x \frac{u^{(n)}(y)}{(x-y)^{\alpha-n+1}}dy, & n-1<\alpha<n,\ n\in N,\ a\le x\le b \\ \dfrac{d^n u(x)}{dx^n}, & \alpha=n,\ n\in N,\ a\le x\le b \end{cases}$$

and

$${}^C D^\alpha_{b^-}u(x) = \begin{cases} \dfrac{(-1)^n}{\Gamma(n-\alpha)}\displaystyle\int_x^b \frac{u^{(n)}(y)}{(x-y)^{\alpha-n+1}}dy, & n-1<\alpha<n,\ n\in N,\ a\le x\le b \\ (-1)^n\dfrac{d^n u(x)}{dx^n}, & \alpha=n,\ n\in N,\ a\le x\le b \end{cases}$$

respectively.

To illustrate the definitions of Caputo fractional derivatives, consider the following example:

Example (1.12):

Let $u(x)=x^2,\ 0\le x\le 2$ and $\alpha=\frac{3}{2}$. The left and right hand Caputo fractional derivatives of u of order $\frac{3}{2}$ are:

$$ {}^{C}D_{0^+}^{\frac{3}{2}}u(x)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{\frac{d^2}{dy^2}(y^2)}{(x-y)^{\frac{3}{2}-2+1}}dy=4\sqrt{\frac{x}{\pi}} $$

and

$$ {}^{C}D_{2^-}^{\frac{3}{2}}u(x)=\frac{(-1)^2}{\Gamma\left(\frac{1}{2}\right)}\int_x^2 \frac{\frac{d^2}{dy^2}(y^2)}{(x-y)^{\frac{3}{2}-2+1}}dy=-4\sqrt{\frac{x-2}{\pi}} $$

respectively.

Next, we give the left hand Caputo fractional derivative of the power function. This proposition appeared in [19]. Here we give the details of its proof.

Proposition (1.13):

$$ {}^{C}D_{0^+}^{\alpha} x^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, & n-1<\alpha<n,\ p>n-1,\ \ p\in \Re \\ \\ 0, & n-1<\alpha<n,\ p\le n-1,\ \ p\in Z \end{cases} $$

Proof:

If $n-1<\alpha<n,\ p\le n-1,\ \ p\in N$, then $\dfrac{d^n}{dx^n}x^p=0$ and this imples that ${}^{C}D_{0^+}^{\alpha} x^p=0.$

If n−1 < α < n, p > n−1, p ∈ R, then

$$ {}^{C}D_{0^+}^{\alpha} x^p = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{\frac{d^n}{dy^n}(y^p)}{(x-y)^{\alpha-n+1}}dy $$

$$ =\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{\Gamma(p+1)}{\Gamma(p-n+1)} y^{p-n}(x-y)^{n-\alpha-1}dy. $$

Let $y=\lambda x,$ then the above equation reduces to:

$$ {}^{C}D_{0^+}^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}\int_0^1 (\lambda x)^{p-n}\,((1-\lambda)x)^{n-\alpha-1}x\,d\lambda $$

$$ =\frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} x^{p-n+n-\alpha-1+1}\int_0^1 (\lambda)^{p-n}\,(1-\lambda)^{n-\alpha-1}d\lambda $$

$$ =\frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} x^{p-n+n-\alpha-1+1}\beta(p-n+1,n-\alpha) $$

Thus

$$ {}^{c}D_{0^+}^{\alpha} x^{p} = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} x^{p-n+n-\alpha-1+1} \beta(p-n+1, n-\alpha) $$

$$ = \frac{\Gamma(p+1)\Gamma(p-n+1)\Gamma(n-\alpha)}{\Gamma(n-\alpha)\Gamma(p-n+1)\Gamma(p-\alpha+1)} x^{p-\alpha} $$

$$ = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}. $$

Remark (1.14):

The left hand Caputo fractional operator ${}^{C}D_{a^+}^{\alpha}$ does not satisfy the semigroup property, that is ${}^{C}D_{a^+}^{\alpha}\,{}^{C}D_{a^+}^{\beta} u(x) \neq {}^{C}D_{a^+}^{\alpha+\beta} u(x)$. To see this, let $u(x) = x^{\frac{1}{2}}$, $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$. To find ${}^{C}D_{0^+}^{\beta} x^{\frac{1}{2}} = {}^{C}D_{0^+}^{\frac{1}{2}} x^{\frac{1}{2}}$, we take n=1. So $p = \frac{1}{2} > n-1 = 0$. Therefore by using proposition (1.13) one can have

$$ {}^{C}D_{0^+}^{\frac{1}{2}} u(x) = {}^{C}D_{0^+}^{\frac{1}{2}} x^{\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma(1)} = \Gamma\left(\frac{3}{2}\right). $$

Hence

$$ {}^{C}D_{0^+}^{\frac{3}{2}} \left({}^{C}D_{0^+}^{\frac{1}{2}} x^{\frac{1}{2}} \right) = \Gamma\left(\frac{3}{2}\right) {}^{C}D_{0^+}^{\frac{3}{2}} (1). $$

Let p=0. To find ${}^{C}D_{0^+}^{\frac{3}{2}}(1)$, we take n=2. So $p = 0 < n-1 = 1$. Therefore by using proposition (1.13) one can have ${}^{C}D_{0^+}^{\frac{3}{2}} \left({}^{C}D_{0^+}^{\frac{1}{2}} x^{\frac{1}{2}} \right) = 0$. But

$$ {}^{C}D_{0^+}^{\frac{3}{2}+\frac{1}{2}}\left(x^{\frac{1}{2}}\right)=\frac{d^2}{dx^2}\left(x^{\frac{1}{2}}\right)=-\frac{1}{4}x^{-\frac{3}{2}}. $$

Therefore ${}^{c}D_{0^+}^{\frac{3}{2}}\,{}^{c}D_{0^+}^{\frac{1}{2}}(x^{\frac{1}{2}}) \neq {}^{c}D_{0^+}^{2}\left(x^{\frac{1}{2}}\right)$.

Remark (1.15):

Another types of definitions for fractional derivatives can be considered namely Hadamard fractional derivative, Erdelyi-Kober fractional derivative, Grunwald-Letnikov fractional derivative, [28], Thomas J. fractional derivative and Nishimoto fractional derivative, [34].

Definition (1.16) , [35]:

Let u be an absolutely continuous function on [a,b], the left and the right hand Riemman-Liouville fractional integrals of u of order $\alpha > 0$, can be defined as:

$$ I_{a^+}^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{u(y)}{(x-y)^{1-\alpha}}dy, \quad a\le x\le b $$

and

$$ I_{b^-}^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_x^b \frac{u(y)}{(y-x)^{1-\alpha}}dy, \quad a\le x\le b $$

respectively.

Remark (1.17):

If $\alpha=1$, then $I_{a^+}^{1}u(x)=\int_a^x u(y)dy, \quad a\le x\le b.$

To illustrate the above definitions of Riemman-Liouville fractional integrations, consider the following example:

Example (1.18):

Let $u(x)=x^3,\ \ 1\le x\le 3$ and $\alpha=\frac{3}{4}$. The left and the right Riemman-Liouville fractional integrals of u of order $\frac{3}{4}$ are:

$$I_{1^+}^{\frac{3}{4}}u(x)=\frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_1^x \frac{y^3}{(x-y)^{\frac{1}{4}}}dy$$

$$=\frac{4(x-1)^{\frac{3}{4}}\left[495x^2(x-1)-315x(x-1)^2-385x^3+77(x-1)^3\right]}{1155\,\Gamma\left(\frac{3}{4}\right)}$$

and

$$I_{3^-}^{\frac{3}{4}}u(x)=\frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_x^3 \frac{y^3}{(y-x)^{\frac{1}{4}}}dy$$

$$=\frac{(3-x)^{\frac{15}{4}}\left(512x^3+1152x^2+3024x+8316\right)}{1155\,\Gamma\left(\frac{3}{4}\right)\left(x^3-9x^2+27x-27\right)}$$

respectively.

Next, we give the left hand Riemman-Liouville fractional integral of the power function.

Proposition (1.19), [36]:

$$I_{0^+}^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} x^{p+\alpha}, \ p>-1, \ x>0.$$

Proof:

It is known that

$$I_{0^+}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x y^p (x-y)^{\alpha-1} dy, \quad x>0.$$

Let $y=\lambda x$, then the above equation reduces to:

$$I_{0^+}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (\lambda x)^p ((1-\lambda)x)^{\alpha-1} x d\lambda = \frac{1}{\Gamma(\alpha)} x^{p+\alpha} \int_0^1 (\lambda)^p (1-\lambda)^{\alpha-1} d\lambda$$

$$= \frac{1}{\Gamma(\alpha)} x^{p+\alpha} \beta(p+1,\alpha) = \frac{\Gamma(p+1)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(p+\alpha+1)} x^{p+\alpha} = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} x^{p+\alpha}, \quad x>0.$$

Remark (1.20), [34]:

Another types of definitions for fractional integrals can be considered namely Kalla and Saxena fractional integral, Kober fractional integral and Saxena fractional integral.

In the rest of this section , we give some properties related to the left hand Caputo fractional derivative, left hand Riemman-Liouville fractional derivative and left hand Riemman-Liouville fractional integral.

Remarks (1.21):

(1) $\ {}^{C}D_{a^+}^{\alpha} u(x) = I_{a^+}^{n-\alpha} \left[\frac{d^n u(x)}{dx^n} \right], \ n-1<\alpha<n,$[36].

(2) $${}_{x}D^{\alpha}_{a^+}u(x) = \frac{d^n}{dx^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{u(y)}{(x-y)^{1-(n-\alpha)}}dy\right]$$
$$= \frac{d^n}{dx^n}\left[I^{n-\alpha}_{a^+}u(x)\right], n-1<\alpha<n, \text{ [36]}.$$

(3) $${}^{C}D^{\alpha}_{a^+}\left(I^{\alpha}_{a^+}u(x)\right) = u(x), \text{ [20]}.$$

(4) $$I^{\alpha}_{a^+}\left({}^{C}D^{\alpha}_{a^+}u(x)\right) = u(x) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(x-a)^k,\ n-1<\alpha<n,\ n\in N, \text{ [27]}.$$

(5) $I^{\alpha}_{a^+}$ satisfy the semigroup property, [36].

1.2 Existence and Uniqueness of the Solutions for the Non-Linear Non-Local Initial Value Problems of Linear Fractional Integro-Differential Equations:

In this section we discuss the existence of the unique solution for special types of the non-linear non-local initial value problems for fractional integro-differential equations. To do this, first consider the linear non-local initial value problem that consists of the linear fractional Fredholm-Volterra integro-differential equation of order α:

$${}^{c}D^{\alpha}_{a^+}u(x) = g(x) + f(x)u(x) + \lambda_1\int_a^b k(x,y)u(y)dy + \lambda_2\int_a^x \ell(x,y)u(y)dy \tag{1.1}$$

together with the linear non-local initial condition:

$$u(a) = \mu_1\int_a^b u(y)dy + \mu_2 \tag{1.2}$$

where $x \in [a, b]$, $0 < \alpha \leq 1$, g, f, $u \in C[a, b]$ and $k: [a, b] \times [a, b] \longrightarrow R$ and $\ell: [a, b] \times [a, b] \longrightarrow R$ are continuous functions, ${}^{c}D^{\alpha}_{a^+}$ is the left hand Caputo fractional derivative of order α, λ_1, λ_2, μ_1, μ_2 are known constants.

To do this we need the following lemma. The proof of this lemma depends on the facts that appeared in [11].

Lemma (1.22):

The non-local initial value problem given by equations (1.1)-(1.2) is equivalent to the following linear integral equation:

$$u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} g(y)dy + \mu_1 \int_a^b u(y)dy + \mu_2 +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)u(y)dy + \frac{\lambda_1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s)u(s)ds\right]dy +$$

$$\frac{\lambda_2}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s)u(s)ds\right]dy, a \le x \le b \qquad (1.3)$$

Proof:

Let u be a solution of the linear non-local initial value problem given by equations (1.1)-(1.2). By taking the left hand Riemman-Liouville fractional integral of order α, $I_{a^+}^{\alpha}$ for both sides of equation (1.1) one can have:

$$I_{a^+}^{\alpha}\left({}^{c}D_{a^+}^{\alpha}u\right)(x) = I_{a^+}^{\alpha}g(x) + I_{a^+}^{\alpha}\big(f(x)u(x)\big) + I_{a^+}^{\alpha}\left(\lambda_1\int_a^b k(x,y)u(y)dy\right) +$$

$$I_{a^+}^{\alpha}\left(\lambda_2\int_a^x \ell(x,y)u(y)dy\right) \qquad (1.4)$$

By using remarks (1.21),(4) and by using the linear non-local initial condition given by equation (1.2) one can have:

$$I_{a^+}^{\alpha}\left({}^{c}D_{a^+}^{\alpha}u\right)(x) = u(x) - u(a)$$

$$= u(x) - \mu_1\int_a^b u(x)dx - \mu_2$$

Therefore equation (1.4) becomes:

$$u(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}g(y)dy\ +\mu_1\int_a^b u(y)dy+\mu_2+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)u(y)dy+\frac{\lambda_1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s)u(s)ds\right]dy+$$

$$\frac{\lambda_2}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s)u(s)ds\right]dy$$

Therefore u is a solution of the linear integral equation given by equation (1.3).

Conversely, let u be a solution of the integral equation given by equation (1.3). Then

$$u(a)=\frac{1}{\Gamma(\alpha)}\int_a^a (a-y)^{\alpha-1}g(y)dy+\mu_1\int_a^b u(y)dy+\mu_2+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^a (a-y)^{\alpha-1}f(y)u(y)dy+\frac{\lambda_1}{\Gamma(\alpha)}\int_a^a (a-y)^{\alpha-1}\left[\int_a^b k(y,s)u(s)ds\right]dy$$

$$+\frac{\lambda_2}{\Gamma(\alpha)}\int_a^a (a-y)^{\alpha-1}\left[\int_a^y \ell(y,s)u(s)ds\right]dy$$

$$=\mu_1\int_a^b u(y)dy+\mu_2$$

Therefore u satisfies the linear non-local initial condition given by equation (1.2). By taking the left hand Caputo fractional derivative of order α of both sides of equation (1.3) one can have:

$${}^{c}D_{a^+}^{\alpha}u(x)=\ {}^{c}D_{a^+}^{\alpha}\left(I_{a^+}^{\alpha}g\right)(x)+\mu_1\ {}^{c}D_{a^+}^{\alpha}\int_a^b u(y)dy+\ {}^{c}D_{a^+}^{\alpha}\mu_2+$$

$${}^{c}D_{a^+}^{\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)u(y)dy\right)+$$

$${}^{c}D_{a^+}^{\alpha}\left(\frac{\lambda_1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s)u(s)ds\right]dy\right)+$$

$$
{}^{c}D^{\alpha}_{a^+}\left(\frac{\lambda_2}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s)u(s)ds\right]dy\right)
$$

$$
= {}^{c}D^{\alpha}_{a^+}\left(I^{\alpha}_{a^+}g\right)(x) + {}^{c}D^{\alpha}_{a^+}\left(I^{\alpha}_{a^+}f(x)u\right)(x) + \lambda_1\ {}^{c}D^{\alpha}_{a^+}I^{\alpha}_{a^+}\left(\int_a^b k(x,y)u(y)dy\right) +
$$

$$
\lambda_2 D^{\alpha}_{a^+}I^{\alpha}_{a^+}\left(\int_a^x \ell(x,y)u(y)dy\right)
$$

By using remarks (1.21), (3) one can get:

$$
{}^{c}D^{\alpha}_{a^+}u(x) = g(x) + f(x)u(x) + \lambda_1\int_a^b k(x,y)u(y)dy + \lambda_2\int_a^x \ell(x,y)u(y)dy,
$$

where $a \le x \le b$. Thus u is a solution of the linear fractional integro-differential equation (1.1).

Next, we are in the position that we can give the following existence and uniqueness theorem. The proof of this theorem is a simple modification of the facts that appeared in [11].

