Republic of Iraq
Ministry of Higher Education
and Scientific Research
Al-Nahrain University
College of Science
Department of Mathematics and
Computer Applications

Accuracy Improvement of Stochastic Linear Multi-Step Methods for Solving Stochastic Ordinary Differential Equations

A Thesis
Submitted to the College of Science / Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By
Nabaa Raheem Kareem
(B.Sc., Math. /College of Science / Al-Nahrain University, 2009)

Supervised by
Asst.Prof.Dr. Fadhel S. Fadhel *Prof.Dr. Alauldin N. Ahmed*

February *Rabeed Al-Thanee*
2013 *1434*

بسم الله الرحمن الرحيم

وَتِلْكَ حُجَّتُنَا آتَيْنَاهَا إِبْرَاهِيمَ عَلَى قَوْمِهِ

نَرْفَعُ دَرَجَاتٍ مَنْ نَشَاءُ إِنَّ رَبَّكَ حَكِيمٌ عَلِيمٌ

صَدَقَ الله العظيم

(سورة الأنعام/ الآية ٨٣)

الاهداء

إلى خاتم الانبياء والمرسلين سيدنا ونبينا

محمد (ص)

إلى من عانت من أجلي الشقاء وخصتني بالدعاء

أمي الحنونة

إلى من بذل قصار جهده ولم يقصر عني طوال مسيرتي

أبي العزيز

إلى الشموع المضيئة الى قرة العين

اخوتي الاعزاء

إلى الايادي التي ساندتني لكي اسير واصل الى ما انا عليه الان

الاستاذين المشرفيين

إلى كل من آفاضوا عليّ بوافر العلم والمعرفة

أساتذتي وأصدقائي

إلى كل من ساعدني وشجعني لأكمال هذا الجهد المتواضع خلال هذه المسيرة الطويلة. أهدي لهم خالص حبي وأحترامي المتواضع.

نبأ

Acknowledgements

Praise to Allah for providing me the willingness, energy and patience to accomplish this work.

I would like to express my sincere gratitude and grateful admiration to my supervisors Asst. Prof. Dr. Fadhel Subhi Fadhel and Prof. Dr. Alauldin N. Ahmed for their over seeing, guidance, interests and suggestions that were very helpful for the completion of this thesis.

I'm so grateful to the College of Science at Al-Nahrain University, which gave me the chance to be one of its students.

I would like also to express my appreciation to the staff members of the Department of Mathematics and Computer Applications at Al-Nahrain University for their support and ongoing help.

Heartfelt thanks goes to my beloved family, in particular my father, mother, brothers and sister, for their love and giving me the facilities during my work.

Finally, I'd like to thank my friends and all those who have participated in any way or another during the work.

Nabaa Raheem Kareem

February, 2013

Abstract

The main objectives of this thesis may be oriented toward three directions.

The first objective is a study, in details, the basic theory of stochastic calculus and study the linear multistep methods for solving stochastic differential equations and prove some results related to this topic, as well as, studying the Itô-Taylor series expansion and its applications.

The second objective is a study the two steps Maruyama method and also the solution of stochastic ordinary differential equations using implicit methods which are treated by using the methods for solving nonlinear algebraic equations resulting from the used implicit method, such as Newton-Raphson method and predictor corrector method, also proposing a new approach for solving stochastic ordinary differential equations using variable step size method have been proposed.

The third objective is to introduce the higher-order Richardson extrapolation method and variable order method for solving stochastic ordinary differential equations, which has the utility of improving the accuracy of the obtained results.

Basic Notations and Abbreviations

$\mathcal{F}$	σ-Algebra
$\mathcal{F}_t$	Filtration, which is an increasing family of σ-algebra fields.
P	Probability measure of $\mathcal{F}$.
Ω	Sample space.
$(\Omega, \mathcal{F}, P)$	Probability space.
X, $X(\omega)$	Random variable.
X_t, $X_t(\omega)$	Stochastic process.
$X.(\omega)$	X as a function of the variables replaced by the dot for fixed ω.
$X_t(.)$	X as a function of the variables replaced by the dot for fixed t.
$X \sim N(\mu, \sigma^2)$	X has a normal distribution with mean μ and variance σ^2.
μ, $E(X)$	The mean or the expected value of X.
σ^2, $Var(X)$	The variance of X.
w.p.1, P-w.p.1	*P* converges with probability one.
$\circ\, dW$	Stratonovich calculus integration symbol.
$C(\mathbb{R}^n, \mathbb{R})$	The space of continuous functions $v : \mathbb{R}^n \longrightarrow \mathbb{R}$.
$C^k(\mathbb{R}^n, \mathbb{R}^n)$	The space of k-times continuously differentiable functions $v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.
$C_P^k(\mathbb{R}^d, \mathbb{R})$	The subspace of functions $v \in C^k(\mathbb{R}^n, \mathbb{R})$ for which all partial derivatives up to order k have polynomial growth, i.e., $\left\|\partial_y^j v(y)\right\| \le K\left(1+\|y\|^{2r}\right)$ where $K > 0$,

	$r \in \{1, 2, \ldots\}$ depending on v when for all $y \in \mathbb{R}^n$ and any partial derivative $\partial_y^j v(y)$ of order $j > k$.
$C^{s-1,s}(J \times \mathbb{R}^n, \mathbb{R}^n)$	The space of (s−1)-times and s-times (for time and stochastic process respectively) continuously differentiable functions $v : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.
f_α	Coefficient function.
$\mathcal{H}$	Hierarchical set.
$-\alpha$	Delete the first component of a multi index α.
$\alpha-$	Delete the last component of a multi index α.
$(0, \delta)$	The interval of absolute stability.
$I_{(.),t}$, $I_{\cdot}^{t,t+h}$	Multiple Itô integrals.
$\mathcal{L}$	All expectation functions, such that $E\left(\int_0^t X_s^2\, ds\right) < \infty$. where E(.) is standing for expectation operators.
$L(\alpha)$	Length of multi index $\alpha = (j_1, j_2, \ldots, j_L)$.
$L(v)$	Length of multi index is zero.
L^0, L^r	The drift operator and the diffusion operator, respectively, where $r = 1, 2, \ldots$
$\mathcal{M}$	The set of all multi indices.
$n(\alpha)$	Number of zero components of a multi index α.
ODE	Ordinary differential equation.
$\mathcal{P}$	Class of measurable functions, such that $P\left(\int_0^t X_s^2\, ds < \infty\right) = 1$.
$L_2(\Omega, \mathbb{R}^n)$	The space of all square integrable functions defined from Ω to $\mathbb{R}^n$.
AB_2	The two-step Adam's Bashforth method.

$\mathcal{R}$	The remainder set.
AM_2	The two-step Adam's Moulton method.
R_n	The remainder of deterministic part
S_n	The remainder of stochastic part
R_n^*	The remainder of deterministic part for the perturbed system.
S_n^*	The remainder of stochastic part for the perturbed system.
R_n^0	The remainder of deterministic part for The Methods with order 1/2.
S_n^0	The remainder of stochastic part for the methods with order 1/2.
L_n	Local error.
D_n	The local error of the perturbed system.
SODE	Stochastic Ordinary Differential Equation.
SLMM's	Stochastic Linear Multi-step Methods.
SLMMM's	Stochastic linear Multi-step Maruyama Methods.
$\|\cdot\|_{L_2}$	The norm of $L_2(\Omega,\mathbb{R}^n)$ space and if $Z\in L_2(\Omega,\mathbb{R}^n)$ then $\|Z\|_{L_2}=(E\|Z\|^2)^{1/2}$.

Contents

Introduction... i

Chapter One: Fundamental Concepts.. 1

1.1 Background of Probability Theory 1

1.1.1 Basic Concepts of Random Variables............................. 2

1.1.2 Basic Concepts of Stochastic Process 5

1.2 Theory of Stochastic Differential Equations 7

1.2.1 Stochastic Integral and their Models................................ 7

1.2.2 Stochastic Differential Equations and their Models....... 11

1.2.3 Some Well-Known Derivatives..................................... 14

1.3 Theory of Stochastic Linear Multi-Step Methods 16

1.3.1 Elementary Numerical Methods..................................... 17

1.3.2 Stochastic Linear Multi-Step Methods........................... 18

1.3.3 Numerical Stability in the Mean-Square Sense.............. 22

1.4 Stochastic Itô-Taylor Series Expansion.................................... 28

Chapter Two: Linear Multi-Step Methods for Solving Stochastic Ordinary Differential Equations............................... 36

2.1 Stochastic Linear Multi-Step Maruyama Methods................... 37

2.1.1 Two-Step Maruyama Methods...................................... 39

2.1.1 (A) Analysis of Local Error for Stochastic Linear Two-step Maruyama Methods....................................... 41

2.1.1 (B) Order of Consistency Conditions for Two-Step Maruyama Scheme 46

2.2 Summary of Some Well Known Methods................................ 48

2.3 Variable Step Size Method for Solving SODE's 50

2.4 Solution of SODE's Using Implicit Methods 53

2.4.1 Newton-Raphson Method... 54

2.4.2 Predictor-Corrector Methods for Solving SODE's......... 56

2.5 Numerical Results 58

Chapter Three: Richardson and Variable Order Methods for Solving Stochastic Ordinary Differential Equations 62

3.1 Convergence Criteria 63

3.1.1 Strong Convergence Criterion....................................... 64

3.1.2 Weak Convergence Criterion 65

3.2 Richardson Extrapolation Method for Solving SODE's........... 67

3.3 Variable Order Methods for Solving SODE's 71

3.4 Numerical Results 75

Conclusions and Recommendations 81

References... 82

Introduction

Stochastic ordinary differential equations (SODE's) constitute an ideal mathematical model for a multitude of phenomena and processes encountered in areas, such as, differential equations, stochastic control, signal processes and mathematical finance, most notably in option pricing (see for example [44] and [28]). Unlike their deterministic counterparts, SODE's do not have explicit solutions, a part from in a few exceptional cases; hence the necessity for a theory of their numerical approximation is important, [21].

A most striking example, where SODE's provide the essential modeling device, is the Nobel Prize-Winning work of Merton in 1973, [31] and Black and Scholes in 1973, [5] about pricing options. The whole financial industries frequently make use of stochastic dynamics to calculate financial quantities, such as, derivative prices and risk measures. The increasing application of SODE's in many models is a major driving force in the development of appropriate numerical methods for the solution of SODE's, [39].

Since only a few specific types of SODE's have explicitly known solutions, the computation of important characteristics such as moments or sample paths is crucial for an effective practical application of SODE's. Therefore, numerical methods those are specific, not only to the considered SODE's, but also for the desired task are required. For these different tasks different types of convergence of a numerical scheme have been considered on recent literatures (see for example [36], [25], [26], [3], [33]).

Similar to the deterministic setting, the order of convergence with respect to the considered criterion of convergence plays a crucial role in the design of numerical algorithms.

Roughly speaking, we can distinguish between two major types of convergence, namely, approximations to the sample paths on the one hand and approximations to the corresponding distributions on the other hand. Usually, these approximations are called strong and weak approximations, respectively.

The numerical methods are based on time discrete approximations. Time discrete approximations for both, the strong and weak convergence criterions, will be presented. Whereas, time discrete approximation which satisfy the strong convergence criterion involves the simulation of sample paths at each step of the discretization time, approximations of some function of the Itô process, such as the first and second moments at a given final time T, [21].

Kloeden et al in 1995 [27] use the extrapolation methods for the weak approximation with two Itô diffusion depending on Euler's scheme to solve certain types of linear SODE's.