Theorem (1.23):

Consider the linear non-local initial value problem given by equations (1.1)-(1.2). If

$$
|\mu_1|(b-a) + \frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(|\lambda_1|K + |\lambda_2|L)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} < 1
$$

where $|f(x)| \le F$, $|k(x,y)| \le K$ and $|\ell(x,y)| \le L$ $\forall\, x,y \in [a,b]$. Then

equations (1.1)-(1.2) have a unique solution.

Proof:

It is known that C[a, b] is a Banach space with respect to the following norm:

$$\|u\|_{C[a,b]} = \sup_{a\le x\le b} |u(x)|.$$

Let A be an operator that is defined by:

$$Au(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} g(y)dy + \mu_1 \int_a^b u(y)dy + \mu_2 +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)u(y)dy + \frac{\lambda_1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s)u(s)ds\right]dy +$$

$$\frac{\lambda_2}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s)u(s)ds\right]dy$$

Then

$$|Au(x) - Av(x)| \le |\mu_1| \int_a^b |u(y)-v(y)|dy +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|f(y)||u(y)-v(y)|dy +$$

$$\frac{|\lambda_1|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |k(y,s)||u(s)-v(s)|ds\right]dy +$$

$$\frac{|\lambda_2|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y |\ell(y,s)||u(s)-v(s)|ds\right]dy$$

$$\le |\mu_1| \int_a^b \|u-v\|_{C[a,b]}\, dy + \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|f(y)|\|u-v\|_{C[a,b]}dy +$$

$$\frac{|\lambda_1|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |k(y,s)|\|u-v\|_{C[a,b]}\, ds\right]dy +$$

$$\frac{|\lambda_2|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y |\ell(y,s)|\|u-v\|_{C[a,b]}ds\right]dy$$

$$= |\mu_1|(b-a)\|u-v\|_{C[a,b]} + \left[\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|f(y)|dy\right]\|u-v\|_{C[a,b]} +$$

$$\left[\frac{|\lambda_1|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left(\int_a^b |k(y,s)|ds\right) + \right.$$

$$\left.\frac{|\lambda_2|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left(\int_a^y |\ell(y,s)|ds\right)dy\right]\|u-v\|_{C[a,b]}$$

$$\leq \left[|\mu_1|(b-a) + \frac{F}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}dy + \right.$$

$$\left.\frac{|\lambda_1|K + |\lambda_2|L}{\Gamma(\alpha)}(b-a)\int_a^x (x-y)^{\alpha-1}dy\right]\|u-v\|_{C[a,b]}$$

$$= \left[|\mu_1|(b-a) + \frac{F}{\alpha\Gamma(\alpha)}(x-a)^\alpha + \right.$$

$$\left.\frac{(|\lambda_1|K + |\lambda_2|L)(b-a)}{\alpha\Gamma(\alpha)}(x-a)^\alpha\right]\|u-v\|_{C[a,b]}$$

$$\leq \left[|\mu_1|(b-a) + \frac{F(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(|\lambda_1|K + |\lambda_2|L)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\right]\|u-v\|_{C[a,b]}$$

Therefore

$$\|Au-Av\|_{C[a,b]} \leq \left[|\mu_1|(b-a) + \frac{F(b-a)^\alpha}{\Gamma(\alpha+1)} + \right.$$

$$\left.\frac{(|\lambda_1|K + |\lambda_2|L)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\right]\|u-v\|_{C[a,b]}$$

Since $|\mu_1|(b-a) + \frac{F(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(|\lambda_1|K+|\lambda_2|L)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} < 1$, then A is contraction operator and by using the Banach fixed point theorem, A has unique fixed point

u. This fixed point is the unique solution of the linear integral equation (1.3). By using lemma (1.22), u is the unique solution of the non-local initial value problem given by equations (1.1)-(1.2).

To illustrate this theorem, we consider the following examples:

Example (1.24):

Consider the linear nonlocal initial value problem that consists of the linear fractional Fredholm integro-differential equation of order $\frac{1}{4}$:

$$ {}^{c}D_{0^+}^{\frac{1}{4}}u(x)=\frac{1}{8}xu(x)+\frac{1}{6}\int_0^1 xyu(y)dy+3x^2,\;\; 0\le x\le 1 \tag{1.5}$$

together with the linear nonlocal initial condition:

$$u(0)=\frac{1}{5}\int_0^1 u(y)dy \tag{1.6}$$

Here $\alpha=\frac{1}{4}, a=0,\; b=1,\; f(x)=\frac{1}{8}x\quad \forall x\in[0,1],\; k(x,y)=xy, \ell(x,y)=0,$

$\lambda_1=\frac{1}{6}, \lambda_2=0, \mu_1=\frac{1}{5}, \mu_2=0$ and $g(x)=3x^2$. Therefore $|f(x)|\le F=\frac{1}{8}\quad \forall\, x\in$ $[0,1]$. Also $|k(x,y)|\le K=1\quad \forall x,y\in[0,1]$. So

$$|\mu_1|(b-a)+\frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(|\lambda_1|K+|\lambda_2|L)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\simeq 0.522<1.$$

By using theorem (1.23), the linear nonlocal initial value problem given by equations (1.5)-(1.6) has a unique solution.

Example (1.25):

Consider the linear nonlocal initial value problem that consists of the linear fractional Fredholm-Volterra integro-differential equation of order $\frac{1}{5}$:

$$ {}^{c}D_{1^+}^{\frac{1}{5}}u(x)=\frac{1}{9}x^2u(x)+\frac{1}{6}\int_1^{\frac{3}{2}}(\sin y)u(y)dy+\frac{1}{5}\int_1^x y^3\,u(y)dy+3e^{-x}, 1\le x\le\frac{3}{2} \tag{1.7}$$

together with the linear nonlocal initial condition:

$$u(1) = \frac{1}{2}\int_1^{\frac{3}{2}} u(y)dy + 7 \qquad (1.8)$$

Here $\alpha = \frac{1}{5}, a = 1,\ b = \frac{3}{2},\ f(x) = \frac{1}{9}x^2 \quad \forall x \in \left[1, \frac{3}{2}\right],\ k(x,y) = \sin y, \ell(x,y) = y^3, \lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{5}, \mu_1 = \frac{1}{2}, \mu_2 = 7$ and $g(x) = 3e^{-x}$. Therefore $|f(x)| \le F = \frac{1}{9}\left(\frac{3}{2}\right)^2 = \frac{1}{4}$, $|k(x,y)| \le K = 1, |\ell(x,y)| \le L = \frac{27}{8} \forall\, x \in \left[1, \frac{3}{2}\right]$. So

$$|\mu_1|(b-a) + \frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(|\lambda_1|K + |\lambda_2|L)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} \simeq 0.886 < 1.$$

By using theorem (1.23), the linear nonlocal initial value problem given by equations (1.7)-(1.8) has a unique solution.

Remark (1.26):

For $\mu_1 = 0$, it is clear that theorem (1.23) can be also used to ensure the existence of the unique solution for the linear local initial value problem that consists of the linear fractional Fredholm-Volterra integro-differential equation (1.1) together with the linear local non-homogeneas initial condition u(a) = μ_2, where μ_2 is a known constant.

Second, we generalize theorem (1.23) to be valid for the non-linear non-local initial value problem that consists of linear fractional Fredholm-Volterra integro-differential equation (1.1) together with the non-linear non-local initial condition:

$$u(a) = \int_a^b w\big(y, u(y)\big)dy \qquad (1.9)$$

where $w: [a,b] \times R \longrightarrow R$ is a continuous function.

But before that we need the following lemma. The proof of this lemma is similar to the proof of lemma (1.22), thus we omitted it.

Lemma (1.27):

The non-linear non-local initial value problem given by equations (1.1),(1.9) is equivalent to the following non-linear integral equation:

$$u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} g(y)dy + \int_a^b w\big(y,u(y)\big)dy +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)u(y)dy + \frac{\lambda_1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s)u(s)ds\right]dy +$$

$$\frac{\lambda_2}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s)u(s)ds\right]dy, a \le x \le b \qquad (1.10)$$

Theorem (1.28):

Consider the non-linear non-local initial value problem given by equations (1.1),(1.9). If the following conditions are satisfied:

(1) w satisfied a Lipschitz condition with respect to the second argument with a Lipschitz constant W:

$$|w(x,u(x)) - w(x,v(x))| \le W|u(x) - v(x)|, \quad a \le x \le b.$$

(2) $W(b-a) + \frac{F}{\Gamma(\alpha+1)}(b-a)^{\alpha} + \frac{|\lambda_1|\ K + |\lambda_2| L}{\Gamma(\alpha+1)}(b-a)^{\alpha+1} < 1.$

where $|f(x)| \le F$, $|k(x,y)| \le K$ and $|\ell(x,y)| \le L \quad \forall\, x,y \in [a,b]$.

Then equations (1.1),(1.9) have a unique solution.

Proof:

Let A be an operator that is defined by:

$$Au(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} g(y)dy + \int_a^b w\big(y,u(y)\big)dy +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)u(y)dy + \frac{\lambda_1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s)u(s)ds\right]dy +$$

$$\frac{\lambda_2}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s)u(s)ds\right]dy, a \le x \le b$$

Then

$$|Au(x)-Av(x)| \le \int_a^b \left|w\big(y,u(y)\big)-w\big(y,v(y)\big)\right|dy +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|f(y)||u(y)-v(y)|dy +$$

$$\frac{|\lambda_1|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |k(y,s)||u(s)-v(s)|ds\right]dy +$$

$$\frac{|\lambda_2|}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^x |\ell(y,s)||u(s)-v(s)|ds\right]dy$$

Thus

$$|Au(x)-Av(x)|$$

$$\le W\int_a^b |u(y)-v(y)|dy + \frac{F}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|u(y)-v(y)|dy +$$

$$\frac{|\lambda_1|K}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |u(s)-v(s)|ds\right]dy +$$

$$\frac{|\lambda_2|L}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |u(s)-v(s)|ds\right]dy$$

$$\le W(b-a)\|u-v\|_{C[a,b]} + \frac{F}{\Gamma(\alpha+1)}\|u-v\|_{C[a,b]}(x-a)^{\alpha} +$$

$$\frac{(|\lambda_1|\,K+|\lambda_2|L)\,(b-a)}{\Gamma(\alpha+1)}\|u-v\|_{C[a,b]}(x-a)^{\alpha}$$

$$\le \left[W(b-a)+\frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(|\lambda_1|\,K+|\lambda_2|L)(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\right]\|u-v\|_{C[a,b]}$$

Therefore

$$\|Au - Av\|_{C[a,b]} \leq \left[W(b-a) + \frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(|\lambda_1|\,K + |\lambda_2|L)\,(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\right]\|u - v\|_{C[a,b]}$$

Since $W(b-a) + \frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(|\lambda_1|\,K+|\lambda_2|L)\,(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} < 1$, then A is a contraction operator and by using the Banach fixed point theorem, A has a unique fixed point u. This fixed point is the unique solution of the non-linear integral equation (1.10). By using lemma (1.27), u is the unique solution of the non-linear non-local initial value problem given by equations (1.1),(1.9).

To illustrate this theorem, consider the following example:

Example (1.29):

Consider the nonlinear nonlocal initial value problem that consists of the linear fractional Fredholm-Volterra integro-differential equation of order $\frac{1}{6}$:

$$ {}^{C}D^{\frac{1}{6}}_{1^{+}}u(x) = \frac{1}{20}u(x) + \frac{1}{3}\int_{0}^{\frac{1}{2}} y^2 u(y)dy + \frac{1}{4}\int_{0}^{x}(x+y)u(y)dy + 6x + 8 \qquad (1.11)$$

together with the nonlinear nonlocal initial condition:

$$u(0) = \int_{0}^{\frac{1}{2}} \sin u(y)dy \qquad (1.12)$$

Here $\alpha = \frac{1}{6}, a = 0,\ b = \frac{1}{2},\ f(x) = \frac{1}{20}\ \ \forall\, x \in \left[0, \frac{1}{2}\right]\ ,\ k(x,y) = y^2,$

$\ell(x,y) = x + y, w(y,u(y)) = \sin u(y), \lambda_1 = \frac{1}{3},\ \lambda_2 = \frac{1}{4}$ and $g(x) = 6x + 8$

Therefore

$|f(x)| \leq F = \frac{1}{20}\ ,\ |k(x,y)| \leq K = \frac{1}{4}, |\ell(x,y)| \leq L = 1.$ Since $|\sin u(x) - \sin v(x)| \leq |u(x) - v(x)|\ \forall\, x \in \left[0, \frac{1}{2}\right]$, this implies that $W = 1$. So

$$W(b-a)+\frac{F}{\Gamma(\alpha+1)}(b-a)^{\alpha}+\frac{|\lambda_1|\,K+|\lambda_2|L}{\Gamma(\alpha+1)}(b-a)^{\alpha+1}\simeq 0.708<1$$

By using theorem (1.28), the nonlinear nonlocal initial value problem given by equations (1.11)-(1.12) has a unique solution.

Third, we extend theorem (1.28) to be valid for the non-linear non-local initial value problem that consists of the non-linear fractional Fredholm-Volterra integro-differential equation of order α:

$$ {}^{c}D^{\alpha}_{a^+}u(x)=f(x,u(x))+\int_a^b k\big(x,y,u(y)\big)dy+\int_a^x \ell(x,y,u(y))\,dy \qquad (1.13)$$

together with the non-linear non-local initial condition

$$u(a)=\int_a^b w(y,u(y))dy \qquad (1.14)$$

where $u\in C[a,b]$, $a\le x\le b$, $0<\alpha\le 1$, $k:[a,b]\times[a,b]\times R\longrightarrow R$,

$\ell:[a,b]\times[a,b]\times R\longrightarrow R$, $f:[a,b]\times R\to R$ and $w:[a,b]\times R\longrightarrow R$ are

continuous functions. To do this we need the following lemma. The proof of this lemma is similar to the previous, thus we omitted it.

Lemma (1.30):

The non-linear non-local initial value problem given by equation (1.13)-(1.14) is equivalent to the non-linear integral equation:

$$u(x)=\int_a^b w(y,u(y))dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y,u(y))dy+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s,u(s))ds\right]dy+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s,u(s))ds\right]dy,\ a\le x\le b$$

Now, we give the following existence and uniqueness theorem which is an extension of theorem (1.28) to be valid for the non-linear non-local initial value problem given by equations (1.13)-(1.14).

Theorem (1.31):

Consider the non-linear non-local initial value problem given by equations (1.13)-(1.14). If the following conditions are satisfied:

(1) f and w satisfy Lipschitz condition with respect to the second argument with Lipschitz constants F and W respectively.

(2) k and ℓ satisfyLipschitz condition with respect to the third argument with Lipschitz constants K and L respectively.

(3) $W(b-a)+\frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(K+L)\,(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}<1$.

Then the non-linear non-local initial value problem given by equations (1.13)-(1.14) has a unique solution.

Proof:

Let A be an operator that is defined by

$$\mathrm{Au(x)}=\int_a^b w\big(y,u(y)\big)dy+\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f\big(y,u(y)\big)dy+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b k(y,s,u(s))ds\right]dy+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y \ell(y,s,u(s))ds\right]dy\,,\ a\leq x\leq b$$

Then

$$|Au(x)-Av(x)|\leq\int_a^b \big|w\big(y,u(y)\big)-w(y,v(y))\big|dy+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|f(y,u(y))-f(y,v(y))|dy+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |k(y,s,u(s)) - k(y,s,v(s))|ds\right]dy +$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^x |\ell(y,s,u(s)) - \ell(y,s,v(s))|ds\right]dy$$

$$\leq W\int_a^b |u(y)-v(y)|dy + \frac{F}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}|u(y)-v(y)|dy +$$

$$\frac{K}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^b |u(s)-v(s)|ds\right]dy +$$

$$\frac{L}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left[\int_a^y |u(s)-v(s)|ds\right]dy$$

$$\leq W(b-a)\|u-v\|_{C[a,b]} + \frac{F}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}dy\,\|u-v\|_{C[a,b]} +$$

$$\frac{(K+L)(b-a)}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\|u-v\|_{C[a,b]}\,dy$$

$$= \left[W(b-a) + \frac{F(x-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(K+L)\,(b-a)(x-a)^{\alpha}}{\Gamma(\alpha+1)}\right]\|u-v\|_{C[a,b]}$$

$$\leq \left[W(b-a) + \frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(K+L)\,(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\right]\|u-v\|_{C[a,b]}$$

Therefore

$$\|Au-Av\|_{c[a,b]} \leq \left[W(b-a) + \frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(K+L)\,(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\right]\|u-v\|_{C[a,b]}$$

Since $W(b-a) + \frac{F(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(K+L)\,(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} < 1$, then A is a contraction operator. Therefore by using the Banach fixed point theorem and lemma (1.30), there exists a unique solution to the non-linear non-local initial value problem given by equations (1.13)-(1.14).