It is well known that Euler's method and most other explicit schemes for solving SODE's work unreliably and sometimes generate large errors, see for instance Milsten et al in 1999 [32], implicit and predictor corrector schemes are designed to achieve improved numerical stability and turn out to be better suited to simulated tasks. Generally, implicit schemes usually cost significant computational time and are sometimes not reliably accomplished, however, this phenomenon can be avoided when using some approximate discrete time schemes, including predictor-corrector methods, [4].

Most numerical schemes converging in the strong sense and further literatures may be found in the monographs of Kloeden and Platen (1999), [25]. It is pointed that the latest development of derivative free strong linear multistep methods (LMM's) (see [9], [10], [11], [12], [13]). They expanded rooted tree theory, well known in the deterministic setting (see [14]).

Al-Tememy N. Z. in 2011 [1] used the LMM's to derive certain types of two steps methods for solving SODE's, as well as, studying the stability and convergence of these methods.

Subhi M. M. in 2012 [42] use Runge-Kutta methods and its modification using variable step size method to solve SODE's using two steps explicit ,implicit and semi-explicit Runge-Kutta methods.

This thesis consist of three chapters. In chapter one, some general concepts, definitions, theorems and illustrative example related to stochastic calculus, theory of SODE's, theory of LMM's, stochastic Itô Taylor series expansion are given for completeness of this work.

In chapter two, some types of stochastic linear multi-step Maruyama methods (SLMMM's) for solving SODE's are studied and derived analytically. Also, in this chapter Newton-Raphson method have been used to solve stochastic implicit LMM's. Finally, in this chapter, the variable step size method for stochastic version has been proposed, as well as, some illustrative examples are considered for comparison purpose.

In chapter three, some illustrative examples have been implemented to the absolute error, strong error, as well a, weak convergence error and introduce the Richardson extrapolation method and variable order method.

Some illustrative examples are given for comparison between the given different schemes and that are proposed in this study.

Finally, the computer programs used in this thesis are coded in MATHCAD 14 computer software.

Chapter One

Fundamental Concepts

This chapter give the background material for the work carried out in this thesis, since there is a number of sources that provides a full details for the background of probability theory and stochastic calculus (for example, see the thesis of Rößler in 2003 [39], Burrage in 1999 [10], the text books of Kloeden and Platen in 1995 [28], Arnold in 1974 [2]).

This chapter consists of four sections. In section (1.1), some basic concepts related to the probability theory are given. In section (1.2), theory of SDE's and their models are given. In section (1.3) theory of SLMM's is given for the sake of numerical solution. Finally, in section (1.4), theory of stochastic Itô-Taylor series expansion was discussed.

1.1 Background of Probability Theory

In this section, some of the most and necessary concepts which are related to the subject of stochastic calculus and this thesis are given for completeness purpose.

1.1.1 Basic Concepts of Random Variables, [10], [28], [2]:

Stochastic calculus is that subject which is concerned with the study of stochastic processes, this involve randomness or noise. Intuitively, this requires knowledge of random variables and probability theory. Therefore, this subsection provides the background definitions and concepts that will be required later in this work, where only those definitions which are of direct relevance to this exposition are given.

Definition (1.1), [10]:

The ***σ-algebra*** $\mathcal{F}$ is a class of subsets of a sample space Ω (which is the set of all possible outcomes of a random experiment) satisfies the following:

1. $\Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$, then $A^c = \{w \in \Omega \mid w \notin A\} \in \mathcal{F}$.
3. For any sequence $\{A_n\} \subseteq \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

The elements of $\mathcal{F}$ are called ***probability measurable sets*** and the pair $(\Omega, \mathcal{F})$ is called a ***probability measurable space.***

Definition (1.2), [10]:

A ***probability measure*** P on $\mathcal{F}$ is a set function which satisfies:

1. $P(\Omega) = 1$.

2. If $A \in \mathcal{F}$, then $P(A) \geq 0$

3. If $A_1, A_2, \ldots, A_n, \ldots$; are mutually exclusive events (that is $A_i \cap A_j = \varnothing$ if $i \neq j$), then:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Definition (1.3), [10]:

A ***probability space*** $(\Omega, \mathcal{F}, P)$, comprises the sample space Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω (called events) and a probability measure P on $\mathcal{F}$.

Definition (1.4), [10]:

If X is a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$, then the ***expected value*** or ***mean value*** μ of X, is:

$$\mu = E(X) = \int_{\Omega} X \, dP$$

provided that the integral exists. That is, the average of X over the entire probability space. For continuous random variables over $\mathbb{R}$, the mean value of X is:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

Definition (1.5), [10]:

A measure of the spread about the mean μ is the ***variance***, which is given by:

$$\mathrm{Var}(X) = E((X - \mu)^2) = E(X^2) - \mu^2$$

the variance is denoted for simplicity by σ^2 and its positive square root σ is called the ***standard deviation*** of X.

Definition (1.6), [10]:

A random variable X is said to be ***Gaussian random variable*** if it has the Gaussian or normal density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

where μ is the mean and σ^2 is the variance of the normal distribution $N(\mu, \sigma^2)$. If $\mu = 0$ and $\sigma^2 = 1$, then the distribution $N(0,1)$ is known as the ***standard Gaussian distribution.***

Infinite sequences may be defined in terms of random variables, then it is important to know how the sequence converges and there is a number of different modes of convergence, which are given in the next definitions:

Definition (1.7), [10]:

A sequence of random variables $\{X_n(\omega)\}$, n = 1, 2, …; is said to be ***converge with probability one*** to $X(\omega)$ if

$$P(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$$

This type of convergence is also called ***almost sure convergence.***

Definition (1.8), [10]:

A sequence of random variables $\{X_n(\omega)\}$, n = 1, 2, …; such that $E(X_n^2) < \infty$, for all $n \in \mathbb{N}$; is said to be ***converges in the mean square*** to $X(\omega)$ if:

$$\lim_{n\to\infty} E(|X_n - X|^2) = 0$$

Definition (1.9), [10]:

A sequence of random variables $\{X_n(\omega)\}$, n = 1, 2, …; is said to be ***converges in probability*** (or ***stochastically***) to $X(\omega)$, if:

$$\lim_{n\to\infty} P(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) = 0,\ \forall\ \varepsilon > 0$$

1.1.2 Basic Concepts of Stochastic Process, [23], [10], [28], [2]:

In many physical applications, there are many processes in which the random variables depends on the space and/or time. Therefore, this introductory material give the main subject of such processes.

A ***stochastic process*** is a family of random variables $X_t(\omega)$ (or briefly X_t) on a probability space (Ω, $\mathcal{F}$, P), which assumes real values and is P-measurable as a function of $\omega \in \Omega$ for each fixed $t \in [t_0, T] \subset [0, \infty)$. The parameter t is interpreted as a time and $X_t(.)$ represents a random variable on the above probability space Ω, while $X.(\omega)$ is called a ***sample path*** or ***trajectory*** of the stochastic process, [10].

Definition (1.10), [10]:

A stochastic process W_t, $t \in [0, \infty)$, is said to be a ***Brownian motion*** or ***Wiener process***, if:

1. $P(\{\omega \in \Omega \mid W_0(\omega) = 0\}) = 1$.
2. For $0 < t_0 < t_1 < \ldots < t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$ …, $W_{t_n} - W_{t_{n-1}}$ are independent.
3. For an arbitrary t and $h > 0$, $W_{t+h} - W_t$ has a Gaussian distribution with mean 0 and variance t.

Remark (1.1), [10]:

In general, a standard Wiener process has the properties that:

$$W_0 = 0 \quad \text{w.p.1},\ E(W_t) = 0,\ Var(W_t - W_s) = t - s$$

for all $0 \le s \le t$; and so the increments are stationary.

Definition (1.11), [10]:

The ***white noise process*** ξ_t is formally defined as the derivative of the Wiener process, i.e.,

$$\xi_t \, dt = dW_t$$

It does not exist as a function of t in the usual sense, since a Wiener process is nowhere differentiable function.

Sometimes, it is called ***Gaussian whit noise***, which is an important example of stochastic process of a purely random process.

1.2 Theory of Stochastic Differential Equations

Theory and models of SODE's are discussed in short in this subsection as an introduction to this topic. Also, in this subsection the Itô formula will be discussed for completeness of the work.

1.2.1 Stochastic Integral and their Models, [39], [24]:

Definition (1.12), [39]:

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)_{t \in I}$, for $I = [0, \infty)$.

Let $\mathcal{L}$ denote the class of all $\mathcal{B}\times\mathcal{F}$-measurable, $\mathcal{F}_t$-adapted processes $X_t : I \times \Omega \longrightarrow \mathbb{R}$, where $\mathcal{B}$ denotes the Borel σ-algebra on I, for which:

$$E\left(\int_0^t X_s^2(\omega)\, ds\right) < \infty, \ \forall\, t > 0 \qquad \text{... (1.1)}$$

holds and the set $\mathcal{P}$ is the class of all $\mathcal{B}\times\ \mathcal{F}$ –measurable, $\mathcal{F}_t$-adapted processes $X_t : I \times \Omega \longrightarrow \mathbb{R}$, satisfying:

$$P\left(\int_0^t X_s^2(\omega)\, ds < \infty\right) = 1, \ \forall\, t > 0 \qquad \text{... (1.2)}$$

It is remarkable that condition (1.1) is stronger and implies condition (1.2). We now consider a series of partitions of the integration interval $[t_0, t]$ given by:

$$t_0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{N_n}^{(n)} = t$$

with the property that they are refinements for increasing n and with:

$$\max_{0\le i\le N_n-1} \left\{t_{i+1}^{(n)} - t_i^{(n)}\right\} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty$$

If we define $\tau_i^{(n)} = \theta\, t_{i+1}^{(n)} + (1-\theta)\, t_i^{(n)}$, for a fixed $\theta \in [0, 1]$, then the series of random variables is called the approximation of the stochastic integral:

$$\sum_{i=0}^{N_n-1} X_{\tau_i^{(n)}} (W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}) \qquad \text{...(1.3)}$$

converges as $n \longrightarrow \infty$ in probability if $X_{\tau_i^{(n)}} \in \mathcal{P}$ and in the mean-square sense if $X_{\tau_i^{(n)}} \in \mathcal{L}$, $\forall$ i = 0, 1, …, N_n–1, n $\in \mathbb{N}$ [24], [25], [44]. Near by the limit does not depend on the choice of the partitions. However, unlike the Riemann-Stieltjes integral, here the selection of θ makes a difference. For $\theta = 0$, which is means that $\tau_i^{(n)}$ is the left end point $t_i^{(n)}$, we have the Itô calculus. The limit of equation (1.3), denoted by:

$$\int_{t_0}^{t} X_s \, dW_s$$

is called the ***Itô stochastic integral***. At Stratonovich calculus, we have to set $\theta = \frac{1}{2}$ and $\tau_i^{(n)}$ described the mid point of $[t_i^{(n)}, \ t_{i+1}^{(n)}]$. Now, the limit of equation (1.3), denoted by:

$$\int_{t_0}^{t} X_s \circ dW_s$$

is called the ***Stratonovich stochastic integral***. For general stochastic integrals with respect to martingales, we refer to [24], [38] and [22].