Introduction:

The Laplace transform is a powerful tool in applied mathematics and engineering. It allows us to transform differential equations into algebraic equations and then by solving these algebraic equations, we can obtain the unknown function can be obtained by using the Inverse Laplace transform, [20].

It is known that the Laplace transform method is one of the most important methods that can be used to solve the local initial value problems that consist of the linear ordinary differential equations with constant coefficients together with linear local initial conditions, [20]. This method can be also used to solve special types of the linear fractional differential equations with constant coefficients without any initial conditions, [41]. Moreover, [40] and [48] used this method to solve special types of the linear local initial value problems for the linear fractional differential equations with constant coefficients and non-constant coefficients respectively.

The aim of this chapter is to use this method to solve special two types of local and non-local problems. The first type is the non-local initial value problems that consist of the n-th order linear Volterra integro-differential equations of difference kernel together with (n-1) linear local initial conditions and one linear non-local initial condition. The second type is the local initial value problems that consist of the linear fractional Volterra integro-differential equations of difference kernel together with local initial conditions.

This chapter consists of three sections:

In section one, we give some basic concepts of Laplace transform.

In section two, we use the Laplace transform method to find the solutions of special types of linear Volterra integro-differential equations of difference kernel together with local and nonlocal linear initial conditions..

In section three, we use the same method to find the solutions of special types of linear fractional Volterra integro-differential equations of difference kernel together with local linear initial conditions.

2.1 Some Basic Concepts of the Laplace Transform:

Recall that, the Laplace transform of a continuous function defined on $[0,\infty)$ which is of exponential order, denoted by $L\{u(x)\}$ or U(s) is defined by:

$$L\{u(x)\} = \int_0^\infty e^{-sx} u(x) dx$$

where s is a complex number for which the above integral converges, [20].

In this case, u(x) is called the inverse Laplace transform of U(s), that is $u(x) = L^{-1}\{U(s)\}$ where L^{-1} is called the inverse Laplace transformation operator.

In this section we give some of the useful properties of the Laplace transform

We start this section by giving the following known properties for the Laplace transform and its inverse, [20]:

(1) L and L^{-1} are linear operators.

(2) Let $U(s) = L\{u(x)\}$ then

$$L\{u^{(m)}(x)\} = s^m U(s) - \sum_{i=0}^{m-1} s^i u^{(m-i-1)}(0), \quad m \in N \tag{2.1}$$

(3) Let $U(s) = L\{u(x)\}$ and $W(s) = L\{w(x)\}$ then

$$L\left\{\int_0^x w(x-y)u(y)dy\right\} = U(s)W(s) \tag{2.2}$$

(4) Let $U(s) = L\{u(x)\}$ then

$$L\left\{\int_0^x u(y)dy\right\} = \frac{U(s)}{s} \tag{2.3}$$

(5) $L\left\{x^{\beta}\right\}=\dfrac{\Gamma(\beta+1)}{s^{\beta+1}}, \quad \beta>-1$ (2.4)

Next, we give some another important properties for the Laplace transform. We start by the following lemma.

<u>Lemma (2.1), [40]:</u>

Let $U(s)=L\{u(x)\}$ then

$$L\left\{I_{0^+}^{\alpha}u(x)\right\}=\frac{U(s)}{s^{\alpha}}, \alpha > 0.$$

<u>Proof:</u>

It is known that

$$L\left\{I_{0^+}^{\alpha}u(x)\right\}=L\left\{\frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}u(y)dy\right\}=\frac{1}{\Gamma(\alpha)}L\left\{\int_0^x (x-y)^{\alpha-1}u(y)dy\right\}.$$

Let $w(x-y)=(x-y)^{\alpha-1}$, then w(x)=$x^{\alpha-1}$. By using equations (2.2),(2.4) one can get:

$$L\left\{I_{0^+}^{\alpha}u(x)\right\}=\frac{1}{\Gamma(\alpha)}U(s)W(s)$$

where $W(s)=L\{x^{\alpha-1}\}=\dfrac{\Gamma(\alpha)}{s^{\alpha}}$. Therefore

$$L\left\{I_{0^+}^{\alpha}u(x)\right\}=\frac{U(s)}{s^{\alpha}}.\ \alpha > 0.$$

<u>Remark (2.2):</u>

If $\alpha=1$, then by using lemma (2.1) one can have:

$$L\left\{I_{0^+}^{1}u(x)\right\}=L\left\{I_{0^+}u(x)\right\}=L\left\{\int_0^x u(y)dy\right\}=\frac{U(s)}{s}.$$

Thus lemma (2.1) is a generalization of equation (2.3).

To illustrate lemma (2.1), consider the following example:

Example (2.3):

Let $u(x) = x^{\beta}$, $\beta > -1$ then

$$L\left\{I_{0^+}^{\alpha} u(x)\right\} = \frac{U(s)}{s^{\alpha}} = \frac{\frac{\Gamma(\beta+1)}{s^{\beta+1}}}{s^{\alpha}} = \frac{\Gamma(\beta+1)}{s^{\alpha+\beta+1}}.$$

On the other hand

$$I_{0^+}^{\alpha} u(x) = I_{0^+}^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}.$$

Thus

$$L\left\{I_{0^+}^{\alpha} u(x)\right\} = L\left\{I_{0^+}^{\alpha} x^{\beta}\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} L\left\{x^{\alpha+\beta}\right\}$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{s^{\alpha+\beta+1}} = \frac{\Gamma(\beta+1)}{s^{\alpha+\beta+1}}.$$

Now, the following lemma gives the Laplace transform for the left hand Caputo fractional derivative. This lemma appeared in [40], here we give the details of its proof.

Lemma (2.4), [40]:

Let $U(s) = L\{u(x)\}$ then

$$L\left\{{}^{C}D_{0^+}^{\alpha} u(x)\right\} = \frac{s^n U(s) - \sum_{i=0}^{n-1} s^i u^{(n-i-1)}(0)}{s^{n-\alpha}}, \ n-1 < \alpha \le n, \ n \in N.$$

provided ${}^{C}D_{0^+}^{\alpha} u(x)$ exists for each $x \ge 0$.

Proof:

By using remarks (1.21), (1), one can get:

$$L\left\{{}^{C}D_{0^+}^{\alpha}u(x)\right\}=L\left\{I_{0^+}^{n-\alpha}u^{(n)}(x)\right\},\ n-1<\alpha\le n,\ n\in N,$$

So, by using lemma (2.1) one can obtain:

$$L\left\{{}^{C}D_{0^+}^{\alpha}u(x)\right\}=L\left\{I_{0^+}^{n-\alpha}u^{(n)}(x)\right\}$$
$$=\frac{L\left\{u^{(n)}(x)\right\}}{s^{n-\alpha}}.$$

By using equation (2.1), the above equation becomes:

$$L\left\{{}^{C}D_{0^+}^{\alpha}u(x)\right\}=\frac{s^{n}U(s)-\sum_{i=0}^{n-1}s^{i}u^{(n-i-1)}(0)}{s^{n-\alpha}},\ n-1<\alpha\le n,\ n\in N,\qquad (2.5)$$

Remark (2.5):

If $\alpha = n$, $n \in N$, then by using lemma (2.4) one can have:

$$L\left\{{}^{c}D_{0^+}^{\alpha}u(x)\right\} = L\left\{u^{(n)}(x)\right\}$$
$$= s^{n}U(s)-\sum_{i=0}^{n-1}s^{i}u^{(n-i-1)}(0)$$

Thus lemma (2.4) is a generalization of equation (2.1).

To illustrate lemma (2.4), consider the following example:

Example (2.6):

Let $u(x)=x^{2}$ and $\alpha=\frac{3}{7}$. Therefore n=1 and

$$L\left\{ {}^{C}D_{0^+}^{\frac{3}{7}} u(x) \right\} = \frac{sU(s) - u(0)}{s^{\frac{4}{7}}} = \frac{2}{s^{\frac{18}{7}}}.$$

On the other hand, by using proposition (1.13) one can have:

$${}^{C}D_{0^+}^{\frac{3}{7}} u(x) = {}^{C}D_{0^+}^{\frac{3}{7}} x^2 = \frac{\Gamma(3)}{\Gamma\left(2 - \frac{3}{7} + 1\right)} x^{2-\frac{3}{7}} = \frac{\Gamma(3)}{\Gamma\left(\frac{18}{7}\right)} x^{\frac{11}{7}}.$$

Then by using equation (2.4) one can get:

$$L\left\{ {}^{C}D_{0^+}^{\frac{3}{7}} u(x) \right\} = L\left\{ {}^{C}D_{0^+}^{\frac{3}{7}} x^2 \right\} = \frac{\Gamma(3)}{\Gamma\left(\frac{18}{7}\right)} L\left\{ x^{\frac{11}{7}} \right\} = \frac{\Gamma(3)}{\Gamma\left(\frac{18}{7}\right)} \frac{\Gamma\left(\frac{18}{7}\right)}{s^{\frac{18}{7}}} = \frac{\Gamma(3)}{s^{\frac{18}{7}}} = \frac{2}{s^{\frac{18}{7}}}.$$

Now, the following lemma gives the Laplace transform for the left hand Riemann-Liouville fractional derivative. This lemma appeared in [40] without proof. Here we give its proof.

Lemma (2.7):

Let $U(s) = L\{u(x)\}$ then

$$L\left\{ {}_{x}D_{0^+}^{\alpha} u(x) \right\} = s^{\alpha} U(s) - \sum_{i=0}^{n-1} s^{i} \left(I_{0^+}^{n-\alpha} u \right)^{(n-i-1)}(0),\ n-1 < \alpha \le n,\ n \in N.$$

Provided ${}_{x}D_{0^+}^{\alpha} u(x)$ exists for each $x \ge 0$.

Proof:

By using remarks (1.21), (2), one can get:

$$L\left\{{}_xD_{0^+}^{\alpha}u(x)\right\}=L\left\{\frac{d^n}{dx^n}\left(I_{0^+}^{n-\alpha}u(x)\right)\right\}$$

$$=L\left\{w^{(n)}(x)\right\},\ n-1<\alpha\le n,\ n\in N,$$

where $w(x)=I_{0^+}^{n-\alpha}u(x)$. Then by using equation (2.1) one can obtain:

$$L\left\{{}_xD_{0^+}^{\alpha}u(x)\right\}=s^nW(s)-\sum_{i=0}^{n-1}s^iw^{(n-i-1)}(0),\ n-1<\alpha\le n,\ n\in N.$$

By using lemma (2.1) one can have:

$$W(s)=L\left\{w(x)\right\}=L\left\{I_{0^+}^{n-\alpha}u(x)\right\}$$

$$=\frac{U(s)}{s^{n-\alpha}}.$$

Therefore

$$L\left\{{}_xD_{0^+}^{\alpha}u(x)\right\}=s^{\alpha}U(s)-\sum_{i=0}^{n-1}s^i\left(I_{0^+}^{n-\alpha}u\right)^{(n-i-1)}(0),\ n-1<\alpha\le n,\ n\in N.$$

Remark (2.8):

If α = n, n ∈ N, then by using lemma (2.7) one can have:

$$L\left\{{}_xD_{0^+}^{\alpha}u(x)\right\}=L\left\{{}_xD_{0^+}^{n}u(x)\right\}$$

$$=s^nU(s)-\sum_{i=0}^{n-1}s^i\left(I_{0^+}^{0}u\right)^{(n-i-1)}(0)$$

$$=s^nU(s)-\sum_{i=0}^{n-1}u^{(n-i-1)}(0)$$

Thus lemma (2.7) is a generalization of equation (2.1).

To illustrate lemma (2.7), consider the following example:

Example (2.9):

Let $u(x)=\sqrt{x}$ and $\alpha=\frac{1}{2}$. Therefore n=1 and

$$L\left\{ {}_xD_{0^+}^{\frac{1}{2}}u(x) \right\} = L\left\{ {}_xD_{0^+}^{\frac{1}{2}}x^{\frac{1}{2}} \right\} = s^{\frac{1}{2}}U(s) - \sum_{i=0}^{0} s^i \left(I_{0^+}^{1-\frac{1}{2}}u \right)^{(1-1)} (0)$$

$$= s^{\frac{1}{2}} \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - s^0 \left(I_{0^+}^{\frac{1}{2}}u \right)^{(0)} (0).$$

By using proposition (1.19) one can have:

$$I_{0^+}^{\frac{1}{2}}u(x) = I_{0^+}^{\frac{1}{2}}x^{\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(2)}x$$

and this implies that $\left(I_{0^+}^{\frac{1}{2}}u \right)(0) = 0$. Thus $L\left\{ {}_xD_{0^+}^{\frac{1}{2}}u(x) \right\} = \frac{\Gamma(\frac{3}{2})}{s}$.

On the other hand, by using proposition (1.8) one can get:

$${}_xD_{0^+}^{\frac{1}{2}}u(x) = {}_xD_{0^+}^{\frac{1}{2}}x^{\frac{1}{2}} = \Gamma\left(\frac{3}{2}\right).$$

Hence

$$L\left\{ {}_xD_{0^+}^{\frac{1}{2}}u(x) \right\} = L\left\{ {}_xD_{0^+}^{\frac{1}{2}}x^{\frac{1}{2}} \right\} = L\left\{ \Gamma\left(\frac{3}{2}\right) \right\} = \frac{\Gamma\left(\frac{3}{2}\right)}{s}.$$

Next, the following lemma gives some properties of the inverse Laplace transform. This lemma appeared in [40]. Here we give the details of its proof.

Lemma (2.10):

Let $\alpha > 0,\ \beta > 0,\ \gamma > 0$ and $a, b \in R$. Then

(1) $$L^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}+a}\right\} = x^{\beta-1}E_{\alpha,\beta}(-ax^{\alpha}),\ \text{for } \left|s^{\alpha}\right| > |a|.$$

(2) $$L^{-1}\left\{\frac{1}{\left(s^{\alpha}+as^{\beta}\right)^{n+1}}\right\} = x^{\alpha(n+1)-1}\sum_{i=0}^{\infty}\left[\frac{(-a)^{i}\binom{n+i}{i}}{\Gamma(i(\alpha-\beta)+\alpha(n+1)\alpha)}x^{i(\alpha-\beta)}\right]$$

where $\alpha \geq \beta$ and $\left|s^{\alpha-\beta}\right| > |a|$.

(3) $$L^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}\right\} = x^{\alpha-\gamma-1}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left[\frac{(-b)^{j}(-a)^{i}\binom{i+j}{i}}{\Gamma(i(\alpha-\beta)+(j+1)\alpha-\gamma)}x^{i(\alpha-\beta)+j\alpha}\right]$$

where $\alpha \geq \beta$, $\left|s^{\alpha-\beta}\right| > |a|$ and $\left|s^{\alpha}+as^{\beta}\right| > |b|$.