Considering Itô and Stratonovich calculus, one may get a simple connection between the solution of an Itô SDE and that of a Stratonovich SDE. Let $(y_t)_{t\in I}$ be the solution of m-dimensional Itô SDE:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(s, y_s)\, ds + \int_{t_0}^{t} g(s, y_s)\, dW_s \qquad \ldots(1.4)$$

where W_t is a m-dimensional Wiener process. Then $(y_t)_{t\in I}$ is also a solution of the SDE:

$$\underline{f}_i(t, y_t) = f_i(t, y_t) - \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{m} g_{ik}(t, y_t) \frac{\partial g_{ik}}{\partial y_{t_i}}(t, y_t) \qquad \ldots(1.5)$$

with respect to Stratonovich calculus, where:

$$y_t = y_{t_0} + \int_{t_0}^{t} \underline{f}(s, y_s)\, ds + \int_{t_0}^{t} g(s, y_s) \circ dW_t \qquad \text{...(1.6)}$$

for i = 1, 2, …, d. Therefore, whichever interpretation of the SDE is appropriate in particular situation, we can always switch to the corresponding SDE in the other calculus. For instance, we can apply the existence and uniqueness theorem for an Itô SDE (1.4) to obtain analogous results for the corresponding Stratonovich SDE (1.5).

One of the main advantages of the Itô calculus in contrast to Stratonovich calculus is the fact that the Itô integrals inherit some good properties of the Wiener process. Let $f : I\times\Omega \longrightarrow \mathbb{R}$, such that $f \in \mathcal{L}$ holds. Then the relation between Itô integration and Lebesgue integration, which is called the ***Itô isometry***, is as follows:

$$E\left[\left(\int_{t_0}^{t} f(s, w)\, dW_s\right)^2\right] = E\left(\int_{t_0}^{t} f^2(s, w)\, ds\right) \qquad \text{…(1.7)}$$

Also the martingale property of a Wiener process carries over to the Itô integral. Let W_t be a Wiener process with respect the filtration $(\mathcal{F}_t)_{t\in I}$ satisfying the usual conditions. Then W_t and the process:

$$\left(\int_{t_0}^{t} f(s, w)\, dW_s\right)_{t\in I} \qquad \text{…(1.8)}$$

are martingales with respect to $(\mathcal{F}_t)_{t\in I}$. Furthermore:

$$E\left(\int_{t_0}^{t} f(s,w)\, dW_s\right) = 0 \qquad \text{...(1.9)}$$

holds for all $t \in I$.

The advantages of Stratonovich calculus is the availability of its rules similar to ordinary integration. However, unlike Itô integrals, Stratonovich integrals are not martingales. We can easily calculate:

$$\int_0^t W_s \circ dW_s = \frac{1}{2} W_t^2$$

whereas for Itô calculus, we have:

$$\int_0^t W_s\, dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t$$

One of the most important tools for the Stochastic calculus and especially for Itô calculus is the Itô formula.

1.2.2 Stochastic Differential Equations and their Models, [39], [10], [28]:

Among the most general models of SODE's is the following:

$$dy_t = f(t, y_t)\, dt + g(t, y_t)\, dW_t, \quad y_{t_0} = y_0 \qquad \text{...(1.10)}$$

where $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel-measurable functions, we call f the ***drift function*** and g the ***diffusion function***.

The SODE given in eq.(1.10) may be written in an equivalent form as:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(s, y_s)\,ds + \int_{t_0}^{t} g(s, y_s)\, dW_s \qquad ...(1.11)$$

However, the second integral given in eq.(1.11) cannot be defined in a following meaning, where W_s is the Wiener process. The variance of the Wiener process satisfies $Var(W_t) = t$, and so this increases as time increases even thought the mean stays at 0. Because of this, typical sample paths of a Wiener process attain larger values in magnitude as time progresses, and consequently the sample paths of the Wiener process are not bounded; hence the second integral in eq.(1.11) cannot be considered as a Riemann-Stieltjes integral. Note that, more general process which has the martingale property can be used in place of W_s, but in this thesis only Wiener process will be used in the formulation of SODE.

Definition (1.13), [39]:

A process y_t, $t \in I$ with values in $\mathbb{R}^d$ is called a ***strong solution of the SODE*** given in eq.(1.10) with respect to the fixed Wiener process W_t, $t \in I$ and the initial condition y_{t_0}, if the following properties hold:

(a) y_t is adapted to the filtration $(\mathcal{F}_t)_{t \in I}$.

(b) y_t has continuous sample paths.

(c) For multi-dimensions given in eq.(1.11), such that for all i = 1, 2, …, d; j = 1, 2, …, m; $m \in \mathbb{N}$ and $t \in I$ satisfy:

$$\int_0^t |f_i(s, y_s)| + g_{ij}^2(s, y_s)\,ds < \infty, \text{ P-w.p.1}$$

(d) y_t satisfy with P-w.p.1 the following stochastic integral equation:

$$y_t = y_{t_0} + \int_0^t f(s, y_s)\,ds + \int_0^t g(s, y_s)\,dW_s, \ \forall\, t \in I$$

<u>Theorem (1.1) (The Existence and Uniqueness Theorem), [39], [28], [2]:</u>

Suppose the functions f and g in eq.(1.10) satisfies the global Lipschitz and linear growth conditions:

$$\| f(t, y_t) - f(t, x_t) \| + \| g(t, y_t) - g(t, x_t) \| \le K \| y_t - x_t \| \quad \ldots(1.12)$$

$$\| f(t, y_t) \|^2 + \| g(t, y_t) \|^2 \le K^2 (1 + \| y_t \|^2) \quad \ldots(1.13)$$

for each $t \in J$, x_t, y_t are stochastic processes in $\mathbb{R}^d$, where K is a positive constant. Let y_{t_0} be a $\mathbb{R}^d$-valued random vector, independent of the Wiener process W_t and with:

$$E(\| y_{t_0} \|^{2L}) < \infty, \text{ for some } L \in \mathbb{R}$$

Then there exists a continuous, adapted process $y = (y_t)_{t \in J}$, which is a unique strong solution of the SODE (1.10) relative to W_t, with initial condition y_{t_0} and each component of y_t belongs to $\mathcal{L}$. Moreover, y_t is square-integrable and for every T > 0, there exists a constant C, depending only on K, T and L, such that:

$$E(\|y_t\|^{2L}) \le (1 + E\| y_{t_0} \|^{2L}) \exp(Ct), \ 0 \le t \le T$$

In contrast to strong solution of SODE's, a notion of solvability for the eq. (1.11) may be defined, which is a weaker condition.

Definition (1.14), [39]:

A ***weak solution of the SODE*** (1.10) is a triple $((\Omega, \mathcal{F}, P), (\mathcal{F}_t)_{t\in J}, (y_t, W_t))$, such that:

(a) $(\Omega, \mathcal{F}, P)$ is a probability space, $(\mathcal{F}_t)_{t\in J}$ is a right-continuous filtration in $\mathcal{F}$ and $\mathcal{F}_0$ contains all P-negligible events in $\mathcal{F}$.

(b) W_t is an m-dimensional Wiener process of $(\mathcal{F}_t)_{t\in J}$ and y_t is a continuous, adapted $\mathbb{R}^d$-values process.

(c) Conditions (c) and (d) of the definition (1.13) are satisfied.

Remark (1.2), [39]:

If $f(t, y_t)$ and $g(t, y_t)$ satisfy the conditions of theorem (1.1), then a solution (weak or strong) of the SODE (1.10) is weakly unique, where weak uniqueness means that any two solutions (weak or strong) satisfy the identical law, i.e., have the same finite-dimensional distributions.

1.2.3 Some Well-Known Dervatives, [19]:

Itô formula in stochastic calculus is the analog of integration by parts in stochastic calculus. The useful range of techniques is practically restricted to those that deal with integral equations,of these by far the most important is that known as Itô's formula, which may be seen as a stochastic chain rule. Let us recall some elementary non-random chain rule; as usual prime may denote differentiation.

1. One variable chain rule: If F = F(v(t)), then:

$$F' = \frac{dF}{dt} = \frac{dF}{dv}\frac{dv}{dt}$$

2. Two variables chain rule: If F = F(x(t), y(t)), then:

$$F' = \frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

In particular, if x(t) = t, then we may obtain, for F = F(t, y(t)):

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial y}dy$$

Itô formula are extremely useful in many topics, particularly in evaluating stochastic integrals.

<u>Theorem (1.2), (Itô Formula), [18]:</u>

Suppose that y_t has a SODE:

$$d\,y_t = f(t,\ y_t)\ dt + g(t,\ y_t)\ dW_t \qquad \text{...(1.14)}$$

for f, g $\in C^{1,2}(J\times\mathbb{R},\ \mathbb{R})$. Assume $F : J\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous and that $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial y_t}$ and $\frac{\partial^2 F}{\partial {y_t}^2}$ exists and are continuous. Set F = F(t, y_t), then F has the stochastic differential:

$$dF = \frac{\partial F}{\partial t}\ dt + \frac{\partial F}{\partial y_t}\ dy_t + \frac{1}{2}\frac{\partial^2 F}{\partial {y_t}^2}g^2\ dt$$

$$dF(t, y_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y_t} f + \frac{1}{2} \frac{\partial^2 F}{\partial y_t{}^2} g^2 \right) dt + \frac{\partial F}{\partial y_t} g \, dW_t \qquad ...(1.15)$$

is called the Itô's formula or Itô's chain rule.

In fact eq.(1.14) is sufficiently general to represent an m-dimensional, d-Wiener process system in which $g(t, y_t))$ is an $m \times d$ matrix and $W_t = (W_t^{(1)}, W_t^{(2)}, \ldots, W_t^{(d)})^T$ is a d-dimensional vector consisting of d independent Wiener processes. By letting the columns of $g(t, y_t)$ be labeled as $g_1(t, y_t)$, $g_2(t, y_t)$, …, $g_d(t, y_t)$; then the m-dimensional d-Wiener process system can also be written as:

$$d y_t = f(t, y_t) \, dt + \sum_{j=1}^{d} g_j(t, y_t) \, dW_t^{(j)}$$

In this case, the component-by-component version of Itô's formula is for k = 1, 2, …, m:

$$dF_k(t, y) = \left(\frac{\partial F_k}{\partial t} + \sum_{i=1}^{m} f_i \frac{\partial F_k}{\partial y} + \frac{1}{2} \sum_{l=1}^{d} \sum_{i,j=1}^{m} g_{il} g_{jl} \frac{\partial^2 F_k}{\partial y_i \partial y_j} \right) dt + \sum_{l=1}^{d} \sum_{i=1}^{m} g_{il} \frac{\partial F_k}{\partial y_i} \, dW_l \qquad ...(1.16)$$

1.3 Theory of Stochastic Linear Multi-Step Methods:

The considered numerical method in this section is the stochastic linear multi-step methods (SLMM's), which was one of the most important of development numerical methods used to give a good numerical accuracy to the approximate solution, [28]. Therefore, this method will be discussed in details in this section.

Also, in this section, some numerical methods with one-step will be studied, and also studying in the mean-square sense numerical stability, of the SLMM's for the approximation of Itô stochastic SODE's, as well as, their general theory and illustrative examples.

1.3.1 Elementary Numerical Methods [28]:

One scheme for stochastic one-step methods which will be often used for evaluating the approximate solution of SODE's will be given, and some definitions for strong and weak approximation will be also given [28].

Let us consider the Itô process y_t satisfying the SODE:

$$dy_t = f(t, y_t)dt + G(t, y_t)dW_t;\ y_t(t_0) = y_0 \qquad \ldots(1.17)$$

for $t \in J$, where $J = [t_0, T]$, $t_0 \in [0, \infty)$, $y_{t_0} \in \mathbb{R}$.