Proof:

(1) It is known that for $\left|s^{\alpha}\right| > |a|$,

$$\frac{s^{\alpha-\beta}}{s^{\alpha}+a} = \left(\frac{1}{s^{\beta}}\right)\left(\frac{1}{1+\frac{a}{s^{\alpha}}}\right) = \frac{1}{s^{\beta}}\sum_{i=0}^{\infty}\left(\frac{-a}{s^{\alpha}}\right)^{i} = \sum_{i=0}^{\infty}\frac{(-a)^{i}}{s^{\alpha i+\beta}}.$$

Therefore

$$L^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}+a}\right\}=L^{-1}\left\{\sum_{i=0}^{\infty}\frac{(-a)^{i}}{s^{\alpha i+\beta}}\right\}$$

$$=\sum_{i=0}^{\infty}(-a)^{i}L^{-1}\left\{\frac{1}{s^{\alpha i+\beta}}\right\}$$

$$=\sum_{i=0}^{\infty}\frac{(-a)^{i}}{\Gamma(\alpha i+\beta)}L^{-1}\left\{\frac{\Gamma(\alpha i+\beta-1+1)}{s^{\alpha i+\beta-1+1}}\right\}$$

$$=\sum_{i=0}^{\infty}\frac{(-a)^{i}}{\Gamma(\alpha i+\beta)}x^{\alpha i+\beta-1}$$

$$=x^{\beta-1}\sum_{i=0}^{\infty}\frac{(-ax^{\alpha})^{i}}{\Gamma(\alpha i+\beta)}$$

$$=x^{\beta-1}E_{\alpha,\beta}(-ax^{\alpha}).$$

(2) It is known that for $|x|<1$,

$$\frac{1}{(1+x)^{n+1}}=\sum_{i=0}^{\infty}\binom{n+i}{i}(-x)^{i}.$$

Therefore

$$\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}}=\frac{1}{\left(s^{\alpha}+\dfrac{as^{\alpha}}{s^{\alpha-\beta}}\right)^{n+1}}$$

$$=\frac{1}{(s^{\alpha})^{n+1}}\frac{1}{\left(1+\dfrac{a}{s^{\alpha-\beta}}\right)^{n+1}}$$

$$=\frac{1}{(s^{\alpha})^{n+1}}\sum_{i=0}^{\infty}\binom{n+i}{i}\left(\frac{-a}{s^{\alpha-\beta}}\right)^{i}.$$

Thus

$$L^{-1}\left\{\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}}\right\}=L^{-1}\left\{\frac{1}{(s^{\alpha})^{n+1}}\sum_{i=0}^{\infty}\binom{n+i}{i}\left(\frac{-a}{s^{\alpha-\beta}}\right)^{i}\right\}$$

$$=\sum_{i=0}^{\infty}\left[(-a)^{i}\binom{n+i}{i}L^{-1}\left\{\frac{1}{s^{i(\alpha-\beta)+\alpha(n+1)}}\right\}\right]$$

$$=\sum_{i=0}^{\infty}\left[\frac{(-a)^{i}\binom{n+i}{i}}{\Gamma(i(\alpha-\beta)+\alpha(n+1))}L^{-1}\left\{\frac{\Gamma(i(\alpha-\beta)+\alpha(n+1)-1+1)}{s^{i(\alpha-\beta)+\alpha(n+1)-1+1}}\right\}\right]$$

$$=\sum_{i=0}^{\infty}\left[\frac{(-a)^{i}\binom{n+i}{i}}{\Gamma(i(\alpha-\beta)+\alpha(n+1))}x^{i(\alpha-\beta)+\alpha(n+1)-1}\right]$$

$$=x^{\alpha(n+1)-1}\sum_{i=0}^{\infty}\left[\frac{(-a)^{i}\binom{n+i}{i}}{\Gamma(i(\alpha-\beta)+\alpha(n+1))}x^{i(\alpha-\beta)}\right].$$

(3) It is known that for $\alpha\geq\beta$, $\left|s^{\alpha-\beta}\right|>|a|$ and $\left|s^{\alpha}+as^{\beta}\right|>|b|$,

$$\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}=\left(\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}}\right).\left(\frac{1}{1+\frac{b}{s^{\alpha}+as^{\beta}}}\right)$$

$$=\left(\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}}\right).\sum_{j=1}^{\infty}\left(\frac{-b}{s^{\alpha}+as^{\beta}}\right)^{j}$$

$$=\sum_{j=1}^{\infty}\frac{s^{\gamma}(-b)^{j}}{(s^{\alpha}+as^{\beta})^{j+1}}$$

Thus

$$\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}=\sum_{j=0}^{\infty}s^{\gamma}(-b)^{j}.\frac{1}{(s^{\alpha}+as^{\beta})^{j+1}}$$

$$=\sum_{j=0}^{\infty}s^{\gamma}(-b)^{j}.\frac{1}{(s^{\alpha})^{j+1}}\sum_{i=0}^{\infty}\binom{i+j}{i}\left(\frac{-a}{s^{\alpha-\beta}}\right)^{i}$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-a)^{i}(-b)^{j}\binom{i+j}{i}\left(\frac{1}{s^{i(\alpha-\beta)+(j+1)\alpha-\gamma}}\right)$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-a)^{i}(-b)^{j}\binom{i+j}{i}}{\Gamma(i(\alpha-\beta)+(j+1)\alpha-\gamma)}\left(\frac{\Gamma(i(\alpha-\beta)+(j+1)\alpha-\gamma-1+1)}{s^{i(\alpha-\beta)+(j+1)\alpha-\gamma-1+1}}\right)$$

Therefore

$$L^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}\right\}$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-a)^{i}(-b)^{j}\binom{i+j}{i}}{\Gamma(i(\alpha-\beta)+(j+1)\alpha-\gamma)}L^{-1}\left\{\frac{\Gamma(i(\alpha-\beta)+(j+1)\alpha-\gamma-1+1)}{s^{i(\alpha-\beta)+(j+1)\alpha-\gamma-1+1}}\right\}$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-a)^{i}(-b)^{j}\binom{i+j}{i}}{\Gamma(i(\alpha-\beta)+(j+1)\alpha-\gamma)}x^{i(\alpha-\beta)+(j+1)\alpha-\gamma-1}$$

$$=x^{\alpha-\gamma-1}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-a)^{i}(-b)^{j}\binom{i+j}{i}}{\Gamma(i(\alpha-\beta)+(j+1)\alpha-\gamma)}x^{i(\alpha-\beta)+j\alpha}.$$

2.2 Laplace Transform Method for Solving Ordinary Non-Local Problems:

In this section we use the Laplace transform method to solve the non-local initial value problems that consist of the linear non-homogenous Volterra integro-differential equations together with non-local initial conditions. To do this, consider the n-th order linear non-homogeneous Volterra integro-differential equation of the difference kernel:

$$u^{(n)}(x)+\sum_{i=0}^{n-1} a_i u^{(i)}(x)=\lambda\int_0^x k(x-y)u(y)dy+g(x),\quad x\geq 0 \tag{2.6}$$

together with the linear local initial conditions:

$$u^{(i)}(x)=c_i,\ \ i=1,2,\ldots,n-1 \tag{2.7}$$

and the linear non-local initial condition:

$$u(0)=\mu_1\int_0^b u(x)dx+\mu_2 \tag{2.8}$$

where $\{c_i\}_{i=1}^{n-1}$, μ_1, μ_2 , λ, $\{a_i\}_{i=0}^{n-1}$ are known constants, b is a known constant such that b > 0 and k, g are continuous functions defined on $[0,\infty)$ which are of exponential order. The Laplace transform method begins by taking the Laplace transform to both sides of equation (2.6) and by using the linearity property of the Laplace transform operator one can obtain:

$$L\{u^{(n)}(x)\}+\sum_{i=0}^{n-1} a_i L\{u^{(i)}(x)\}=\lambda L\left\{\int_0^x k(x-y)u(y)dy\right\}+L\{g(x)\},\quad x\geq 0$$

By using equations (2.1)-(2.2), the above equation becomes:

$$s^nU(s)-\sum_{i=0}^{n-1} s^i u^{(n-i-1)}(0)+a_0U(s)+\sum_{i=1}^{n-1} a_i\left\{s^iU(s)-\sum_{j=0}^{i-1} s^j u^{(i-j-1)}(0)\right\}=$$
$$K(s)U(s)+G(s) \tag{2.9}$$

Where $U(s)=L\{u(x)\}$, $K(s)=L\{k(x)\}$ and $G(s)=L\{g(x)\}$. Assume that $u(0)=c_0$, where c_0 is unknown constant. By substituting $u(0)=c_0$ and equation (2.7) into equation (2.9) one can have:

$$\left[s^n+\sum_{i=0}^{n-1}a_is^i-\lambda K(s)\right]U(s)=\sum_{k=0}^{n-1}s^kc_{n-k-1}+\sum_{i=1}^{n-1}a_i\sum_{j=0}^{i-1}s^ju^{(i-j-1)}(0)+G(s).$$

Therefore

$$u(x)=L^{-1}\{U(s)\}=L^{-1}\left\{\frac{\displaystyle\sum_{k=0}^{n-1}s^kc_{n-k-1}+\sum_{i=1}^{n-1}a_i\sum_{j=0}^{i-1}s^ju^{(i-j-1)}(0)+G(s)}{\displaystyle s^n+\sum_{i=0}^{n-1}a_is^i-\lambda K(s)}\right\}.$$

The obtained solution u depends on x and on the unknown constant c_0. Then this solution must satisfy the non-local initial condition given by equation (2.8). So, the value of the unknown constant c_0 can be determined by solving the algebraic equation:

$$c_0=\mu_1\int_0^b u(x)dx+\mu_2.$$

To illustrate this method, consider the following examples:

Example (2.11):

Consider the nonlocal initial value problem that consists of the second order linear non-homogeneous Volterra integro-differential equation of the difference kernel:

$$u''(x)+2u'(x)+u(x)=\int_0^x(x-y)u(y)dy+x^3-\frac{x^5}{20}+6x^2+6x,\quad x\geq 0 \qquad (2.10)$$

together with the linear local initial conditions:

$$u'(0)=0 \tag{2.11}$$

and the linear nonlocal initial condition:

$$u(0)=\int_0^1 u(x)dx-\frac{1}{4} \tag{2.12}$$

We use the Laplace transform method to solve this nonlocal initial value problem. To do this, we take the Laplace transform to both sides of equation (2.10) and by using the linearity property of the Laplace transform operator one can obtain:

$$L\{u''(x)\}+2L\{u'(x)\}+L\{u(x)\}=$$
$$L\left\{\int_0^x (x-y)u(y)dy\right\}+L\{x^3\}-L\left\{\frac{x^5}{20}\right\}+L\{6x^2\}+L\{6x\} \tag{2.13}$$

Let u(0) = c_0. Therefore

$$L\{u''(x)\}=s^2U(s)-su(0)-u'(0)=s^2U(s)-sc_0,$$
$$L\{u'(x)\}=sU(s)-u(0)=sU(s)-sc_0.$$

Since k(x-y) = x-y, thus k(x) = x and this implies that k(s) = $\frac{1}{s}$.

moreover, since $f(x)=x^3-\frac{x^5}{20}+6x^2+6x$, then $F(s)=\frac{6}{s^4}-\frac{6}{s^6}+\frac{12}{s^3}+\frac{6}{s^2}$.

So, equation (2.13) takes the form:

$$\left(s^2+2s+1\right)U(s)-(s+2)c_0=\frac{1}{s^2}U(s)+\frac{6}{s^4}-\frac{6}{s^6}+\frac{12}{s^3}+\frac{6}{s^2}.$$

Thus

$$U(s)=\frac{s^6(s+2)c_0+6s^2-6+12s^3+6s^4}{s^4(s^4+2s^3+s^2-1)}$$
$$=\frac{s^6(s+2)c_0+6s^2-6+12s^3+6s^4}{s^4(s^2+s-1)((s^2+s+1)}.$$

After simple computations, the above equation can be rewritten as:

$$U(s)=\frac{6}{s^4}+\frac{c_o}{2}\left[\frac{1}{s^2+s-1}\right]+c_0\left[\frac{s+\frac{1}{2}}{s^2+s+1}\right]$$

$$=\frac{6}{s^4}+\frac{c_o}{2}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]+c_0\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$=\frac{6}{s^4}+\frac{c_o}{\sqrt{5}}\left[\frac{\frac{\sqrt{5}}{2}}{\left(s+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]+c_0\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right].$$

Hence

$$u(x)=L^{-1}\left\{\frac{6}{s^4}+\frac{c_o}{\sqrt{5}}\left[\frac{\frac{\sqrt{5}}{2}}{\left(s+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]+c_0\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\right\}$$

$$=x^3+\frac{c_0}{\sqrt{5}}e^{-\frac{1}{2}x}\sinh\left(\frac{\sqrt{5}}{2}x\right)+c_0e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right),\ x\geq 0 \qquad (2.14)$$

By substituting this solution into equation (2.12) one can get the linear algebraic equation:

$$c_0-\int_0^1\left\{x^3+\frac{c_0}{\sqrt{5}}e^{-\frac{1}{2}x}\sinh\left(\frac{\sqrt{5}}{2}x\right)+c_0e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right)\right\}dx=-\frac{1}{4}$$

After simple computations, the above algebraic equation becomes:

$$\frac{c_0}{4}e^{-\frac{\sqrt{5}+1}{2}} - c_0 + \frac{c_0}{4}e^{\frac{\sqrt{5}-1}{2}} - c_0\frac{\sqrt{5}}{20}e^{-\frac{\sqrt{5}+1}{2}} + c_0\frac{\sqrt{5}}{20}e^{\frac{\sqrt{5}-1}{2}} -$$
$$c_0\frac{1}{2}e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right) + c_0\frac{\sqrt{3}}{2}e^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right) = 0.$$

which has the solution $c_0 = 0$. By substituting $c_0 = 0$ into equation (2.14) one can get:

$$u(x) = x^3,\ x \geq 0$$

which is the exact solution of the nonlocal initial value problem given by equations (2.10)-(2.12).

2.3 Laplace Transform Method for Solving the Local Initial Value Problems for Fractional Volterra Integro-Differential Equations:

It is known that the Laplace transform method can be used to some types of fractional local initial value problems involving Caputo differential operator, [40], [41].

In this section we use the Laplace transform method to solve the fractional local initial value problems that consist of the linear non-homogenous Volterra integro-differential equations together with local initial conditions. To do this, consider the linear fractional Volterra integro-differential equation of the difference kernel of order α of the second kind:

$$^{C}D^{\alpha}_{0^+}u(x) + au(x) = \lambda\int_0^x k(x-y)u(y)dy + g(x),\quad n-1<\alpha \leq n \tag{2.15}$$

together with the linear local initial conditions:

$$u^{(i)}(x) = c_i,\ \ i=0,1,...,n-1 \tag{2.16}$$

where $\{c_i\}_{i=1}^{n-1}$, μ_1, μ_2 a, λ are known constants, k, g are continuous functions defined on $[0,\infty)$ which are of exponential order. The Laplace transform method begins by taking the Laplace transform to both sides of equation (2.15) and by using the linearity property of the Laplace transform operator one can obtain:

$$L\left\{{}^{C}D_{0^+}^{\alpha}u(x)\right\}+aL\{u(x)\}=\lambda L\left\{\int_0^x k(x-y)u(y)dy\right\}+L\{g(x)\},\quad 0<\alpha\le 1$$

By using equations (2.2), (2.5), the above equation becomes:

$$s^{\alpha}U(s)-\sum_{i=0}^{n-1}s^{i+\alpha-n}u^{(n-i-1)}(0)+aU(s)=\lambda K(s)U(s)+G(s) \tag{2.17}$$

where $U(s)=L\{u(x)\}$, $K(s)=L\{k(x)\}$ and $G(s)=L\{g(x)\}$. By substituting equation (2.16) into equation (2.17) one can have:

$$\left[s^{\alpha}+a-\lambda K(s)\right]U(s)=\sum_{i=0}^{n-1}s^{i+\alpha-n}c_{n-i-1}+G(s).$$

Therefore

$$u(x)=L^{-1}\{U(s)\}=L^{-1}\left\{\frac{\sum_{i=0}^{n-1}s^{i+\alpha-n}c_{n-i-1}+G(s)}{s^{\alpha}+a-\lambda K(s)}\right\} \tag{2.18}$$

To illustrate this method, consider the following examples:

Example (2.12):

Consider the local initial value problem that consists of the fractional linear Volterra integro-differential equation of the difference kernel of order α:

$${}^{C}D_{0^+}^{\alpha}u(x)+au(x)=\lambda\int_0^x (x-y)^{\beta}u(y)dy,\quad n-1<\alpha\le n,\beta>-1 \tag{2.19}$$

together with the linear local initial conditions:

$u^{(i)}(0) = c_i$, i=0,1,...,n−1.

We use the Laplace transform method to solve this local initial value problem. By using equation (2.18), the solution of the above local initial value problem takes the form:

$$u(x) = L^{-1}\{U(s)\} = L^{-1}\left\{\frac{\sum_{i=0}^{n-1} s^{i+\alpha-n} c_{n-i-1}}{s^{\alpha} + a - \lambda\frac{\Gamma(\beta+1)}{s^{\beta+1}}}\right\}$$

$$= L^{-1}\left\{\frac{\sum_{i=0}^{n-1} s^{i+\alpha-n+\beta+1} c_{n-i-1}}{s^{\alpha+\beta+1} + as^{\beta+1} - \lambda\Gamma(\beta+1)}\right\} \tag{2.20}$$

If $0<\alpha \le 1$, then n=1. In this case, equation (2.20) becomes

$$u(x) = L^{-1}\left\{\frac{s^{\alpha+\beta} c_0}{s^{\alpha+\beta+1} + as^{\beta+1} - \lambda\Gamma(\beta+1)}\right\}.$$

By using lemma (2.10), (3), the solution of this local problem is

$$u(x) = c_0 \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left[\frac{(\lambda\Gamma(\beta+1))^j(-a)^i \binom{i+j}{i}}{\Gamma(i\alpha+(j+1)(\alpha+\beta+1)-\alpha-\beta)} x^{i\alpha+j(\alpha+\beta+1)}\right]$$

where $|s^{\alpha}| > |a|$ and $\left|s^{\beta+1}(s^{\alpha}+a)\right| > |\lambda\Gamma(\beta+s)|$.

In this case, if λ=0, then equations (2.19) become

$$^{C}D_{0^+}^{\alpha}u(x)+au(x)=0,\quad 0<\alpha\le 1$$

together with the linear local initial condition:

$$u(0)=c_0$$

By using lemma (2.10), (1), this local initial value problem has the solution

$$u(x)=c_0L^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+a}\right\}=c_0E_{\alpha}(-ax^{\alpha}),\left|s^{\alpha}\right|>\left|a.\right|$$

This solution is a generalization to the solution that appeared in [25].

If $1<\alpha\le 2$, then n=2. In this case, equation (2.20) becomes

$$u(x)=L^{-1}\left\{\frac{s^{\alpha+\beta-1}c_1+s^{\alpha+\beta}c_0}{s^{\alpha+\beta+1}+as^{\beta+1}-\lambda\Gamma(\beta+1)}\right\}.$$

By using lemma (2.10), (3), the solution of this local problem is

$$u(x)=c_0x\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left[\frac{(\lambda\Gamma(\beta+1))^{j}(-a)^{i}\binom{i+j}{i}}{\Gamma(i+(j+1)(\alpha+\beta+1)-\alpha-\beta+1)}x^{i+j(\alpha+\beta+1)}\right]+$$

$$c_1x^{2}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\left[\frac{(\lambda\Gamma(\beta+1))^{j}(-a)^{i}\binom{i+j}{i}}{\Gamma(i\alpha+(j+1)(\alpha+\beta+1)-\alpha-\beta+1)}x^{i\alpha+j(\alpha+\beta+1)}\right]$$

In this case, if λ=0, then equation (2.19) become

$$^{C}D_{0^+}^{\alpha}u(x)+au(x)=0,\quad 1<\alpha\le 2$$

together with the local linear initial conditions:

$$u(0)=c_0,\ u'(0)=c_1$$

By using lemma (2.10), (1) this local initial value problem has the solution

$$u(x)=c_1L^{-1}\left\{\frac{s^{\alpha-2}}{s^{\alpha}+a}\right\}+c_0L^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+a}\right\}=c_1xE_{\alpha,2}(-ax^{\alpha})+c_0E_{\alpha}(-ax^{\alpha}).$$

Introduction:

It is known that, when a Taylor series is truncated to a finite number of terms the result is a Taylor polynomial. This Taylor polynomial is used to approximate functions numerically, [45].

Taylor expansion method is an approach which based on approximating the unknown function in terms of Taylor's polynomial and can be used to solve the linear Fredholm integral equations of the second kind, [26] and the local initial value problems for the first order Fredholm-Volterra integro-differential equations of the second kind, [31].

The aim of this chapter is to present a method named as the generalized Taylor expansion method. This method depends on approximating the unknown function in terms of the generalized Taylor's formula and can be used to solve the non-local initial value problems for the linear fractional Fredholm-Volterra integro-differential equations of order α, where $0<\alpha \leq 1$.

This chapter consists of three sections:

In section one, we give some basic concepts of generalized Taylor formula.

In section two, we use Taylor expansion method for solving the local initial value problems that consist of the first order linear Fredholm-Volterra integro-differential equations together with the linear non-local initial conditions.

In section three, we use the generalized Taylor expansion method for solving the non-local initial value problems that consist of the linear fractional Fredholm-Volterra integro-differential equations together with the linear non-local initial conditions.

3.1 Some Basic Concepts of Generalized Taylor Formula:

In this section we give some basic concepts of the generalized Taylor formula with some illustrative examples. This generalized Taylor formula for a function u defined on x > a is obtained in terms of its left hand Caputo fractional derivatives evaluated only at the initial point a of the independent variable x, $\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u\right)(a)$, i=0,1,….

We start this section by giving the generalized mean value theorem. But before that we need the following lemma.

Lemma (3.1), (Generalized Mean Value Theorem for Integrals), [12]:

Let u be a continuous function on [a,x], v is an integrable function on [a,x] and v ≥ 0, then there exists a number $\xi \in$ [a,x] such that

$$\int_a^x u(x)v(x)dx = u(\xi)\int_a^x v(x)dx$$

Theorem (3.2), (Generalized Mean Value Theorem), [46]:

Suppose that u ∈ C [a, b] and ${}^{C}D_{a^+}^{\alpha}u \in C[a,b]$ for $0<\alpha\leq 1$, then

$$u(x)=u(a)+\frac{1}{\Gamma(\alpha+1)}\left(\left({}^{C}D_{a^+}^{\alpha}\right)u\right)(\xi)(x-a)^{\alpha}$$

where $a\leq\xi\leq x$.

Proof:

It is known that

$$I_{a^+}^{\alpha}\left({}^{C}D_{a^+}^{\alpha}u\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}\left(\left({}^{C}D_{a^+}^{\alpha}\right)u\right)(y)dy.$$

By using lemma (3.1) one can have:

$$
\begin{aligned}
I_{a^+}^{\alpha}\left({}^{C}D_{a^+}^{\alpha}u\right)(x) &= \frac{1}{\Gamma(\alpha)}\left(\left({}^{C}D_{a^+}^{\alpha}\right)u\right)(\xi)\int_{a}^{x}(x-y)^{\alpha-1}dy \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\left({}^{C}D_{a^+}^{\alpha}\right)u\right)(\xi)(x-a)^{\alpha} \quad \forall x\in[a,b]
\end{aligned}
$$

where $a \le \xi \le b$.

By using remarks (1.21), (4) one can get:

$$
\begin{aligned}
I_{a^+}^{\alpha}\left({}^{C}D_{a^+}^{\alpha}u(x)\right) &= u(x)-u(a) \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\left({}^{C}D_{a^+}^{\alpha}\right)u\right)(\xi)(x-a)^{\alpha} \quad \forall x\in[a,b].
\end{aligned}
$$

Therefore

$$
u(x) = u(a) + \frac{1}{\Gamma(\alpha+1)}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)u\right)(\xi)(x-a)^{\alpha}.
$$

Remark (3.3):

If α=1, then the generalized mean value theorem reduces to the classical mean value theorem.

Next, before we give the generalized Taylor theorem, we need the following generalized lemma.

Lemma (3.4), [48]:

Suppose that $\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\in C[a,b]$ and $\left({}^{C}D_{a^+}^{\alpha}\right)^{i+1}u\in C[a,b]$ for $0 < \alpha \le 1$, and i=0,1,…, then for each x∈ [a, b]:

$$
\left(I_{a^+}^{i\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x) - \left(I_{a^+}^{(i+1)\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{(i+1)}u\right)(x) = \frac{1}{\Gamma(i\alpha+1)}\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)(x-a)^{i\alpha}
$$

Proof:

Consider

$$\left(I_{a^+}^{i\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x)-\left(I_{a^+}^{(i+1)\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{(i+1)}u\right)(x)$$

$$=I_{a^+}^{i\alpha}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x)-\left(I_{a^+}^{\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{(i+1)}u\right)(x)\right)$$

$$=I_{a^+}^{i\alpha}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x)-\left(I_{a^+}^{\alpha}\,{}^{C}D_{a^+}^{\alpha}\right)\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x)\right)$$

Since $0<\alpha\leq 1$, then n=1. So by using remarks (1.21), (4) one can get:

$$\left(I_{a^+}^{i\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x)-\left(I_{a^+}^{(i+1)\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{(i+1)}u\right)(x)$$

$$=I_{a^+}^{i\alpha}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x)-\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(x)+\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)\right)$$

$$=I_{a^+}^{i\alpha}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)\right)$$

$$=\frac{1}{\Gamma(i\alpha)}\int_a^x (x-y)^{i\alpha-1}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)\right)dy$$

$$=\frac{1}{\Gamma(i\alpha)}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)\right)\int_a^x (x-y)^{i\alpha-1}dy$$

$$=\frac{1}{\Gamma(i\alpha+1)}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)\right)(x-a)^{i\alpha},\quad i=0,1,\ldots.$$

Remark (3.5):

If $\alpha=1$, then lemma (3.4) reduces to the classical equation:

$$I_{a^+}^{i}u^{(i)}(x)-I_{a^+}^{(i+1)}u^{(i+1)}(x)=\frac{1}{i!}u^{(i)}(a)(x-a)^{i},\quad i=0,1,\ldots,$$

Theorem (3.6), (Generalized Taylor Formula), [46]:

Suppose that $\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u \in C[a,b]$ for $0 < \alpha \leq 1$, and i=0,1,…,N+1 then for each x∈ [a, b]:

$$u(x) = \sum_{i=0}^{N} \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u\right)(a)}{\Gamma\left(i\alpha+1\right)}(x-a)^{i\alpha} + \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1} u\right)(\xi)}{\Gamma\left((N+1)\alpha+1\right)}(x-a)^{(N+1)\alpha} \qquad (3.1)$$

where $a \leq \xi \leq b$.

Proof:

By using lemma (3.4), one can have:

$$\sum_{i=0}^{N}\left[\left(I_{a^+}^{i\alpha}({}^{C}D_{a^+}^{\alpha})^{i} u\right)(x) - \left(I_{a^+}^{(i+1)\alpha}({}^{C}D_{a^+}^{\alpha})^{i+1} u\right)(x)\right] = \sum_{i=0}^{N}\frac{\left(({}^{C}D_{a^+}^{\alpha})^{i} u\right)(a)}{\Gamma(i\alpha+1)}(x-a)^{i\alpha}$$

Therefore

$$u(x) - \left(I_{a^+}^{(N+1)\alpha}({}^{C}D_{a^+}^{\alpha})^{N+1} u\right)(x) = \sum_{i=0}^{N}\frac{\left(({}^{C}D_{a^+}^{\alpha})^{i} u\right)(a)}{\Gamma(i\alpha+1)}(x-a)^{i\alpha} \qquad (3.2)$$

By using lemma (3.1), one can obtain:

$$\begin{aligned}\left(I_{a^+}^{(N+1)\alpha}\left({}^{C}D_{a^+}^{\alpha}\right)^{(N+1)} u\right)(x) &= \frac{1}{\Gamma((N+1)\alpha)}\int_{a}^{x}(x-y)^{(N+1)\alpha-1}\left(\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1} u\right)(y)\right)dy \\ &= \frac{1}{\Gamma((N+1)\alpha)}\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1} u\right)(\xi)\int_{a}^{x}(x-y)^{(N+1)\alpha-1}dy \\ &= \frac{1}{\Gamma((N+1)\alpha+1)}\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1} u\right)(\xi)(x-a)^{(N+1)\alpha}, \qquad (3.3)\end{aligned}$$

By substituting equation (3.3) into equation (3.2), one can get:

$$u(x)=\sum_{i=0}^{N}\frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)}{\Gamma\left(i\alpha+1\right)}(x-a)^{i\alpha}+\frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1}u\right)(\xi)}{\Gamma\left((N+1)\alpha+1\right)}(x-a)^{(N+1)\alpha}$$

Remarks (3.7), [46]:

(1) For $\alpha = 1$, theorem (3.6) reduces to the classical Taylor formula of u about a:

$$u(x)=\sum_{i=0}^{N}\frac{u^{(i)}(a)}{i!}(x-a)^{i}+\frac{u^{(N+1)}(\xi)}{(N+1)!}(x-a)^{(N+1)},\quad a\le\xi\le b$$

(2) suppose that $\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\in C[a,b]$ for $0 < \alpha \le 1$, and i=0,1,…, then the generalized Taylor series for u takes the form:

$$\sum_{i=0}^{\infty}\frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)}{\Gamma\left(i\alpha+1\right)}(x-a)^{i\alpha}$$

(3) Suppose that $\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\in C[a,b]$ for $0 < \alpha \le 1$, and i=0,1,…,N+1, then

$$u(x)\cong u_N(x)=\sum_{i=0}^{N}\frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)}{\Gamma\left(i\alpha+1\right)}(x-a)^{i\alpha}$$

Furthermore, the error term $R_N(x)$ has the form:

$$R_N(x)=\frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N+1}u\right)(\xi)}{\Gamma\left((N+1)\alpha+1\right)}(x-a)^{(N+1)\alpha},\quad a\le\xi\le b$$

Example (3.8):

Let u(x) = k, $x\in [a, b]$, where k is a known constant, then

$\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\in C[a,b]$ for $0 < \alpha \le 1$, and i=0,1,…. Thus

$$u(x) \cong u_N(x) = \sum_{i=0}^{N} \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^i u\right)(a)}{\Gamma(i\alpha+1)}(x-a)^{i\alpha}, \quad a \le x \le b, \ 0<\alpha \le 1$$

But $\left(\left({}^{C}D_{a^+}^{\alpha}\right)^i u\right)(a) = 0$, i=1,2,..., then $u(x) \cong u_0(x) = u(a) = k$. In this case $R_0(x) = 0, \ \forall x \in [a,b]$.