The drift and diffusion functions are given respectively as

$f : J\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$; $G = \{g_1, g_2, \ldots, g_n\} : J\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and f, g_r for $r = 1, 2, \ldots, m$; are continuous functions. Using an m-dimensional Wiener process W_t the problem (1.17) is understood as a stochastic integral equation:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(s, y_s)\, ds + \sum_{r=1}^{m} \int_{t_0}^{t} g_r(s, y_s)\, dW_r(s),\ t \in J \qquad \ldots(1.18)$$

In order to avoid confusions encountered in applying the numerical methods, y_t will be replaced for simplicity by y.

1.3.2 Stochastic Linear Multi-Step Methods, [40], [6], [8]:

We start this subsection by the following notations and definitions:

Let $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ the corresponding matrix norm. The mean-square norm of a vector valued square integrable random variable $Z \in L_2(\Omega, \mathbb{R}^n)$ will be defined by:

$$\| Z \|_{L_2} = (E\,| Z |^2)^{1/2}.$$

Let us denote by $C^{s-1,s}$ the class of all functions $V(t, y(t))$: $J\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ having continuous partial derivatives up to order $s - 1$ with respect to the first variable and continuous partial derivatives of order s with respect to the second variable. Moreover, let C^k be the class of functions V satisfying a linear growth condition of the form:

$$| V(t, y) | \le k(1+| y |^2)^{1/2},\ \forall\ t \in J,\ y \in \mathbb{R}^n \qquad \ldots(1.19)$$

where k is a positive constant.

Furthermore, we introduce the notation

$$I^{t,t+h}_{r_1,r_2,\ldots,r_j}(V) = \int_t^{t+h}\int_t^{s_1}\cdots\int_t^{s_{j-1}} V(s_j, y(s_j))\,dW_{r_1}(s_j)\cdots dW_{r_j}(s_1) \qquad \ldots(1.20)$$

where $r_j \in \{0, 1, \ldots, m\}$ and $dW_0(s) = ds$ for general multiple stochastic Itô integrals (see [40]).

If $V \equiv 1$ we write $I^{t,t+h}_{r_1,r_2,\ldots,r_j}$, note that the integral $I^{t,t+h}_{1}$ is one wiener process the increment $\Delta W = W(t + h) - W(t)$ of the scalar Wiener process W.

The next lemma presents the order of the multiple stochastic integrals.

Lemma (1.1), [40], [8]:

If $V \in C^k$ is any function and for any $t \in J$, $h > 0$, such that $t + h \in J$, then:

$$E\left(I^{t,t+h}_{r_1,r_2,\ldots,r_j}(V)\middle|\mathcal{F}_t\right) := 0\text{; if } r_i \neq 0 \text{ for some } i \in \{1,2,\ldots,j\} \quad \ldots(1.21)$$

$$\left\|E\left(I^{t,t+h}_{r_1,r_2,\ldots,r_j}(V)\middle|\mathcal{F}_t\right)\right\|_{L_2} \leq \left\|I^{t,t+h}_{r_1,\ldots,r_j}(V)\right\|_{L_2} = O\left(h^{i_1+\frac{i_2}{2}}\right) \quad \ldots(1.22)$$

where i_1 is the number of zero indices r_{i_1} and i_2 the number of non-zero indices r_{i_2}

Now, we consider a stochastic linear k-step method for the approximation of the solution of the SODE (1.17), for $n = k, k+1, \ldots, N$, $N \in \mathbb{N}$; which takes the form:

$$\sum_{j=0}^{k}\alpha_j y_{n-j} = h\sum_{j=0}^{k}\beta_j f(t_{n-j}, y_{n-j}) + \sum_{j=1}^{k}G_j(t_{n-j}, y_{n-j})I^{t_{n-j},t_{n-j+1}} \quad \ldots(1.23)$$

where we may set without loss of generality $\alpha_0 = 1$ and require given initial and starting values $y_0, y_1, \ldots, y_{k-1} \in L_2(\Omega, \mathbb{R}^n)$, such that y_n is $\mathcal{F}_{t_n}$-measurable for n = 0, 1, …, k – 1, [5].

As in the deterministic case, usually only $y_0 = y(t_0)$ is given by the initial value problem and the values $y_1, y_2, \ldots, y_{k-1}$ need to be computed numerically. This can be done by any suitable one-step method, where one has to be careful to achieve the desired accuracy. Every diffusion term $G_j(t, y)I^{t,t+h}$ is a finite sum of terms each containing an appropriate function $\mathcal{G}$ of t and y multiplied by a multiple Wiener integral (1.20) over [t, t + h], i.e., it takes the general form:

$$G_j(t,y)I^{t,t+h} = \sum_{r=1}^{m} \mathcal{G}^r(t,y)I_r^{t,t+h} + \sum_{\substack{r_1,r_2=0 \\ r_1+r_2>0}}^{m} \mathcal{G}^{r_1,r_2}(t,y)I_{r_1,r_2}^{t,t+h} + \ldots \quad \ldots(1.24)$$

where the Wiener process is m-dimension. If $\beta_0 = 0$, then the SLMM (1.23) is said to be explicit, otherwise it is implicit.

Finally, consider the autonomous SODE:

$$dy(t)= f(y(t))dt + G(y(t))dw(t);\ y(t_0) = y_0 \quad \ldots(1.25)$$

then the SLMM for the approximation of the solution of the SODE (1.17), with n = k, k+1, …, N; takes the form:

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = h\sum_{j=0}^{k} \beta_j f(y_{n-j}) + \sum_{j=1}^{k} G_j(y_{n-j}) I^{t_{n-j},t_{n-j+1}} \quad \ldots(1.26)$$

with $G_j(y) = \sum_{r=1}^{m} \mathcal{G}^{r}(y) I_{r}^{t,t+h} + \sum_{\substack{r_1,r_2=0 \\ r_1+r_2>0}}^{m} \mathcal{G}^{r_1,r_2}(y) I_{r_1,r_2}^{t,t+h} + \cdots$

Next, an example for two-step stochastic method will be given:

Example (1.1), [7], [8]:

The implicit two-step method (Milne-Simpson method) for

n = 2, 3, …, N; takes the form:

$$y_n - y_{n-2} = h\left(\frac{1}{3}f(t_n, y_n) + \frac{4}{3}f(t_{n-1}, y_{n-1}) + \frac{1}{3}f(t_{n-2}, y_{n-2})\right) + \sum_{r=1}^{m} g_r(t_{n-1}, y_{n-1}) I_r^{t_{n-1},t_n} + \sum_{r=1}^{m} g_r(t_{n-2}, y_{n-2}) I_r^{t_{n-2},t_{n-1}}$$

for this method one has:

$$\alpha_0 = 1 \ \alpha_1 = 0 \ \alpha_2 = -1; \ \beta_0 = \frac{1}{3}; \ \beta_1 = \frac{4}{3}; \ \beta_2 = \frac{1}{3}$$

$$G_1(t, y) I^{t,t+h} = \sum_{r=1}^{m} g_r(t_{n-1}, y_{n-1}) I_r^{t_{n-1},t_n}$$

$$G_2(t, y) I^{t,t+h} = \sum_{r=1}^{m} g_r(t_{n-2}, y_{n-2}) I_r^{t_{n-2},t_{n-1}}$$

Definition (1.15), [8]:

The ***local error*** of the SLMM (1.23) for the approximation of the solution of the SODE (1.17) for n = k, k+1, …, N; may be as:

$$L_n = \begin{cases} \sum_{j=0}^{k} \alpha_j y(t_{n-j}) - h\sum_{j=0}^{k} \beta_j f(t_{n-j}, y(t_{n-j})) - \\ \sum_{j=1}^{k} G_j(t_{n-j}, y(t_{n-j}))I^{t_{n-j},t_{n-j+1}}, \text{ for } n = k, k+1, ..., N & \ldots(1.27) \\ y(t_n) - y_n, \qquad \text{for } n = 0,1,..., k-1 \end{cases}$$

and represent the local error in the following form:

$$L_n := R_n + S_n =: R_n + \sum_{j=1}^{k} S_{j,n-j+1}, \; n = k, k+1, \ldots, N \qquad \ldots(1.28)$$

where each $S_{j,n-j+1}$ is $\mathcal{F}_{t_{n-j+1}}$-measurable with $E(S_{j,n-j+1} \mid \mathcal{F}_{t_{n-j}}) = 0$, $\forall$ n = k, k + 1, …, N; j = 1, 2, …, k.

Also, $R_n = L_n$; $S_n = 0$, $R_n = E(L_n \mid \mathcal{F}_{t_{n-k}})$; $S_n = L_n - R_n$ …(1.29)

$$S_{j,n-j+1} = E\left(L_n - R_n - \sum_{i=j+1}^{k} S_{i,n-i+1} \mid \mathcal{F}_{t_{n-j+1}} \right) \qquad \ldots(1.30)$$

1.3.3 Numerical Stability in the Mean-Square Sense,[8],[17],[29]:

With the numerical stability property one can estimate the influence of any perturbations of the right-hand side of the discrete scheme on the global solution of that discrete scheme. Sources of perturbations may be the local error or round-off errors or defects in the approximate solution of the implicit schemes.

The stability concept is often called zero-stability, or in honor of Dahlquist stability, also D-stability, for further discussions we refer the reader to the deterministic literature [17], [29]. The mean-square stability estimate of the global error is based on the mean-square norm and on the conditional mean of the perturbations D_n of the right-hand side of the perturbed system (1.31). Its solution is denoted by $\tilde{y}_n$.

In our analysis, we thus consider the following discrete system has the perturbed form of (1.23) for n = k, k + 1, …, N.

$$\sum_{j=0}^{k}\alpha_j\tilde{y}_{n-j} = h\sum_{j=0}^{k}\beta_j f(t_{n-j},\tilde{y}_{n-j}) + \sum_{j=1}^{k}G_j(t_{n-j},\tilde{y}_{n-j})I^{t_{n-j},t_{n-j+1}} + D_n \qquad \ldots(1.31)$$

with initial and starting values $\tilde{y}_n = y_n + D_n$, n = 0, 1, …, k – 1.

It is supposed that the perturbations D_n are $\mathcal{F}_{t_n}$-measurable and that $D_n \in L_2(\Omega, \mathbb{R}^n)$.