Example (3.9):

Let $u(x) = k_1 x + k_2, \ x \in [a, b]$, where k_1, k_2 are known constants, then $\left({}^{C}D_{a^+}^{\alpha}\right)^i u \in C[a,b]$ for $0 < \alpha \le 1$, and i=0,1,…. Thus

$$u(x) \cong u_N(x) = \sum_{i=0}^{N} \frac{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^i u\right)(a)}{\Gamma(i\alpha+1)}(x-a)^{i\alpha}, \quad a \le x \le b, \ 0<\alpha \le 1$$

Therefore, if $a = 0, \alpha = \frac{1}{2}$ then by using proposition (1.13) one can have:

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right) u\right)(x) = \frac{k_1\sqrt{x}}{\Gamma\left(\frac{3}{2}\right)} = \frac{2k_1\sqrt{x}}{\sqrt{\pi}}$$

and

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^2 u\right)(x) = k_1.$$

So

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^i u\right)(x) = 0, \ \text{i=3,4,....}$$ Hence

$$u(x)\cong u_2(x)=u(0)+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)}{\Gamma\left(\frac{3}{2}\right)}\sqrt{x}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^2 u\right)(0)}{\Gamma(2)}x$$
$$=k_2+k_1x,\ \ 0\le x\le b.$$

In this case $R_2(x)=0$. On the other hand, if $a=1,\alpha=\frac{1}{2},$ then

$$\left(\left({}^{C}D_{1^+}^{\frac{1}{2}}\right)u\right)(x)=\frac{2k_1\sqrt{x-1}}{\sqrt{\pi}},\ \left(\left({}^{C}D_{1^+}^{\frac{1}{2}}\right)^2 u\right)(x)=k_1 \text{ and}$$

$$\left(\left({}^{C}D_{1^+}^{\frac{1}{2}}\right)^i u\right)(x)=0, i=3,4,\ldots. \text{Hence}$$

$$u(x)\cong u_2(x)=u(1)+\frac{\left(\left({}^{C}D_{1^+}^{\frac{1}{2}}\right)u\right)(1)}{\Gamma\left(\frac{3}{2}\right)}\sqrt{x-1}+\frac{\left(\left({}^{C}D_{1^+}^{\frac{1}{2}}\right)^2 u\right)(1)}{\Gamma(2)}(x-1)$$
$$=k_2+k_1+k_1(x-1)$$
$$=k_2+k_1x,\ \ 1\le x\le b.$$

In this case $R_2(x)=0$. Moreover, if $\alpha=\frac{1}{3},$ then

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{3}}\right)u\right)(x)=\frac{3k_1x^{\frac{2}{3}}}{2\,\Gamma\left(\frac{2}{3}\right)},\ \ \left(\left({}^{C}D_{1^+}^{\frac{1}{3}}\right)u\right)(x)=\frac{3k_1(x-1)^{\frac{2}{3}}}{2\,\Gamma\left(\frac{2}{3}\right)},$$

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{3}}\right)^{2} u\right)(x)=\frac{3\sqrt{3}\ \Gamma\left(\frac{2}{3}\right)k_1 x^{\frac{1}{3}}}{2\ \pi},\ \left(\left({}^{C}D_{1^+}^{\frac{1}{3}}\right)^{2} u\right)(x)=\frac{3\sqrt{3}\ \Gamma\left(\frac{2}{3}\right)k_1 (x-1)^{\frac{1}{3}}}{2\ \pi},$$

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{3}}\right)^{3} u\right)(x)=k_1,\ \left(\left({}^{C}D_{1^+}^{\frac{1}{3}}\right)^{3} u\right)(x)=k_1.$$

Hence

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{3}}\right)^{i} u\right)(x)=\ \left(\left({}^{C}D_{1^+}^{\frac{1}{3}}\right)^{i} u\right)(x)=0,\ i=4,5,....$$

Thus

$$\begin{aligned} u(x) &\cong u_3(x)\\ &=u(0)+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{3}}\right)u\right)(0)}{\Gamma\left(\frac{4}{3}\right)}x^{\frac{1}{3}}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{3}}\right)^{2} u\right)(0)}{\Gamma\left(\frac{5}{3}\right)}x^{\frac{2}{3}}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{3}}\right)^{3} u\right)(0)}{\Gamma(2)}x\\ &=k_2+k_1x,\ \ 0\le x\le b \end{aligned}$$

and

$$\begin{aligned} u(x) &\cong u_3(x)\\ &=u(1)+\frac{\left(\left({}^{C}D_{1^+}^{\frac{1}{3}}\right)u\right)(1)}{\Gamma\left(\frac{4}{3}\right)}(x-1)^{\frac{1}{3}}+\frac{\left(\left({}^{C}D_{1^+}^{\frac{1}{3}}\right)^{2} u\right)(1)}{\Gamma\left(\frac{5}{3}\right)}(x-1)^{\frac{2}{3}}+\\ &\qquad \frac{\left(\left({}^{C}D_{1^+}^{\frac{1}{3}}\right)^{3} u\right)(1)}{\Gamma(2)}(x-1) \end{aligned}$$

$$=k_2+k_1+k_1(x-1)$$
$$=k_2+k_1x,\ \ 1\le x\le b.$$

In the above two cases $R_3(x)=0$.

Example (3.10):

Let $u(x)=x^3$, a=0 and $\alpha=\frac{1}{2}$, then

$$u(x)\cong u_N(x)=\sum_{i=0}^{N}\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{i}u\right)(0)}{\Gamma\left(\frac{i}{2}+1\right)}x^{\frac{i}{2}},\ \ 0\le x\le b$$

By using proposition (1.13) one can have:

$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(x)=\dfrac{16x^{\frac{5}{2}}}{5\sqrt{2}}$ and this implies that $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)=0,$

$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{2}u\right)(x)=3x^2$ and this implies that $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{2}u\right)(0)=0,$

$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{3}u\right)(x)=\dfrac{8x^{\frac{3}{2}}}{\sqrt{\pi}}$ and this implies that $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{3}u\right)(0)=0,$

$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{4}u\right)(x)=6x$ and this implies that $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{4}u\right)(0)=0,$

$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{5}u\right)(x)=\dfrac{12\sqrt{x}}{\sqrt{\pi}}$ and this implies that $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{5}u\right)(0)=0,$

$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{6} u\right)(x) = 6$ and this implies that $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{6} u\right)(0) = 6$

and $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{i} u\right)(x) = 0,\ i=7,8,\ldots$ Hence

$$u(x) \cong u_6(x) = \sum_{i=0}^{6} \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{i} u\right)(0)}{\Gamma\left(\frac{i}{2}+1\right)} x^{\frac{i}{2}}$$

$$= x^3, \quad 0 \le x \le b.$$

In the this case $R_6(x) = 0$.

3.2 The Classical Taylor Expansion Method for Solving Linear Integro-Differential Equations with Non-Local Conditions:

Recall that the classical Taylor expansion method is used to solve the local initial value problem for the linear integro-differential equations, [31].

In this section we use the same method to solve the non-local initial value problem that consists of the linear first order Fredholm-Volterra integro-differential equation of the second kind:

$$u'(x) = g(x) + \lambda_1 \int_a^b k(x,y)u(y)dy + \lambda_2 \int_a^x \ell(x,y)u(y)dy, \quad a \le x \le b \tag{3.4}$$

together with the linear non-local initial condition:

$$u(a) = \mu_1 \int_a^b u(y)dy + \mu_2 \tag{3.5}$$

where g, u $\in$ C[a, b], $k:[a,b]\times[a,b]\longrightarrow R$, $\ell:[a,b]\times[a,b]\longrightarrow R$ are continuous functions, μ_1, $\mu_2,\lambda_1,\lambda_2$ are known constants. To do this, we assume that the solution u of the non-local initial value problem given by equations (3.4)-(3.5) can be approximated by the classical Taylor's polynomial of degree N about c:

$$u(x)\cong u_N(x)=\sum_{i=0}^{N}\frac{1}{i!}u^{(i)}(c)(x-c)^i,\quad a\le c\le b \tag{3.6}$$

By substituting equation (3.6) into equations (3.4)-(3.5), one can have:

$$u'(x)=g(x)+\sum_{i=0}^{N}\frac{1}{i!}u^{(i)}(c)\left[\lambda_1\int_a^b k(x,y)(y-c)^i\,dy+\lambda_2\int_a^x \ell(x,y)(y-c)^i\,dy\right]$$

and

$$\sum_{i=0}^{N}\frac{1}{i!}u^{(i)}(c)(a-c)^i=\mu_1\sum_{i=0}^{N}\frac{1}{i!}u^{(i)}(c)\int_a^b (y-c)^i\,dy+\mu_2$$

$$=\mu_1\sum_{i=0}^{N}\frac{1}{i!}u^{(i)}(c)\left[\frac{(b-c)^{i+1}-(a-c)^{i+1}}{i+1}\right]+\mu_2$$

So

$$u'(c)=g(c)+\sum_{i=0}^{N}\frac{1}{i!}u^{(i)}(c)\left[\lambda_1\int_a^b k(c,y)(y-c)^i\,dy+\lambda_2\int_a^c \ell(c,y)(y-c)^i\,dy\right] \tag{3.7}$$

and

$$\sum_{i=0}^{N}\frac{1}{i!}\left[(a-c)^i-\frac{\mu_1}{i+1}\left\{(b-c)^{i+1}-(a-c)^{i+1}\right\}\right]u^{(i)}(c)=\mu_2 \tag{3.8}$$

Let $a_{i,0}=-\frac{\lambda_1}{i!}\int_a^b k(c,y)(y-c)^i\,dy-\frac{\lambda_2}{i!}\int_a^c \ell(c,y)(y-c)^i\,dy,\ i=0,1,\ldots,N$

and

$$f_i = \frac{1}{i!}\left[(a-c)^i - \frac{\mu_1}{i+1}\left\{(b-c)^{i+1} - (a-c)^{i+1}\right\} \right], \quad i=0,1,\dots,N.$$

Then equations (3.7)-(3.8) become:

$$\sum_{\substack{i=0 \\ i\neq 1}}^{N} a_{i,0} u^{(i)}(c) + (1+a_{1,0})u'(c) = g(c) \tag{3.9}$$

and

$$\sum_{i=0}^{N} f_i u^{(i)}(c) = \mu_2 \tag{3.10}$$

Moreover

$$u^{(j+1)}(x) = g^{(j)}(x) + \lambda_1 \sum_{i=0}^{N} \frac{1}{i!} u^{(i)}(c) \int_a^b \frac{\partial^j}{\partial x^j}\{k(x,y)\}(y-c)^i dy +$$
$$\lambda_2 \sum_{i=0}^{N} \frac{1}{i!} u^{(i)}(c) \frac{d^j}{dx^j}\left\{ \int_a^x \{\ell(x,y)\}(y-c)^i dy \right\}, \; j=1,2,\dots,N-1$$

So

$$u^{(j+1)}(c) = g^{(j)}(c) + \lambda_1 \sum_{i=0}^{N} \frac{1}{i!} u^{(i)}(c) \int_a^b \left(\frac{\partial^j}{\partial x^j}\{k(x,y)\} \right)\Bigg|_{x=c} (y-c)^i dy +$$
$$\lambda_2 \sum_{i=0}^{N} \frac{1}{i!} u^{(i)}(c) \left(\frac{d^j}{dx^j}\left\{ \int_a^x \ell(x,y)(y-c)^i dy \right\} \right)\Bigg|_{x=c}, \; j=1,2,\dots,N-1$$

$$\tag{3.11}$$

Let

$$a_{i,j}=\frac{1}{i!}\left[-\lambda_1\int_a^b\left(\frac{\partial^j}{\partial x^j}\{k(x,y)\}\right)\Bigg|_{x=c}(y-c)^i dy-\lambda_2\left(\frac{d^j}{dx^j}\left\{\int_a^x \ell(x,y)(y-c)^i dy\right\}\right)\Bigg|_{x=c}\right]$$

where $i=0,1,...,N,\ \ j=1,2,...,N-1$.

Then equation (3.11) becomes:

$$u^{(j+1)}(c)+\sum_{i=0}^{N}a_{i,j}u^{(i)}(c)=g^{(j)}(c),\ j=1,2,...,N-1 \tag{3.12}$$

Thus, by evaluating equation (3.12) at each j = 1, 2,…, N−1 and by using equations (3.9)-(3.10), one can have the following linear system of N + 1 equations with (N + 1) unknown $\{u^{(i)}(c)\}_{i=0}^{N}$:

$$AU=B \tag{3.13}$$

where

$$A=\begin{bmatrix} f_0 & f_1 & f_2 & f_3 & \cdots & f_{N-1} & f_N \\ a_{0,0} & 1+a_{1,0} & a_{2,0} & a_{3,0} & \cdots & a_{N-1,0} & a_{N,0} \\ a_{0,1} & a_{1,1} & 1+a_{2,1} & a_{3,1} & \cdots & a_{N-1,1} & a_{N,1} \\ a_{0,2} & a_{1,2} & a_{2,2} & 1+a_{3,2} & \cdots & a_{N-1,2} & a_{N,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{0,N-2} & a_{1,N-2} & a_{2,N-2} & a_{3,N-2} & \cdots & 1+a_{N-1,N-2} & a_{N,N-2} \\ a_{0,N-1} & a_{1,N-1} & a_{2,N-1} & a_{3,N-1} & \cdots & a_{N-1,N-1} & 1+a_{N,N-1} \end{bmatrix},$$

$$U=\begin{bmatrix} u(c) \\ u'(c) \\ u''(c) \\ u'''(c) \\ \vdots \\ u^{(N-1)}(c) \\ u^{(N)}(c) \end{bmatrix} \text{ and } B=\begin{bmatrix} \mu_2 \\ g(c) \\ g'(c) \\ g''(c) \\ \vdots \\ g^{(N-2)}(c) \\ g^{(N-1)}(c) \end{bmatrix}.$$

By solving the above linear system of equations, one can get the values of $\{u^{(i)}(c)\}_{i=0}^{N}$. These values are substituted into equation (3.6) to get the approximated solution of the non-local initial value problem given by equations (3.4)-(3.5).

Example (3.11):

Consider the nonlocal initial value problem that consists of the linear first order Fredholm-Volterra integro-differential equation of the second kind:

$$u'(x) = 2 + 2x - \frac{11}{12}x^2 + \frac{(20+70x)}{12}x^3 + \int_0^1 x^2 y u(y)dy + \int_0^x (x+y)u(y)dy \tag{3.14}$$

together with the linear nonlocal initial condition:

$$u(0) = \int_0^1 u(y)dy - \frac{4}{3} \tag{3.15}$$

Here $g(x) = 2 + 2x - \frac{11}{12}x^2 + \frac{(20+70x)}{12}x^3$, $0 \le x \le 1$, a=0, b=1,

$\lambda_1 = \lambda_2 = \mu_1 = 1$, $\mu_2 = \frac{-4}{3}$, k(x,y)=x^2y and ℓ(x,y)=x+y.

We use the classical Taylor expansion method to solve this linear nonlocal initial value problem.

To do this first, let N = 1 and c = 1, then the solution u can be approximated as a Taylor polynomial of degree 1 about c=1:

$$u(x) \cong u_1(x) = u(1) + u'(1)(x-1), \quad 0 \le x \le 1 \tag{3.16}$$

In this case:

$$f_0 = \frac{1}{0!}\left[(a-c)^0 - \mu_1\{(b-c)-(a-c)\}\right] = 0,$$
$$f_1 = \frac{1}{1!}\left[(a-c)^1 - \frac{\mu_1}{2}\{(b-c)^2-(a-c)^2\}\right] = \frac{-1}{2},$$

$$a_{0,0} = -\frac{\lambda_1}{0!}\int_a^b k(c,y)dy - \frac{\lambda_2}{0!}\int_a^c \ell(c,y)dy = -\int_0^1 ydy - \int_0^1 (1+y)dy = -2,$$

$$a_{1,0} = -\frac{\lambda_1}{1!}\int_a^b k(c,y)(y-c)dy - \frac{\lambda_2}{i!}\int_a^c \ell(c,y)(y-c)dy$$
$$= -\int_0^1 y(y-1)dy - \int_0^1 (1+y)(y-1)dy = \frac{5}{6}$$

and $g(c) = g(1) = \frac{5}{6}$. Therefore the system given by equation (3.13) takes the form:

$$\begin{bmatrix} 0 & \frac{-1}{2} \\ -2 & \frac{11}{6} \end{bmatrix}\begin{bmatrix} u(1) \\ u'(1) \end{bmatrix} = \begin{bmatrix} \frac{-4}{3} \\ \frac{5}{6} \end{bmatrix}$$

which has the solution $u(1)=\frac{73}{36}$ and $u'(1)=\frac{8}{3}$. By substituting these values into equation (3.16) one can have:

$$u(x) \cong u_1(x) = \frac{-23}{36} + \frac{8}{3}x$$

Since

$$u_1'(x) - 2 - 2x + \frac{11}{12}x^2 - \frac{(20+70x)}{12}x^3 - \int_0^1 x^2 y u_1(y)dy - \int_0^x (x+y)u_1(y)dy =$$
$$\frac{(x-1)(21x^3+x^2+48x-24)}{36} \neq 0 \ \forall x,\ 0 \le x \le 1,$$

then we must increase the value of N. So, let N=2, therefore the solution u can be approximated as a Taylor polynomial of degree 2 about c=1:

$$u(x) \cong u_2(x) = u(1) + u'(1)(x-1) + \frac{u''(1)}{2!}(x-1)^2, \quad 0 \le x \le 1 \tag{3.17}$$

In this case:

$$f_0 = 0,\ f_1 = \frac{-1}{2},\ f_2 = \frac{1}{2!}\left[(a-c)^2 - \frac{\mu_1}{3}\left\{(b-c)^3 - (a-c)^3\right\}\right] = \frac{1}{3},$$

$$a_{0,0} = -2,\ a_{1,0} = \frac{5}{6},\ a_{2,0} = -\frac{1}{2!}\int_0^1 y(y-1)^2\,dy - \frac{1}{2!}\int_0^1 (1+y)(y-1)^2\,dy = -\frac{1}{4}$$

$$a_{0,1} = -\int_0^1 \left(\frac{\partial}{\partial x}\left\{x^2 y\right\}\right)\Bigg|_{x=1} dy - \left(\frac{d}{dx}\left\{\int_0^x (x+y)\,dy\right\}\right)\Bigg|_{x=1} = -4,$$

$$a_{1,1} = -\int_0^1 \left(\frac{\partial}{\partial x}\left\{x^2 y\right\}\right)\Bigg|_{x=1} (y-1)\,dy - \left(\frac{d}{dx}\left\{\int_0^x (x+y)(y-1)\,dy\right\}\right)\Bigg|_{x=1} = \frac{5}{6}$$

$$a_{2,1} = \frac{-1}{2!}\left[\int_0^1 \left(\frac{\partial}{\partial x}\left\{x^2 y\right\}\right)\Bigg|_{x=1} (y-1)^2\,dy + \left(\frac{d}{dx}\left\{\int_0^x (x+y)(y-1)^2\,dy\right\}\right)\Bigg|_{x=1}\right] = -\frac{1}{4}$$

and $g(c) = g(1) = \frac{5}{6}$ and $g'(c) = g'(1) = \frac{-43}{6}$. Therefore the system given by equation (3.13) takes the form:

$$\begin{bmatrix} 0 & \frac{-1}{2} & \frac{1}{3} \\ -2 & \frac{11}{6} & \frac{-1}{4} \\ -4 & \frac{5}{6} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} u(1) \\ u'(1) \\ u''(1) \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{6} \\ \frac{-43}{6} \end{bmatrix}$$

which has the solution $u(1) = 3, u'(1) = 4,\ u''(1)=2$. By substituting these values into equation (3.17) one can have:

$$u(x) \cong u_2(x) = x^2 + 2x, \quad 0 \le x \le 1$$

which is the exact solution of the nonlocal initial value problem given by equations (3.14)-(3.15).

Second, we try to solve this example with another value of c. To do this, let $c = \frac{1}{2}$, and N=2, then

$$u(x) \cong u_2(x) = u\left(\frac{1}{2}\right) + u'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2!}u''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2, \quad 0 \le x \le 1 \tag{3.18}$$

In this case

$f_0 = 0,\ f_1 = \frac{-1}{2},\ f_2 = \frac{1}{12},\ a_{0,0} = \frac{-1}{2},\ a_{1,0} = \frac{1}{16},\ a_{2,0} = -\frac{7}{384},\ a_{0,1} = -2,$

$a_{1,1} = \frac{1}{24},\ a_{2,1} = \frac{-1}{24},\ g(c) = g\left(\frac{1}{2}\right) = \frac{485}{192}$ and $g'(c) = g'\left(\frac{1}{2}\right) = \frac{-11}{24}.$

Therefore the system given by equation (3.13) takes the form:

$$\begin{bmatrix} 0 & \frac{-1}{2} & \frac{1}{12} \\ \frac{-1}{2} & \frac{17}{16} & \frac{-7}{384} \\ -2 & \frac{1}{24} & \frac{23}{24} \end{bmatrix} \begin{bmatrix} u\left(\frac{1}{2}\right) \\ u'\left(\frac{1}{2}\right) \\ u''\left(\frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{485}{192} \\ \frac{-11}{24} \end{bmatrix}$$

which has the solution $u\left(\frac{1}{2}\right) = \frac{5}{4},\ u'\left(\frac{1}{2}\right) = 3,\ u''\left(\frac{1}{2}\right) = 2$. By substituting these values into equation (3.18) one can have:

$$u(x) \cong u_2(x) = x^2 + 2x, \quad 0 \le x \le 1$$

which is the exact solution of the nonlocal initial value problem given by equations (3.14)-(3.15).

3.3 The Generalized Taylor Expansion Method for Solving Linear Fractional Integro-Differential Equations with Non-Local Conditions:

In this section we introduced a method named as the generalized Taylor expansion method for solving the non-local initial value problem that consists of the linear Fredholm-Volterra fractional integro- differential equation of order α of the second kind:

$$ {}^{C}D_{a^+}^{\alpha}u(x)=g(x)+\lambda_1\int_a^b k(x,y)u(y)dy+\lambda_2\int_a^x \ell(x,y)u(y)dy,\quad 0<\alpha\le 1 \qquad (3.19)$$

together with the linear non-local initial condition:

$$u(a)=\mu_1\int_a^b u(y)dy+\mu_2 \qquad (3.20)$$

where $g,\ u\in C[a,b]$, $k:[a,b]\times[a,b]\longrightarrow R$, $\ell:[a,b]\times[a,b]\longrightarrow R$ are continuous functions, ${}^{C}D_{a^+}^{\alpha}u$ is the left hand Caputo fractional derivative of u of order α and $\mu_1,\ \mu_2,\lambda_1,\lambda_2$ are known constants.

To do this, we assume that the solution u of the non-local initial value problem given by equations (3.19)-(3.20) can be approximated as a generalized Taylor's formula:

$$u(x)\cong u_N(x)=\sum_{i=0}^{N}\frac{\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^i u\right)(a)}{\Gamma(i\alpha+1)}(x-a)^{i\alpha},\quad a\le x\le b \qquad (3.21)$$

By substituting equation (3.21) into equations (3.19)-(3.20), one can have:

$$ {}^{C}D_{a^+}^{\alpha}u(x)=g(x)+\sum_{i=0}^{N}\frac{1}{\Gamma(i\alpha+1)}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^i u\right)(a)\left[\lambda_1\int_a^b k(x,y)(y-a)^{i\alpha}dy+\lambda_2\int_a^x \ell(x,y)(y-a)^{i\alpha}dy\right]$$

and

$$u(a)=\mu_1\sum_{i=0}^{N}\frac{1}{\Gamma(i\alpha+1)}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)\int_a^b (y-a)^{i\alpha}dy+\mu_2$$

$$=\mu_1\sum_{i=0}^{N}\frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)+\mu_2$$

So

$$\left({}^{C}D_{a^+}^{\alpha}u\right)(a)=g(a)+\sum_{i=0}^{N}\frac{1}{\Gamma(i\alpha+1)}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)\left[\lambda_1\int_a^b k(a,y)(y-a)^{i\alpha}dy\right] \tag{3.22}$$

and

$$\left[1-\mu_1(b-a)\right]u(a)-\mu_1\sum_{i=1}^{N}\frac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)=\mu_2 \tag{3.23}$$

Let $a_{i,0}=\dfrac{-\lambda_1}{\Gamma(i\alpha+1)}\displaystyle\int_a^b k(a,y)(y-a)^{i\alpha}dy$, i=0,1,...,N, $f_0=1-\mu_1(b-a)$

and $f_i=-\mu_1\dfrac{(b-a)^{i\alpha+1}}{\Gamma(i\alpha+2)}$, i=1,2,...,N.

Then equations (3.22)-(3.23) become:

$$\sum_{\substack{i=0\\ i\neq 1}}^{N}a_{i,0}\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)+(1+a_{1,0})\left({}^{C}D_{a^+}^{\alpha}u\right)(a)=g(a) \tag{3.24}$$

and

$$\sum_{i=0}^{N}f_i\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)=\mu_2 \tag{3.25}$$

Let

$$m_i(x)=\int_a^b k(x,y)(y-a)^{i\alpha}dy,\ p_i(x)=\int_a^x \ell(x,y)(y-a)^{i\alpha}dy,\ i=0,1,\ldots,N$$

Then

$$\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j+1}u\right)(x)=\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}g\right)(x)+$$
$$\sum_{i=0}^{N}\left[\frac{\lambda_1\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}m_i\right)(x)+\lambda_2\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}p_i\right)(x)}{\Gamma(i\alpha+1)}\right]\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)$$

where $j=1,2,\ldots,N-1$.

So,

$$\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j+1}u\right)(a)=\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}g\right)(a)+$$
$$\sum_{i=0}^{N}\left[\frac{\lambda_1\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}m_i\right)(a)+\lambda_2\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}p_i\right)(a)}{\Gamma(i\alpha+1)}\right]\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a) \tag{3.26}$$

Let

$$a_{i,j}=-\frac{\lambda_1\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}m_i\right)(a)+\lambda_2\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}p_i\right)(a)}{\Gamma(i\alpha+1)},\ i=0,1,\ldots,N,\ j=1,2,\ldots,N-1.$$

Then equation (3.26) becomes:

$$\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j+1}u\right)(a)+\sum_{i=0}^{N}a_{i,j}\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i}u\right)(a)=\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{j}g\right)(a),\ j=1,2,\ldots,N-1 \tag{3.27}$$

Thus, by evaluating equation (3.27) at each j=1, 2,…, N−1 and by using equations (3.24)-(3.25), one can have the following linear system of N+1 equations with (N+1) unknowns $\left\{\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)\right\}_{i=0}^{N}$:

$$AU=B \tag{3.28}$$

where

$$A=\begin{bmatrix} f_0 & f_1 & f_2 & f_3 & \cdots & f_{N-1} & f_N \\ a_{0,0} & 1+a_{1,0} & a_{2,0} & a_{3,0} & \cdots & a_{N-1,0} & a_{N,0} \\ a_{0,1} & a_{1,1} & 1+a_{2,1} & a_{3,1} & \cdots & a_{N-1,1} & a_{N,1} \\ a_{0,2} & a_{1,2} & a_{2,2} & 1+a_{3,2} & \cdots & a_{N-1,2} & a_{N,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{0,N-2} & a_{1,N-2} & a_{2,N-2} & a_{3,N-2} & \cdots & 1+a_{N-1,N-2} & a_{N,N-2} \\ a_{0,N-1} & a_{1,N-1} & a_{2,N-1} & a_{3,N-1} & \cdots & a_{N-1,N-1} & 1+a_{N,N-1} \end{bmatrix},$$

$$U=\begin{bmatrix} u(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)u\right)(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{2}u\right)(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{3}u\right)(a) \\ \vdots \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)u\right)^{(N-1)}(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)u\right)^{(N)}(a) \end{bmatrix} \text{ and } B=\begin{bmatrix} \mu_2 \\ g(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)g\right)(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{2}g\right)(a) \\ \vdots \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)g\right)^{(N-2)}(a) \\ \left(\left({}^{C}D_{a^+}{}^{\alpha}\right)g\right)^{(N-1)}(a) \end{bmatrix}$$

By solving the above linear system of equations, one can get the values of $\left\{\left(\left({}^{C}D_{a^+}{}^{\alpha}\right)^{i}u\right)(a)\right\}_{i=0}^{N}$. These values are substituted into equation (3.21) to get the approximated solution of the non-local initial value problem given by equations (3.19)-(3.20).

Remark (3.12):

If $\mu_1 = 0$, then the initial value problem given by equations (3.19)-(3.20) is local. In this case, $u(a)=\mu_2$, where μ_2 is a known constant. So in the generalized Taylor formula given by equation (3.21), the values of $\left\{\left(\left({}^{C}D_{a^+}^{\alpha}\right)^{i} u\right)(a)\right\}_{i=1}^{N}$ can be obtained by solving the N×N linear system:

$$AU=B \qquad (3.29)$$

where

$$A=\begin{bmatrix} 1+a_{1,0} & a_{2,0} & a_{3,0} & \cdots & a_{N-1,0} & a_{N,0} \\ a_{1,1} & 1+a_{2,1} & a_{3,1} & \cdots & a_{N-1,1} & a_{N,1} \\ a_{1,2} & a_{2,2} & 1+a_{3,2} & \cdots & a_{N-1,2} & a_{N,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1,N-2} & a_{2,N-2} & a_{3,N-2} & \cdots & 1+a_{N-1,N-2} & a_{N,N-2} \\ a_{1,N-1} & a_{2,N-1} & a_{3,N-1} & \cdots & a_{N-1,N-1} & 1+a_{N,N-1} \end{bmatrix},$$

$$U=\begin{bmatrix} \left(\left({}^{C}D_{a^+}^{\alpha}\right)u\right)(a) \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)^{2}u\right)(a) \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)^{3}u\right)(a) \\ \vdots \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N-1}u\right)(a) \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N}u\right)(a) \end{bmatrix} \text{ and } B=\begin{bmatrix} g(a)-\mu_2 a_{0,0} \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)g\right)(a)-a_{0,1}\mu_2 \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)^{2}g\right)(a)-a_{0,2}\mu_2 \\ \vdots \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N-2}g\right)(a)-a_{0,N-2}\mu_2 \\ \left(\left({}^{C}D_{a^+}^{\alpha}\right)^{N-1}g\right)(a)-a_{0,N-1}\mu_2 \end{bmatrix}.$$

To illustrate this method, consider the following examples:

Example (3.13):

Consider the nonlocal initial value problem that consists of the fractional linear Fredholm-Volterra integro-differential equation of order $\frac{1}{2}$:

$$ {}^{C}D_{0^+}^{\frac{1}{2}}u(x) = -\frac{3}{2}-\frac{25}{4}x^2+\frac{16}{\sqrt{\pi}}x^{\frac{5}{2}}-\frac{23}{4}x^5+\int_0^1(x^2+y)u(y)dy+\int_0^x(3x+2y)u(y)dy \tag{3.30} $$

together with the nonlocal linear initial condition:

$$ u(0) = 2\int_0^1 u(y)dy - \frac{7}{2} \tag{3.31} $$

Here $g(x) = -\frac{3}{2}-\frac{25}{4}x^2+\frac{16}{\sqrt{\pi}}x^{\frac{5}{2}}-\frac{23}{4}x^5,\ 0 \le x \le 1$, a=0, b=1,

$\lambda_1 = \lambda_2 = 1,\ \mu_1 = 2,\ \mu_2 = \frac{-7}{2}$, $k(x,y)=x^2+y$ and $\ell(x,y)=3x+2y$.