Remark (1.3), [8]:

It is useful to represent the perturbations in the form:

$$D_n = R_n^* + S_n^* = R_n^* + \sum_{j=1}^{k}S_{j,n-j+1}^*, \; n = k, k+1, \ldots, N \qquad \ldots(1.32)$$

where each $S_{j,n-j+1}^*$ $\forall$ j = 1, 2, …, k is $\mathcal{F}_{t_{j,n-j+1}}$-measurable with $E(S_{j,n-j+1}^* | \mathcal{F}_{t_{n-j}}) = 0$, $\forall$ n = k, k+1, …, N; j = 1, 2, …, k; where * refers to the perturbed system. The representation (1.32) is not unique and one extreme possibility is $R_n^* = D_n$ and $S_n^* = 0$, another more useful one, is given by:

$$R_n^* = E(D_n \mid \mathcal{F}_{t_{n-k}}),\ \ S_n^* = D_n - R_n^*$$

$$S_{j,n-j+1}^* = E\left(D_n - R_n^* - \sum_{i=j+1}^{k} S_{i,n-j+1}^* \mid \mathcal{F}_{t_{n-j+1}} \right),\ j=k,k-1,\ldots \quad \ldots(1.33)$$

This construction guarantees the required measurability conditions in (1.32). As an example, one obtains for k = 2:

$$R_n^* = E(D_n \mid \mathcal{F}_{t_{n-2}}),\ S_{2,n-1}^* = E(D_n - R_n^* \mid \mathcal{F}_{t_{n-1}}),$$

$$S_{1,n}^* = D_n - R_n^* - S_{2,n-1}^*$$

Here, in the hypothetical case that $D_n = C_0 I_r^{t_{n-1},t_n} + C_1 I_r^{t_{n-2},t_{n-1}} + C_2$, we have $R_n^* = C_2$, $S_{2,n-1}^* = C_1 I_r^{t_{n-2},t_{n-1}}$, $S_{1,n}^* = C_0 I_r^{t_{n-1},t_n}$

Now, the precise definition of mean square stability and some other notions will be given next:

<u>Definition (1.16), [8]:</u>

The SLMM (1.23) is said to be ***numerically stable in the mean square sense*** if there exist constants $h_0 > 0$ and $S > 0$, such that for all step sizes $h < h_0$ and for all $\mathcal{F}_{t_n}$ measurable perturbations $D_n \in L_2(\Omega, \mathbb{R}^n)$, n = 0, 1, …, N, all their representations (1.32), the following inequality holds:

$$\max_{n=0,\ldots,N} \| y_n - \tilde{y}_n \|_{L_2} \le S \left\{ \max_{n=0,\ldots,k-1} \| D_n \|_{L_2} + \max_{n=k,\ldots,N} \left(\frac{\| R_n \|_{L_2}}{h} + \frac{\| S_n \|_{L_2}}{h^{1/2}} \right) \right\}$$

…(1.34)

where $(y_n)_{n=1}^{N}$ and $(\tilde{y}_n)_{n=1}^{N}$ are the solutions of the SLMM (1.23) and the perturbed discrete system (1.31), respectively.

Definition (1.17), [8]:

A function $f : J\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to satisfies the ***uniform Lipschitz condition*** with respect to x if there exists a positive constant C_f, such that:

$$|f(t,x) - f(t,y)| \leq C_f\,|x - y|,\ \forall\ x,y \in \mathbb{R}^n,\ t \in J = [t_0,T] \qquad \ldots(1.35)$$

Definition (1.18), [8]:

The characteristic polynomial of (1.23) is given by:

$$\rho(r) = \alpha_0 r^k + \alpha_1 r^{k-1} + \ldots + \alpha_k \qquad \ldots(1.36)$$

and the SLMM (1.23) is said to ***fulfill Dahlquist's root condition*** if:

(i) The roots of $\rho(r)$ lie on or within the unit circle;

(ii) The roots on the boundary of the unit circle are simple.

The next theorem is of great importance, which is given and proved in the corresponding references.

Theorem (1.3), [8]:

The SLMM (1.23) is numerically stable in the mean-square sense for every continuous f and G_j satisfying (1.35) respectively, if and only if its characteristic polynomial $\rho(r)$ (1.36) satisfies Dahlquist's root condition.

Now, to study the mean square stability of two step methods, consider the methods given in example (1.1) and their stability in the next example:

Example (1.2), [7], [8]:

When back to Example (1.1), the method may be rewritten in the form:

$$y_n - y_{n-2} = h\left(\frac{1}{3}f(t_n, y_n) + \frac{4}{3}f(t_{n-1}, y_{n-1}) + \frac{1}{3}f(t_{n-2}, y_{n-2})\right) + \sum_{r=1}^{m} g_r(t_{n-1}, y_{n-1}) I_r^{t_{n-1},t_n} + \sum_{r=1}^{m} g_r(t_{n-2}, y_{n-2}) I_r^{t_{n-2},t_{n-1}}$$

here; $k = 2$, $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = -1$ and by Definition (1.18) the characteristic polynomial is given by:

$$\rho(r) = r^2 - 1$$

which have the roots $r_1 = 1$ and $r_2 = -1$ which lies on and inside the unit circle. Then $\rho(r)$ satisfies the Dahlquist's root condition.

Also, by using Theorem (1.3), we have this method is numerically stable in the mean-square sense.

Definition (1.19), [40], [8]

The SLMM (1.23) for the approximate solution of the SODE (1.17) is said to be ***mean-square consistent*** if the local error L_n satisfies:

$$h^{-1} \| E(L_n \mid \mathcal{F}_{t_{n-k}}) \|_{L_2} \longrightarrow 0 \text{ for } h \longrightarrow 0 \text{ and } h^{1/2} \| L_n \|_{L_2} \longrightarrow 0$$

for $h \longrightarrow 0$ or we call the SLMM (1.23) for the approximation of the solution of the SODE (1.17) mean-square consistent of order $p > 0$, if the local error L_n satisfies:

$$\| E(L_n \mid \mathcal{F}_{t_{n-k}}) \|_{L_2} \le \bar{C} h^{p+1} \text{ and } \| L_n \|_{L_2} \le C h^{p+\frac{1}{2}}, \; n = k, k+1, \ldots, N$$

with constants C, $\bar{C} > 0$ only depending on the SODE and its solution.

It must be remind the reader that consistency is only concerned with the local error. In the case that we disregard other sources of errors in (1.31) we only have to deal with perturbations $D_n = L_n$.

Lemma (1.2), [1]:

The SLMM (1.23) is mean-square consistent of order p if and only if there exists constants C, $\bar{C} > 0$, such that $\| R_n \|_{L_2} \le \bar{C} h^{p+1}$ and $\| S_n \|_{L_2} \le C h^{p+\frac{1}{2}}$, n = k, k+1, …, N, for any representation (1.32) of the local error $D_n = L_n$; $0 < h \le 1$.

Definition (1.20), [40], [8]:

The SLMM (1.23) for the approximation of the solution of the SODE (1.17) is said to be ***mean-square convergent*** if the global error $y(t_n) - y_n$ (where the global error means the accumulation of the local error up to the grid point t_n) satisfies:

$$\max_{n=0,1,\ldots,N} \| y(t_n) - y_n \|_{L_2} \longrightarrow 0 \text{ as } h \longrightarrow 0 \text{ and } t_n \text{ is fixed}$$

or equivalently, the SLMM (1.23) is said to be mean-square convergent with order $p > 0$ if the global error satisfies:

$$\max_{n=1,\ldots,N} \| y(t_n) - y_n \|_{L_2} \leq C.h^p \text{ as } h \longrightarrow 0$$

with constant $C > 0$ which is independent of the step-size h.

Theorem (1.4), [8]:

A mean-square consistent SLMM (1.23) for the approximation of the solution of SODE (1.17) is mean-square convergent for all continuous f and G_j satisfying (1.35), respectively if and only if it is numerically stable in the mean-square sense and if, in addition, it is mean-square consistent with order $p > 0$, then the SLMM (1.23) is mean-square convergent with order p.

1.4 Stochastic Itô- Taylor Series Expansion, [39], [10], [28]

Taylor series expansion is well-known for deterministic functions where they turn out to be useful tool, especially in numerical analysis. This idea can be carried over the stochastic setting by applying the Itô formula. Thus, following Platen and Wagner [37], stochastic Taylor formula will be gotten, which represents a generalization of the deterministic Taylor formula.

With deterministic differential equation methods, a numerical method may be derived by comparing the expansion of the method and the solution of the ordinary differential equation in a Taylor series; and exactly the same procedure can take place in the stochastic setting, using a stochastic version of Taylor series. The Itô -Taylor expansion was first established by Platen and Wanger in 1982 [37], and full details are given by Kloeden and Platen in 1995 [28]. It allows y_t (or any function of y_t to be expanded about the point y_{t_0} up to the required degree of accuracy) in terms of multiple stochastic integrals along with function evaluations at y_{t_0}. In order to derive the expansion, the Itô formula is applied successively to the SODE (1.14) as it is represented in the autonomous integral form:

$$y_t = y_{t_0} + \int_{t_0}^{t} f(y_s)\, ds + \int_{t_0}^{t} g(y_s)\, dW_s \qquad \ldots(1.37)$$

From the stochastic chain rule of eq. (1.15) in autonomous form:

$$F(y_t) - F(y_{t_0}) = \int_{t_0}^{t} \left(\frac{dF}{dy} f + \frac{1}{2} \frac{d^2F}{dy^2} g^2 \right) ds + \int_{t_0}^{t} \frac{dF}{dy} g\, dW_s$$

$$= \int_{t_0}^{t} L^0 F(y_s)\, ds + \int_{t_0}^{t} L^1 F(y_s)\, dW_s \qquad \ldots(1.38)$$

where the operators L^0 and L^1 for scalar problems are given by:

$$L^0F(y) = \frac{dF}{dy} f + \frac{1}{2}\frac{d^2F}{dy^2} g^2 \quad \text{and} \quad L^1F(y) = \frac{dF}{dy} \; g$$

Applying the Itô formula given by (1.38) for f and g in (1.37), then one application give:

$$y_t = y_{t_0} + \int_{t_0}^{t} \left(f(y_{t_0}) + \int_{t_0}^{s} L^0(f(y_u)du + \int_{t_0}^{s} L^1(f(y_u)\, dW_u \right) ds +$$

$$\int_{t_0}^{t} \left(g(y_{t_0}) + \int_{t_0}^{s} L^0(g(y_u)du + \int_{t_0}^{s} L^1(g(y_u)dW_u \right) dW_s \qquad \ldots(1.39)$$

Consequently, by applying the Itô formula and using L^0f, L^1f, L^0g and L^1g, the Itô-Taylor expansion will be derived next.

Remark (1.4), [39]:

The above discussion is given for one-dimensional autonomous SODE's, and we shall consider next the non-autonomous SODE's will be considered, and deriving its related stochastic Taylor series expansion. Let X_t be the solution of the Itô SODE in general form:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s,\, X_s)\, ds + \int_{t_0}^{t} b(s,\, X_s)\, dW_s \qquad \ldots(1.40)$$

and let $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ with $f \in C^{1,2}(J \times \mathbb{R}\, ,\, \mathbb{R})$. By applying the Itô formula, getting for $Y_t = f(t,\, X_t)$, the following equation:

$$y_t = y_{t_0} + \int_{t_0}^{t} \left(\frac{\partial f}{\partial t}(s, X_s) + a(s, X_s)\frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} b^2(s,\, X_s) \right.$$

$$\left. \frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds + \int_{t_0}^{t} b(s,\, X_s)\frac{\partial f}{\partial x}(s,\, X_s)\, dW_s$$

For simplicity of notations, the operators $L^0. = \frac{\partial .}{\partial t} + a\frac{\partial .}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 .}{\partial x^2}$ and $L^1. = b\frac{\partial .}{\partial x}$ are introduced and rewriting the above mentioned equations as:

$$y_t = y_{t_0} + \int_{t_0}^{t} L^0 f(s, X_s)\, ds + \int_{t_0}^{t} L^1 f(s, X_s)\, dW_s$$

and by applying the Itô formula (1.15) to the functions f = a and f = b in (1.39), getting:

$$X_t = X_{t_0} + \int_{t_0}^{t}\left(a(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 a(u, X_u)\,du + \int_{t_0}^{s} L^1 a(u, X_u)\,dW_u\right)ds + \int_{t_0}^{t}\left(b(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 b(u, X_u)\,du + \int_{t_0}^{s} L^1 b(u, X_u)\,dW_u\right)dW_s \quad \ldots(1.41)$$

which may be also written as:

$$X_t = X_{t_0} + a(t_0, X_{t_0})\int_{t_0}^{t} ds + b(t_0, X_{t_0})\int_{t_0}^{t} dW_s + R$$

where R denotes the remainder. Continuing in this way by applying the Itô formula to the functions $f = L^i a$ and $f = L^i b$, for i = 0, 1 in (1.41) to get the Itô-Taylor series expansion.

In order to describe the stochastic Taylor series expansion, a multi-dimensional and for multi Wiener process setting, the following terminology will be used:

A multiple Itô integral is given by:

$$I_{(j_1,j_2,\ldots,j_L),t} = \int_0^t \int_0^{S_L} \ldots \int_0^{S_2} dW_{S_1}^{j_1} \ldots dW_{S_L}^{j_L} \qquad \ldots(1.42)$$

whereas $j_i \in \{0, 1, \ldots, m\}$ for m-Wiener processes, and where $dW_{S_i}^0 = ds_i$. For more explanation to this context, we start with the definition of multi-indices and hierarchical sets which provide an efficient notation in the following. Let:

$$\mathcal{M} = \{\alpha = (j_1,j_2,\ldots,j_L) \in \{0,1,\ldots,m\}^L : L \in \mathbb{N}\} \cup \{v\} \qquad \ldots(1.43)$$

be set of all ***multi-indices***. The length $L(\alpha)$ of a multi-index $\alpha = (j_1, j_2, \ldots, j_L)$, where $j_i \in \{0, 1, .., m\}$, $i \in \{0, 1, .., L\}$ and $m = 1, 2, \ldots$ be defined as:

$$L(\alpha) = L \in \{1, 2, ...\} \qquad \ldots(1.44)$$

Where v is the multi-index of length 0, such that:

$$L(v) = 0 \qquad \ldots(1.45)$$

Thus, for example $L((1, 0)) = 2$ and $L((1, 0, 1)) = 3$.

In addition let $n(\alpha)$ denote the number of components of a multi-index α, which are equal to 0, such that:

$$n(\alpha) = n \qquad \ldots(1.46)$$

where n is the number of zero components of α, for example $n((1, 0, 1)) = 1$, $n((0, 1, 0)) = 2$, $n((0, 0)) = 2$.

Now, for $\alpha = (j_1, j_2, \ldots, j_L) \in \mathcal{M}$ with $L = L(\alpha) \geq 1$, define:

$$-\alpha = (j_2, j_3, \ldots, j_L) \text{ and } \alpha- = (j_1, j_2, \ldots, j_{L-1}) \qquad \ldots(1.47)$$

by deleting the first and the last components of α, respectively. For example:

$$-(1, 0) = (0),\ (1, 0)- = (1),\ -(0, 1, 1) = (1, 1),\ (0, 1, 1)- = (0, 1)$$

A subset $\mathcal{H} \subset \mathcal{M}$ is called a ***hierarchical set*** if $\mathcal{H} \neq \emptyset$ and if:

$$\sup_{\alpha\in\mathcal{H}} L(\alpha) < \infty \ \text{ and } \ -\alpha \in \mathcal{H}, \text{ for each } \alpha \in \mathcal{H} \setminus \{v\} \qquad \ldots(1.48)$$

For example, the sets $\{v\}$, $\{v, (0), (1)\}$, $\{v, (0), (1), (1, 1)\}$ are hierarchical sets.

The corresponding ***remainder set*** $\mathcal{R}(\mathcal{H})$ for the hierarchical set $\mathcal{H}$ is defined as:

$$\mathcal{R}(\mathcal{H}) = \{\alpha \in \mathcal{M} \setminus \mathcal{H} : -\alpha \in \mathcal{H}\} \qquad \ldots(1.49)$$

For example:

$\mathcal{R}(\{v\}) = \{(0), (1)\}$, $\mathcal{R}(\{v, (0), (1)\}) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, and $\mathcal{R}(\{v, (0), (1), (1, 1)\}) = \{(0, 0), (0, 1), (1, 0), (0, 1, 1), (1, 1, 1)\}$

and consists of all the next following multi-indices with respect to the given hierarchical set $\mathcal{H}$.

We are now able to define multiple stochastic integrals. Let us introduce three classes of adapted right continuous stochastic processes $(f_t)_{t\in J}$ with left hand limits. We say:

$$f \in H_v \ \text{ if } |f(t, \omega)| < \infty, \ \text{P-w.p.1 for each } t \geq 0 \qquad \ldots(1.50)$$

and we say for each $t \geq 0$, $f \in H_{(0)}$ if f satisfies condition given by:

$$P\left(\int_0^t |f(s,\omega)|\, ds < \infty\right) = 1, \text{ w.p.1} \qquad \ldots(1.51)$$

Furthermore, define $f \in H_{(j)}$ for each $j \in \{1, 2, \ldots, m\}$ if $f \in \mathcal{P}$ holds, (back to definition (1.12) in (1.2)).

$$\int_0^t |f(s,\omega)|^2\, ds < \infty, \text{ w.p.1 and } t \geq 0 \qquad \ldots(1.52)$$

In addition, we write $H_{(j)} = H_{(1)}$ for each $j \in \{2, 3, \ldots, m\}$ if $m \geq 2$.

Now, let ρ and τ be two stopping times with:

$$0 \leq \rho(\omega) \leq \tau(\omega) \leq T, \text{ P-w.p.1} \qquad \ldots(1.53)$$

For a multi-index $\alpha = (j_1, j_2, \ldots, j_L) \in \mathcal{M}$ and a process $f \in H_\alpha$, we define the ***multiple Itô integral*** $I_\alpha^{\rho,\tau}\left[f(.)\right]$ with respect to the m-dimensional Wiener process $W = (W^1, W^2, \ldots, W^m)$ recursively by:

$$I_\alpha^{\rho,\tau}[f(.)] = \begin{cases} f(\tau), & \text{if } L = 0 \\ \int_\rho^\tau I_{\alpha-}^{\rho,s}[f(.)]ds, & \text{if } L \geq 1 \text{ and } j_L = 0 \\ \int_\rho^\tau I_{\alpha-}^{\rho,s}[f(.)]dW_s^{j_L}, & \text{if } L \geq 1 \text{ and } j_L \geq 1 \end{cases} \qquad \ldots(1.54)$$

Here, we note the H_α with $\alpha = (j_1, j_2, \ldots, j_L)$ and $L \geq 2$ describes the totality of adapted right continuous process f with left hand limits, such that the integral process $(I_{\alpha-}^{\rho,\tau}\left[f(.)\right])_{t\in J}$ considered as a function of t satisfies $I_{\alpha-}^{\rho,\cdot}\left[f(.)\right] \in H_{(jL)}$. If the integrand is constant, i.e., $f(t,\omega) \equiv 1$, we abbreviate $I_\alpha^{\rho,\tau}\left[f(.)\right]$ as I_α if the limits ρ and τ are obvious from the context. In the following, we denote $W_t^0 = t$, $dW_t^0 = dt$ and

$$I_{\alpha,t} = I_\alpha^{0,t}\left[f(.)\right] \text{ when } \rho = 0 \text{ and } \tau = t.$$

As an illustration of this terminology, consider the following examples:

$$I_v^{0,t}\left[f(.)\right] = f(t),\ \ I_{(0)}^{\tau_i,\tau_{i+1}}\left[f(.)\right] = \int_{\tau_i}^{\tau_{i+1}} f(s)\,ds,$$

$$I_{(1)}^{\rho,\tau}\left[f(.)\right] = \int_\rho^\tau f(s)\,dW_s^1,\ I_{(0,1)}^{0,t}\left[f(.)\right] = \int_0^t \int_0^{S_2} f(S_1)\,dS_1\,dW_{S_1}^1$$

<u>Theorem (1.5) (The Itô -Taylor Expansion), [39], [28]:</u>

Let $\mathcal{H} \subseteq \mathcal{M}$ be a hierarchical set, let ρ and τ be two stopping times with $t_0 \le \rho(\omega) \le \tau(\omega) \le T < \infty$ *P-w.p.1* and let $f : J \times \mathbb{R}^d \longrightarrow \mathbb{R}$, then for the solution $(X_t)_{t\in J}$ of the Itô SODE (1.40). The Itô - Taylor expansion:

$$f(\tau, X_\tau) = \sum_{\alpha\in\mathcal{H}} I_\alpha[f_\alpha(\rho,\ X_\rho)]_{\rho,\tau} + \sum_{\alpha\in\mathcal{R}(\mathcal{H})} I_\alpha[f_\alpha(.,\ X_.)]_{\rho,\tau} \quad \ldots(1.55)$$

holds, provided that all of the derivatives of f, a and b and all of the multiple Itô integrals appearing in (1.55) exist. Similarly, to get theorem of the Stratonovich-Taylor expansion (for more details see [10], [28]).

Chapter Two

Linear Multi-Step Methods for Solving Stochastic Ordinary Differential Equations

From the variety of SLMM's, those methods which only include information on the increments of the driving Wiener process will be considered. Analogously to the Euler-Maruyama scheme, such methods will be called the stochastic linear multi-step Maruyama methods (SLMMM's), [40].

As an example for the SLMMM's is the two-step Maruyama methods which have conditions for their mean-square consistency. These conditions allow determination of the parameters for the stochastic part from the parameters of the deterministic part and reduce to those of the underlying deterministic schemes when there is no noise, [8].

This chapter consists of five section, in section (2.1), the derivation of SLMMM's is given according to the style of Buckwar and Winkler [6], [8]. In section (2.2) summary of some well known methods have been introduced. In section (2.3), the variable step size method will be introduced which was given for solving SLMM's in order to improve the accuracy of the numerical results. In Section (2.4) was prepared to study the solution of SODE's using implicit methods. In section (2.5), numerical examples illustrating the discussed numerical methods given in this chapter are given, with its comparison with the exact solution.

2.1 Stochastic Linear Multi-Step Maruyama Methods, [40],[6],[8]

As it is known a LMMM's with one Wiener process takes the form for all n = k, k+1, …, N:

$$\sum_{j=0}^{k}\alpha_j y_{n-j} = h\sum_{j=0}^{k}\beta_j f(t_{n-j}, y_{n-j}) + \sum_{j=1}^{k}\gamma_j g(t_{n-j}, y_{n-j}) I_1^{t_{n-j}, t_{n-j+1}} \quad \ldots(2.1)$$

For drift and diffusion coefficients f and g which are continuous and satisfy (1.35), theorem (1.3) may be applied and the SLMMM's (2.1) is mean-square stable if the coefficients α_0, α_1, …, α_k satisfy the Dahlquist's root condition. If, in addition, eq. (2.1) is mean-square consistent of order p, which is in general requires more smoothness of the coefficients functions then eq. (2.1) is mean-square converge of the same order. Thus, we will be concerned with mean-square consistency of eq.(2.1) and derive order conditions in terms of the coefficients α_0, α_1, …, α_k; β_0, β_1, β_2, …, β_k and γ_1, γ_2, …, γ_k.

The local error of eq. (2.1) is given by:

$$L_n := \begin{cases} \sum_{j=0}^{k} \alpha_j y(t_{n-j}) - h \sum_{j=0}^{k} \beta_j f(t_{n-j}, y(t_{n-j})) - \\ \sum_{j=1}^{k} \gamma_j g(t_{n-j}, y(t_{n-j})) I^{t_{n-j}, t_{n-j+1}}, \text{ for } n = k, k+1, \ldots, N \\ y(t_n) - y_n, \qquad \text{for } n = 0, 1, \ldots, k-1 \end{cases} \quad \ldots(2.2)$$

In general, the mean-square order of convergence will be only ½, since the only information about the driving noise process that the Maruyama-type schemes include are the Wiener increments. We note that the simple Euler-Maruyama method would suffice to obtain the same order of convergence. However, convergence is an asymptotic property, i.e., it holds for h $\longrightarrow$ 0 and a result concerning the order of convergence may not provide sufficient information about the size of the actual error that arise for reasonable choices of the step-size, [8].