We use the generalized Taylor expansion method to solve this fractional linear nonlocal initial value problem. To do this, let N =1, then equation (3.21) takes the form:

$$ u(x) \cong u_1(x) = u(0) + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)}{\Gamma\left(\frac{3}{2}\right)}\sqrt{x},\ 0 \le x \le 1 \tag{3.32} $$

In this case:

$$ f_0 = 1-\mu_1(b-a) = 1-2 = -1,\ f_1 = -\mu_1\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} = \frac{-2}{\Gamma\left(\frac{5}{2}\right)} = \frac{-8}{3\sqrt{\pi}}, $$

$$ a_{0,0} = \frac{-1}{\Gamma(1)}\int_0^1 y dy = \frac{-1}{2},\ a_{1,0} = \frac{-1}{\Gamma\left(\frac{3}{2}\right)}\int_0^1 y^{\frac{3}{2}}dy = \frac{-2}{5\Gamma\left(\frac{3}{2}\right)} = \frac{-4}{5\sqrt{\pi}}. $$

and g(0) = $\frac{-3}{2}$. Then the system given by equation (3.28) takes the form:

$$\begin{pmatrix} -1 & \frac{-8}{3\sqrt{\pi}} \\ -\frac{1}{2} & \frac{5\sqrt{\pi}-4}{5\sqrt{\pi}} \end{pmatrix}\begin{pmatrix} u(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0) \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{3}{2} \end{pmatrix}$$

which has the solution:

$$u(0) = \frac{3(35\sqrt{\pi}+12)}{2(15\sqrt{\pi}+8)} \cong 3.21087$$

and

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0) = \frac{1}{4} - \frac{2}{15\sqrt{\pi}+8} \cong 0.192174.$$

By substituting these values into equation (3.32) one can have:

$$u(x) \cong u_1(x) = 3.21087 + 0.216846\sqrt{x}, \quad 0 \le x \le 1$$

By substituting this approximated solution into equation (3.30) one can have:

$${}^{C}D_{0^+}^{\frac{1}{2}}u_1(x) + \frac{3}{2} + \frac{25}{4}x^2 - \frac{16}{\sqrt{\pi}}x^{\frac{5}{2}} + \frac{23}{4}x^5 - \int_0^1 (x^2+y)u_1(y)dy - \int_0^x (3x+2y)u_1(y)dy \cong 5.52233 - 5.625x^2 - 92.625x^{\frac{5}{2}} + 5.75x^5$$

Since the right hand side of the above equation does not equal zero for each x∈[0,1], so we must increase the value of N. Therefore, let N=2, then equation (3.21) takes the form:

$$u(x) \cong u_2(x) = u(0) + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^2 u\right)(0)}{\Gamma(2)}x, \quad 0 \le x \le 1 \tag{3.33}$$

In this case:

$$f_0=-1,\ f_1=\frac{-8}{3\sqrt{\pi}},\ f_2=-1,\ a_{0,0}=\frac{-1}{2},\ a_{1,0}=\frac{-4}{5\sqrt{\pi}},\ a_{2,0}=\frac{-1}{3}.$$

$$m_0(x)=x^2+\frac{1}{2},\ m_1(x)=\frac{2}{3}x^2+\frac{2}{5},\ m_2(x)=\frac{1}{2}x^2+\frac{1}{3},$$

$$p_0(x)=4x^2,\ p_1(x)=\frac{14}{5}x^{\frac{5}{2}},\ p_2(x)=\frac{13}{6}x^3.$$

Therefore

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)m_0\right)(x)=\frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}},\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)m_1\right)(x)=\frac{16}{9\sqrt{\pi}}x^{\frac{3}{2}},$$

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)m_2\right)(x)=\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}},\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)p_0\right)(x)=\frac{32}{3\sqrt{\pi}}x^{\frac{3}{2}},$$

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)p_1\right)(x)=\frac{21\sqrt{\pi}}{8}x^{2},\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)p_2\right)(x)=\frac{104}{15\sqrt{\pi}}x^{\frac{5}{2}}.$$

So $a_{0,1}=a_{1,1}=a_{2,1}=0$. Moreover

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)g\right)(x)=\frac{-50x^{\frac{3}{2}}}{3\sqrt{\pi}}+15x^2-\frac{1472x^{\frac{9}{2}}}{63\sqrt{\pi}},\quad 0\le x\le 1.$$

So $\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)g\right)(0)=0.$

Thus the system given by equation (3.28) takes the form:

$$\begin{pmatrix} -1 & \dfrac{-8}{3\sqrt{\pi}} & -1 \\ \dfrac{-1}{2} & \dfrac{5\sqrt{\pi}-4}{5\sqrt{\pi}} & \dfrac{-1}{3} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(0) \\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)u\right)(0) \\ \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^2 u\right)(0) \end{pmatrix} = \begin{pmatrix} -\dfrac{7}{2} \\ -\dfrac{3}{2} \\ 0 \end{pmatrix}$$

which has the solution:

$$u(0) = \frac{3(35\sqrt{\pi}+12)}{2(15\sqrt{\pi}+8)} \cong 3.21087$$

$$\left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)u\right)(0) = \frac{1}{4} - \frac{2}{15\sqrt{\pi}+8} \cong 0.192174 \text{ and } \left(\left({}^{C}D_{o^+}^{\frac{1}{2}}\right)^2 u\right)(0) = 0.$$

By substituting these values into equation (3.33) one can have:

$$u(x) \cong u_2(x) = 3.21087 + 0.216846\sqrt{x}, \quad 0 \le x \le 1.$$

Since $u_2(x)=u_1(x)$, so we must increase the value of N. By continuing in this manner one can get for N=6, equation (3.21) takes the form:

$$\begin{aligned} u(x) &\cong u_6(x) \\ &= u(0) + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0)}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}} + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^2 u\right)(0)}{\Gamma(2)} x + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^3 u\right)(0)}{\Gamma\left(\frac{5}{2}\right)} x^{\frac{3}{2}} + \\ &\quad \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^4 u\right)(0)}{\Gamma(3)} x^2 + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^5 u\right)(0)}{\Gamma\left(\frac{7}{2}\right)} x^{\frac{5}{2}} + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^6 u\right)(0)}{\Gamma(4)} x^3, \end{aligned} \tag{3.34}$$

where $0 \le x \le 1$

Then after simple computations and by using the Mathcad software package, the system given by equation (3.28) takes the form:

$$\begin{pmatrix} -1 & \frac{-8}{3\sqrt{\pi}} & -1 & \frac{-16}{15\sqrt{\pi}} & \frac{-1}{3} & \frac{-32}{105\sqrt{\pi}} & \frac{-1}{12} \\ \frac{-1}{2} & 1-\frac{4}{5\sqrt{\pi}} & \frac{-1}{3} & \frac{-8}{21\sqrt{\pi}} & \frac{-1}{8} & \frac{-16}{135\sqrt{\pi}} & \frac{-1}{30} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -10 & \frac{-8}{3\sqrt{\pi}} & -1 & \frac{-16}{15\sqrt{\pi}} & \frac{-1}{3} & 1-\frac{32}{105\sqrt{\pi}} & \frac{-1}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{2}u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{3}u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{4}u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{5}u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{6}u\right)(0) \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{3}{2} \\ 0 \\ 0 \\ 0 \\ -\frac{25}{2} \\ 30 \end{pmatrix}$$

which has the solution:

$$u(0)=1,\ \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{i}u\right)(0)=0,\ i=1,2,3,4,5 \text{ and } \left(\left({}^{C}D_{0^+}^{\frac{1}{2}}\right)^{6}u\right)(0)=30.$$

By substituting these values into equation (3.34) one can have:

$$u(x) \cong u_6(x)=1+5x^3, \quad 0 \le x \le 1$$

By substituting this approximated solution into equations (3.30)-(3.31) one can have:

$$ {}^{C}D_{0^+}^{\frac{1}{2}}u_6(x)+\frac{3}{2}+\frac{25}{4}x^2-\frac{16}{\sqrt{\pi}}x^{\frac{5}{2}}+\frac{23}{4}x^5-\int_0^1(x^2+y)u_6(y)dy- $$

$$ \int_0^x(3x+2y)u_6(y)dy=0 $$

and

$$ u_6(0)=2\int_0^1 u_6(y)dy-\frac{7}{2}. $$

Therefore u_6 is the exact solution of the linear nonlocal initial value problem given by equations (3.30)-(3.31).

Example (3.14):

Consider the local initial value problem that consists of the fractional linear Fredholm-Volterra integro-differential equation of order $\frac{1}{4}$:

$$ {}^{C}D_{0^+}^{\frac{1}{4}}u(x)=\frac{8}{\Gamma\left(\frac{3}{4}\right)}x^{\frac{3}{4}}-\frac{3}{2}x+\frac{1}{2}x^2-x^3-3x^4+\int_0^1 xyu(y)dy+\int_0^x(x^2+y)u(y)dy \tag{3.35} $$

together with the local initial condition:

$$ u(0)=-1 \tag{3.36} $$

Here $g(x)=\frac{8}{\Gamma\left(\frac{3}{4}\right)}x^{\frac{3}{4}}-\frac{3}{2}x+\frac{1}{2}x^2-x^3-3x^4$, $0\le x\le 1$, a=0, b=1,

$\lambda_1=\lambda_2=1$, $\mu_2=-1$, k(x,y)=xy and ℓ(x,y)=x^2+y.

We use the generalized Taylor expansion method to solve this fractional linear local initial value problem. To do this, let N =1, then equation (3.21) takes the form:

$$u(x) \cong u_1(x) = -1 + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)u\right)(0)}{\Gamma\left(\frac{5}{4}\right)} x^{\frac{1}{4}}, \quad 0 \le x \le 1 \tag{3.37}$$

In this case:

$$a_{0,0} = \frac{-\lambda_1}{\Gamma(1)} \int_0^1 k(0,y)dy = 0,$$

$$a_{1,0} = \frac{-1}{\Gamma\left(\frac{5}{4}\right)} \int_0^1 k(0,y) y^{\frac{1}{4}} dy = 0 \text{ and } g(0) = 0.$$

Then the system given by equation (3.29) takes the form:

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)u\right)(0) = 0.$$

Therefore

$$u(x) \cong u_1(x) = -1, \quad 0 \le x \le 1$$

By substituting this approximated solution into equation (3.35) one can have:

$${}^{C}D_{0^+}^{\frac{1}{4}} u_1(x) - \frac{8}{\Gamma\left(\frac{3}{4}\right)} x^{\frac{3}{4}} + \frac{3}{2}x - \frac{1}{2}x^2 + x^3 + 3x^4 - \int_0^1 xyu_1(y)dy - \int_0^x (x^2+y)u_1(y)dy =$$

$$-\frac{8}{\Gamma\left(\frac{3}{4}\right)} x^{\frac{3}{4}} + 2x + 2x^3 + 3x^4.$$

Since the right hand side of the above equation does not equal zero, so we must increase the value of N. Therefore, let N=2, then equation (3.21) takes the form:

$$u(x) \cong u_2(x) = -1 + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)u\right)(0)}{\Gamma\left(\frac{5}{4}\right)} x^{\frac{1}{4}} + \frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^2 u\right)(0)}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}}, \quad 0 \le x \le 1$$

(3.38)

In this case:

$$a_{0,0} = 0,\ a_{1,0} = 0,\ a_{2,0} = \frac{-1}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 k(0,y) y^{\frac{1}{2}} dy = 0$$

$$m_0(x) = \frac{x}{2},\ m_1(x) = \frac{4}{9}x,\ m_2(x) = \frac{2}{5}x,$$

$$p_0(x) = x^3 + \frac{1}{2}x^2,\ p_1(x) = \frac{4x^{\frac{9}{4}}(9x+5)}{45},\ p_2(x) = \frac{2x^{\frac{5}{2}}(5x+3)}{15}.$$

Therefore

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)m_0\right)(x) = \frac{2}{3\Gamma\left(\frac{3}{4}\right)} x^{\frac{3}{4}},\ \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)m_1\right)(x) = \frac{16}{27\Gamma\left(\frac{3}{4}\right)} x^{\frac{3}{4}},$$

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)m_2\right)(x) = \frac{8}{15\ \Gamma\left(\frac{3}{4}\right)} x^{\frac{3}{4}},\ \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)p_0\right)(x) = \frac{16x^{\frac{7}{4}}(24x+11)}{231\ \Gamma\left(\frac{3}{4}\right)},$$

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)p_1\right)(x) = \frac{\pi\sqrt{2}(39x^3+20x^2)}{128\ \Gamma\left(\frac{3}{4}\right)},\ \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)p_2\right)(x)$$

$$= \frac{8\sqrt{2}\Gamma\left(\frac{3}{4}\right)}{585\sqrt{\pi}} x^{\frac{9}{4}}(70x+39).$$

So $a_{0,1} = a_{1,1} = a_{2,1} = 0$. Moreover

$$\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)g(x)=\frac{12}{\sqrt{\pi}}x^{\frac{1}{2}}-\frac{2048}{385\,\Gamma\left(\frac{3}{4}\right)}x^{\frac{15}{4}}-\frac{128}{77\,\Gamma\left(\frac{3}{4}\right)}x^{\frac{11}{4}}+\frac{16}{12\,\Gamma\left(\frac{3}{4}\right)}x^{\frac{7}{4}}-\frac{2}{\Gamma\left(\frac{3}{4}\right)}x^{\frac{3}{4}},\quad 0\le x\le 1$$

So $\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)g\right)(0)=0.$

Thus the system given by equation (3.29) takes the form:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)u\right)(0)\\ \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^{2}u\right)(0)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

which has the solution: $\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)u\right)(0)=\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^{2}u\right)(0)=0.$

By substituting these values into equation (3.38) one can have:

$$u(x)\cong u_2(x)=u_1(x)=-1,\quad 0\le x\le 1.$$

So we must increase the value of N. By continuing in this manner one can get for N=4, equation (3.21) takes the form:

$$u(x)\cong u_4(x)=-1+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)u\right)(0)}{\Gamma\left(\frac{5}{4}\right)}x^{\frac{1}{4}}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^{2}u\right)(0)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}}+$$

$$\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^{3}u\right)(0)}{\Gamma\left(\frac{7}{4}\right)}x^{\frac{3}{4}}+\frac{\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^{4}u\right)(0)}{\Gamma(2)}x,\quad 0\le x\le 1 \qquad (3.39)$$

Then after simple computations and by using the Mathcad software package, the system given by equation (3.29) takes the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^2 u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^3 u\right)(0) \\ \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^4 u\right)(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \end{pmatrix}$$

which has the solution:

$$\left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^i u\right)(0) = 0,\ \text{i=1,2,3} \text{ and } \left(\left({}^{C}D_{0^+}^{\frac{1}{4}}\right)^4 u\right)(0) = 6.$$

By substituting these values into equation (3.39) one can have:

$$u(x) \cong u_4(x) = -1 + 6x, \quad 0 \le x \le 1$$

By substituting this approximated solution into equations (3.35)-(3.36) one can have:

$${}^{C}D_{0^+}^{\frac{1}{4}} u_4(x) - \frac{8}{\Gamma\left(\frac{3}{4}\right)} x^{\frac{3}{4}} + \frac{3}{2}x - \frac{1}{2}x^2 + x^3 + 3x^4 - \int_0^1 xyu_4(y)dy - \int_0^x (x^2+y)u_4(y)dy = 0$$

and

$$u_4(0) = -1.$$

Therefore u_4 is the exact solution of the linear local initial value problem given by equations (3.35)-(3.36).

Conclusions and Recommendations

From this work, one can concludes the following aspects:

(1) The existence and uniqueness of the solution for the linear non-local initial value problem is a generalization of the existence and the uniqueness for the solution of the linear local initial value problem.

(2) The classical Taylor expansion method that depends on approximating the unknown function as a Taylor polynomial centered at the left endpoint of its domain is a special case of the generalized Taylor expansion method.

(3) The generalized Taylor expansion method like the classical Taylor expansion method gave more accurate results as N increases.

(4) The Laplace transform method is so difficult to use it to solve the linear non-local fractional integro-differential equations of order α, $\alpha \notin \mathrm{N}$ since we get functions of s that can not find its Laplace inverse.

(5) It is known that there is no an explicit form for the Laplace transform of the Fredholm integral operator, so the Laplace transform method fails to be used to solve the non-local initial value problems for the linear Fredholm-Volterra integro-differential equations.

(6) The generalized Taylor expansion method can be also used to solve systems of linear fractional integro-differential equations with non-local initial conditions.

For future works, the following problems may be recommended:

(1) Discuss the existence and the uniqueness of the solution for the non-linear fractional integro-differential equations with non-linear non-local boundary conditions via fixed point theorems.

(2) Use the generalized Taylor expansion method to solve the non-local problems for some types of non-linear fractional integro-differential equations.

(3) Devote the study of the non-local delay problems.

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المستخلص

الهدف من هذا العمل هو دراسة بعض الانواع من المسائل اللامحليه الكسورية . هذه الانواع هي مسائل القيم الابتدائية اللامحلية الكسورية التي تتضمن معادلات فريدهولم، فولتيرا، فريدهولم-فولتيرا التكاملية التفاضلية الكسورية الخطية واللاخطية مع الشروط الابتدائية اللامحلية الخطية واللاخطية من نوع التكامل.

تتضمن هذه الدراسة وجود ووحدانية الحل لبعض معادلات فريدهولم-فولتيرا التكاملية التفاضلية الكسورية الخطية واللاخطية مع الشروط الابتدائية اللامحلية الخطية واللاخطية.

وكذلك تم استعمال طريقة تحويل لابلاس لحل بعض انواع المسائل اللامحلية الخطية الكسورية مع بعض الامثلة التوضيحية.

اضافة الى ذلك تم تقديم طريقة توسيع معمم تيلر لحل مسائل القيم الابتدائية اللامحلية التي تتضمن معادلات فريدهولم-فولتيرا التكاملية التفاضلية الكسورية الخطية مع الشروط الابتدائية اللامحلية الخطية ، مع بعض الامثلة التوضيحية.

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