From the deterministic theory, it is known that for a linear multi-step method:

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = h \sum_{j=0}^{k} \beta_j f(t_{n-j}, y_{n-j}), \text{ for } n = k, k+1, \ldots, N \qquad \ldots(2.3)$$

when applied to $y'(t) = f(t, y(t))$, the local error is of order $p + 1$ for sufficiently smooth function f, if:

$$\sum_{j=0}^{k} \alpha_j = 0 \text{ and } \sum_{j=0}^{k} \alpha_j (k-j)^q = q \sum_{j=0}^{k} \beta_j (k-j)^{q-1}, \text{ for } q=1,2,\ldots,p \quad \ldots(2.4)$$

Let the coefficients of the scheme (2.1) be normalized in such a way that $\alpha_0 = 1$ for all n. Again, we emphasize that if $\beta_0 = 0$, then the scheme (2.1) is explicit, otherwise it is implicit.

Finally, consider the autonomous SODE of the form:

$$dy_t = f(y_t)dt + g(y_t)dW_t$$

then the SLMMM's for the above autonomous SODE, will be:

$$\sum_{j=0}^{k}\alpha_j y_{n-j} = h\sum_{j=0}^{k}\beta_j f(y_{n-j}) + \sum_{j=1}^{k}\gamma_j g(y_{n-j}) I_1^{t_{n-j},t_{n-j+1}} \text{, for } n = k, k+1, \dots, N \quad \dots(2.5)$$

In the next subsection, the two-step Maruyama scheme will be considered and derive the consistency conditions for this scheme. We establish a representation of the local error L_n in term of certain multiple stochastic integrals obtained by the Itô-Taylor expansion. It turns out that the consistency condition is guaranteed under the above conditions for deterministic order 1 and additional conditions that determine the method parameters γ_1 and γ_2.

2.1.1 Two-Step Maruyama Methods, [40], [8]:

Consider the Itô process y_t satisfying the SODE with one Wiener process:

$$dy_t = f(t, y_t)dt + g(t, y_t)dW_t; \; y_{t_0} = y_0 \quad \dots(2.6)$$

for $t \in J$, where $J = [t_0, T]$, $t_0 \in [0, \infty)$, $y_0 \in \mathbb{R}$

where f and g are the drift and diffusion functions respectively, then, a linear two-steps Maruyama methods, with one Wiener process, for n = 2, 3, …, N , will take the form:

$$\sum_{j=0}^{2}\alpha_j y_{n-j} = h\sum_{j=0}^{2}\beta_j f(t_{n-j}, y_{n-j}) + \sum_{j=1}^{2}\gamma_j g(t_{n-j}, y_{n-j})\Delta W_{n-j} \qquad \ldots(2.7)$$

and when $\alpha_0 = 1$ and $I_1^{t_{n-j}, t_{n-j+1}} = W(t_{n-j+1}) - W(t_{n-j}) = \Delta W_{n-j}$, and

1- If $\beta_0 = 0$ then the explicit two-step Maruyama methods is given by:

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = h[\beta_1 f(t_{n-1}, y_{n-1}) + \beta_2 f(t_{n-2}, y_{n-2})] + [\gamma_1 g(t_{n-1}, y_{n-1})\Delta W_{n-1} + \gamma_2 g(t_{n-2}, y_{n-2})\Delta W_{n-2}], \text{ for } n = 2, 3, \ldots, N \qquad \ldots(2.8)$$

   where y_0 is given by the initial condition and the starting value y_1 need to be computed numerically, which may be calculated by any suitable one-step method, such as the simple Euler-Maruyama method:

$$y_{n-1} = y_{n-2} + hf(t_{n-2}, y_{n-2}) + g(t_{n-2}, y_{n-2})\Delta W_{n-2}, \ n = 2, 3, \ldots, N \qquad \ldots(2.9)$$

   where y_{n-1} will be called the supporting value.

2- If $\beta_0 \neq 0$ then the implicit two-step Maruyama methods is given by:

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = h[\beta_0 f(t_n,y_n) + \beta_1 f(t_{n-1},y_{n-1}) + \beta_2 f(t_{n-2},y_{n-2})]$$
$$+ [\gamma_1 g(t_{n-1},y_{n-1})\Delta W_{n-1} + \gamma_2 g(t_{n-2},y_{n-2})\Delta W_{n-2}], \text{ for } n = 2, 3, \ldots, N \quad \ldots(2.10)$$

where also y_0 is given by the initial condition and the starting values y_1, y_2, need to be computed numerically, the value y_1 may be evaluated by any suitable one-step method. In addition the value y_2 may be evaluated by the explicit two-step Maruyama method. It is remarkable that, the combination of an explicit and implicit technique is called a predictor-corrector method and we will call y_{n-1} and y_n for n = 2, 3,…, N in eq. (2.10) the supporting values.

2.1.1 (A) Analysis of Local Error for Stochastic Linear Two-step Maruyama Methods, [40], [8]:

The local error of the two-step Maruyama method (2.7) for the SODE (2.6) which is given by:

$$L_n := \begin{cases} \sum_{j=0}^{2} \alpha_j y(t_{n-j}) - h \sum_{j=0}^{2} \beta_j f(t_{n-j}, y(t_{n-j})) - \\ \sum_{j=1}^{2} \gamma_j g(t_{n-j}, y(t_{n-j}))\Delta W_{n-j}, & \text{for } n = 2,\ldots,N \\ y(t_n) - y_n, & \text{for } n = 0,1 \end{cases} \quad \ldots(2.11)$$

and we remind the reader for the representation (1.28) of the local error. In the context of two-step schemes the local error representation (1.28) reduces to:

$L_n = R_n + S_n = R_n + S_{1,n} + S_{2,n-1}$, for n = 2, …, N …(2.12)

One useful choice is provided by:

$$R_n = E\left(L_n \mid \mathcal{F}_{t_{n-2}}\right), S_{2,n-1} = E\left(L_n - R_n \mid \mathcal{F}_{t_{n-2}}\right), \; S_{1,n} = L_n - R_n - S_{2,n-1}$$

see also the discussion in Remark (1.3). In the hypothetical case that:

$$L_n = C_0\Delta W_{n-1} + C_1\Delta W_{n-2} + C_2$$

holds, we have:

$$R_n = C_2;\; S_{2,n-1} = C_1\Delta W_{n-2};\; S_{1,n} = C_0\Delta W_{n-1}$$

Applying the Itô-formula on the corresponding intervals to the drift coefficient f, as well as, to the diffusion coefficient g yields for $s \in [t_{n-j}, t_{n-j+1}]$; j = 1, 2

$$f(s, y(s)) = f(t_{n-j}, y(t_{n-j})) + I_0^{t_{n-j},s}(L^0 f) + I_1^{t_{n-j},s}(L^1 f) \quad \ldots(2.13)$$

$$g(s, y(s)) = g(t_{n-j}, y(t_{n-j})) + I_0^{t_{n-j},s}(L^0 g) + I_1^{t_{n-j},s}(L^1 g) \quad \ldots(2.14)$$

and tracing back the values of the drift coefficient to the point s = t_{n-1} and j = 2, to obtain:

$$f(t_{n-1}, y(t_{n-1})) = f(t_{n-2}, y(t_{n-2})) + I_0^{t_{n-2},t_{n-1}}(L^0 f) + I_1^{t_{n-2},t_{n-1}}(L^1 f) \quad \ldots(2.15)$$

or

$$f(t_n, y(t_n)) = f(t_{n-1}, y(t_{n-1})) + I_0^{t_{n-1},t_n}(L^0 f) + I_1^{t_{n-1},t_n}(L^1 f)$$

or

$$f(t_n, y(t_n)) = f(t_{n-2}, y(t_{n-2})) + I_0^{t_{n-2},t_{n-1}}(L^0 f) + I_1^{t_{n-2},t_{n-1}}(L^1 f) + I_0^{t_{n-1},t_n}(L^0 f) + I_1^{t_{n-1},t_n}(L^1 f) \quad \text{….(2.16)}$$

by analyzing the local error L_n given by eq. (2.11) of the scheme (2.7) for the SODE (2.6), one can derive the consistency conditions for scheme (2.7). The following lemma has this result, which is given here with its proof for completeness.

Lemma (2.1), [40], [8]:

Assume that the coefficients f, g of the SODE (2.6) belong to the class $C^{1,2}$ with $L^0 f$, $L^0 g$, $L^1 f$, $L^1 g \in C^k$. Then the local error (2.11) of the stochastic two-step Maruyama scheme (2.7) allows the representation:

$$L_n = R_n^0 + S_{1,n}^0 + S_{2,n-1}^0, \text{ for } n = 2, 3, \dots, N \quad \text{…(2.17)}$$

where $R_n^0, S_{j,n}^0$, j = 1, 2 are $\mathcal{F}_{t_n}$ -measurable with $E\left(S_{j,n}^0 \mid \mathcal{F}_{t_{n-j}}\right) = 0$ and

$$R_n^0 = \left[\sum_{j=0}^{2} \alpha_j\right] y(t_{n-2}) + \left[2\alpha_0 + \alpha_1 - \sum_{j=0}^{2} \beta_j\right] h f(t_{n-2}, y(t_{n-2})) + \tilde{R}_n^0$$

$$S_{1,n}^0 = [\alpha_0 - \gamma_1]\, g(t_{n-1}, y(t_{n-1}))\Delta W_{n-1} + \tilde{S}_{1,n}^0$$

$$S_{2,n-1}^0 = [(\alpha_0 + \alpha_1) - \gamma_2]\, g(t_{n-2}, y(t_{n-2}))\Delta W_{n-2} + \tilde{S}_{2,n-1}^0$$

with:

$$\|\tilde{R}_n^0\|_{L_2} = O(h^2)\,;\ \|\tilde{S}_{1,n}^0\|_{L_2} = O(h)\,;\ \|\tilde{S}_{2,n-1}^0\|_{L_2} = O(h) \quad \text{…(2.18)}$$

Proof:

To derive a representation of the local error of the form (2.17), the deterministic parts are evaluated and resumed at the point (t_{n-2}, $y(t_{n-2})$) and separate the stochastic terms carefully over the different subintervals $[t_{n-2}, t_{n-1}]$ and $[t_{n-1}, t_n]$. This ensures the independence of the random variables. It does make the calculations more complicated. Since:

$$\sum_{j=0}^{2}\alpha_j y(t_{n-j}) = \alpha_0(y(t_n)-y(t_{n-1}))+(\alpha_0+\alpha_1)(y(t_{n-1})-y(t_{n-2}))+\left(\sum_{j=0}^{2}\alpha_j\right)y(t_{n-2})$$

then the local error for the two-step Maruyama methods (2.11) may be expressed as:

$$L_n = \alpha_0(y(t_n) - y(t_{n-1})) + (\alpha_0 + \alpha_1)(y(t_{n-1}) - y(t_{n-2})) + \sum_{j=0}^{2}\alpha_j y(t_{n-2}) - h\sum_{j=0}^{2}\beta_j f(t_{n-j}, y(t_{n-j})) - \sum_{j=1}^{2}\gamma_j g(t_{n-j}, y(t_{n-j}))\,\Delta W_{n-j}$$

The SODE (2.6) implies the identity:

$$y(t_{n-1}) - y(t_{n-2}) = \int_{t_{n-2}}^{t_{n-1}} f(s, y(s))ds + \int_{t_{n-2}}^{t_{n-1}} g(s, y(s))dW(s)$$

i.e.,

$$y(t_{n-1}) - y(t_{n-2}) = I_0^{t_{n-2},t_{n-1}}(f) + I_1^{t_{n-2},t_{n-1}}(g)$$

Applying the Itô formula eq. (2.13) and (2.14) for $I_0^{t_{n-2},t_{n-1}}(f)$ and $I_1^{t_{n-2},t_{n-1}}(g)$, respectively, to obtain:

$$y(t_{n-1})-y(t_{n-2})=hf(t_{n-2},y(t_{n-2}))+I_{00}^{t_{n-2},t_{n-1}}(L^0 f)+I_{10}^{t_{n-2},t_{n-1}}(L^1 f)$$
$$+g(t_{n-2},y(t_{n-2}))I_1^{t_{n-2},t_{n-1}}+I_{01}^{t_{n-2},t_{n-1}}(L^0 g)+I_{11}^{t_{n-2},t_{n-1}}(L^1 g) \qquad \ldots(2.19)$$

Also:

$$y(t_n)-y(t_{n-1})=\int_{t_{n-1}}^{t_n} f(s,y(s))ds+\int_{t_{n-1}}^{t_n} g(s,y(s))dw(s)$$
$$=I_0^{t_{n-1},t_n}(f)+I_1^{t_{n-1},t_n}(g)$$

Applying the Itô formula (2.13) and (2.14) for $I_0^{t_{n-1},t_n}(f)$ and $I_1^{t_{n-1},t_n}(g)$, respectively, to obtain:

$$y(t_n)-y(t_{n-1})=hf(t_{n-1},y(t_{n-1}))+I_{00}^{t_{n-1},t_n}(L^0 f)+I_{10}^{t_{n-1},t_n}(L^1 f)$$
$$+g(t_{n-1},y(t_{n-1}))I_1^{t_{n-1},t_n}+I_{01}^{t_{n-1},t_n}(L^0 g)+I_{11}^{t_{n-1},t_n}(L^1 g)$$

and, additionally using (2.15), yields to:

$$y(t_n)-y(t_{n-1})=h\left\{f(t_{n-2},y(t_{n-2}))+I_0^{t_{n-2},t_{n-1}}(L^0 f)+I_1^{t_{n-2},t_{n-1}}(L^1 f)\right\}+I_{00}^{t_{n-1},t_n}(L^0 f)$$
$$+I_{10}^{t_{n-1},t_n}(L^1 f)+g(t_{n-1},y(t_{n-1}))I_1^{t_{n-1},t_n}$$
$$+I_{01}^{t_{n-1},t_n}(L^0 g)+I_{11}^{t_{n-1},t_n}(L^1 g) \qquad \ldots(2.20)$$

Inserting eqs.(2.19) and (2.20) and the expansions (2.15); (2.16) into the local error formula (2.11) and reordering the terms, and letting:

$$I_1^{t_{n-1},t_n}=W(t_n)-W(t_{n-1})=\Delta W_{n-1}\ ;\ I_1^{t_{n-2},t_{n-1}}=W(t_{n-1})-W(t_{n-2})=\Delta W_{n-2}$$

and from this, yields to:

$$L_n = \left[\sum_{j=0}^{2}\alpha_j\right]y(t_{n-2}) + \left[2\alpha_0 + \alpha_1 - \sum_{j=0}^{2}\beta_j\right]hf(t_{n-2}, y(t_{n-2})) + \tilde{R}_n^0$$

$$+[\alpha_0 - \gamma_1]g(t_{n-1}, y(t_{n-1}))\Delta W_{n-1} + \tilde{S}_{1,n}^0$$

$$+[(\alpha_0 + \alpha_1) - \gamma_2]g(t_{n-2}, y(t_{n-2}))\Delta W_{n-2} + \tilde{S}_{2,n-1}^0 \qquad \ldots(2.21)$$

where:

$$\begin{aligned}\tilde{R}_n^0 &= \alpha_0\left\{h\, I_0^{t_{n-2},t_{n-1}}(L^0 f) + I_{00}^{t_{n-1},t_n}(L^0 f)\right\} + (\alpha_0 + \alpha_1) I_{00}^{t_{n-2},t_{n-1}}(L^0 f)\\ &\quad - h\beta_0\left\{I_0^{t_{n-2},t_{n-1}}(L^0 f) + I_0^{t_{n-1},t_n}(L^0 f)\right\} - h\beta_1 I_0^{t_{n-2},t_{n-1}}(L^0 f) \qquad \ldots(2.22)\end{aligned}$$

$$\tilde{S}_{1,n}^0 = \alpha_0 I_{11}^{t_{n-1},t_n}(L^1 g) - h\beta_0 I_1^{t_{n-1},t_n}(L^1 f) + \alpha_0 I_{01}^{t_{n-1},t_n}(L^0 g) + \alpha_0 I_{10}^{t_{n-1},t_n}(L^1 f) \qquad \ldots(2.23)$$

$$\begin{aligned}\tilde{S}_{2,n-1}^0 &= (\alpha_0 + \alpha_1) I_{11}^{t_{n-2},t_{n-1}}(L^1 g) + (\alpha_0 + \alpha_1) I_{10}^{t_{n-2},t_{n-1}}(L^1 f)\\ &\quad + (\alpha_0 + \alpha_1) I_{01}^{t_{n-2},t_{n-1}}(L^0 g) + h(\alpha_0 - \beta_0 - \beta_1) I_1^{t_{n-2},t_{n-1}}(L^1 f) \qquad \ldots(2.24)\end{aligned}$$

Finally, the estimates (2.18) are derived by means of Lemma (1.1), where the first terms in (2.23) and (2.24) determine the order h. ■

2.1.1 (B) Order of Consistency Conditions for Two-Step Maruyama Scheme, [6], [41]:

The following corollary give the order of consistency conditions for the scheme (2.7) have been proved to be of order ½, which is given in literatures without details of the proof.

Corollary (2.1), [40], [8]:

Let the coefficients f and g of the SODE (2.6) satisfy the assumptions of Lemma (2.1) and suppose that they are Lipschitz with respect to their first variable. Let the coefficients of the stochastic linear two-step Maruyama scheme (2.7) satisfy the Dahlquist's root condition and the consistency conditions:

$$\sum_{j=0}^{2}\alpha_j = 0\ ,\ 2\alpha_0 + \alpha_1 = \sum_{j=0}^{2}\beta_j\ ;\ \alpha_0 = \gamma_1\ ;\ \alpha_0 + \alpha_1 = \gamma_2 \qquad \ldots(2.25)$$

Then the global error of the scheme (2.7) applied to (2.6) allows the expansion

$$\max_{n=0,\ldots,N} \| y(t_n) - y_n \|_{L_2} = O\left(h^{\frac{1}{2}}\right) + O\left(\max_{n=0,1} \| y(t_n) - y_n \|_{L_2}\right)$$

Proof:

By Lemma (2.1), we have the representation (2.17) for the local error (2.11). Applying the consistency conditions (2.25), yields to:

$$R_n^0 = \tilde{R}_n^0\ ,\ S_{1,n}^0 = \tilde{S}_{1,n}^0\ ,\ S_{2,n-1}^0 = \tilde{S}_{2,n-1}^0, \text{ for } n = 2, 3, \ldots, N$$

As the scheme (2.7) satisfies the Dahlquist's root condition, then by Theorem (1.3) it is numerically stable in the mean-square sense. Then $\tilde{y}_n = y(t_n)$ and $D_n = L_n$ in the stability inequality (1.34), and the assertion follows from the stability inequality (1.34)

$$\max_{n=0,1,\ldots,N} \| y(t_n) - y_n \|_{L_2} \le S\left\{ \max_{n=2,\ldots,N}\left(\frac{\| R_n^0 \|_{L_2}}{h} + \frac{\| S_n^0 \|_{L_2}}{h^{1/2}} \right) + \max_{n=0,1} \| L_n \|_{L_2} \right\}$$

from eq. (2.11) and for n = 0, 1, the local error $L_n = y(t_n) - y_n$ and also using (2.18) in the stability inequality, with S = 1

$$\max_{n=0,1,\ldots,N} \| y(t_n) - y_n \|_{L_2} \le \left\{ \max_{n=2,\ldots,N} \left(\frac{O(h^2)}{h} + \frac{O(h)}{h^{1/2}} \right) + \max_{n=0,1} \| y(t_n) - y_n \|_{L_2} \right.$$

$$\max_{n=0,1,\ldots,N} \| y(t_n) - y_n \|_{L_2} \le \left\{ \max_{n=2,\ldots,N} \left(O(h) + O\left(h^{1/2}\right) \right) + \max_{n=0,1} \| y(t_n) - y_n \|_{L_2} \right.$$

$$\max_{n=0,1,\ldots,N} \| y(t_n) - y_n \|_{L_2} = O\left(h^{1/2}\right) + O\left(\max_{n=0,1} \| y(t_n) - y_n \|_{L_2} \right) \quad \blacksquare$$

2.2 Summary of Some Well Known Methods, [1]

Using the analysis of the local truncation error for the deterministic case, one may obtain a number of equations less than the number of coefficients and hence will give infinite number of solutions. To drive certain methods, the coefficients which satisfy the consistency and zero-stability will be considered. While in the stochastic case, the analysis of local error for each step have been used which will give certain consistency conditions for each step which are also less than the number of coefficients and we get an infinite number of solutions.

In deriving certain method, select the coefficients which satisfy consistency conditions also select these coefficients $\alpha_0, \alpha_1, \ldots, \alpha_k$ which satisfy the Dahlquist root condition, hence the method is mean-square consistency of order p and then by Theorem (1.3), the method is numerically stable in the mean-square sense. By Theorem (1.4), we get the method is mean-square convergent with order p.

Certain classification of linear multi-step methods may be considered, namely:

a- If the characteristic polynomial (1.36) of the methods takes the roots r = 1, r = 0 then the methods are called of Adams-methods and if they are explicit then they are called of Adam's -Bashforth type while if they are implicit then they are called of Adam's-Moulton type.

b- If the characteristic polynomial (1.36) of the methods takes the roots r = 1; r = −1; r = 0 then the methods are called of Nystrom type if they are explicit and of Milne-Simpson if they are implicit, as in example (1.1).

Some models for an explicit linear multi-step methods which are found in literatures, are:

The two-step Adam's-Bashforth method (AB_2) for n = 2,3, …, N, N $\in \mathbb{N}$; with one Wiener process, has the form:

$$y_n - y_{n-1} = h\left[\frac{3}{2}f(t_{n-1}, y_{n-1}) - \frac{1}{2}f(t_{n-2}, y_{n-2})\right] + g(t_{n-1}, y_{n-1})\Delta W_{n-1} \quad \ldots(2.26)$$

where:

$$t \in J = [t_0, T]; h = \frac{T - t_0}{N}; \Delta W_{n-1} = W(t_n) - W(t_{n-1})$$

Also, some models for an implicit linear multi-step methods which are found in literatures, are:

The two-step Adam's-Moulton method (AM_2), for n = 2, 3, …, N; $N \in \mathbb{N}$ with one Wiener process, has the form:

$$y_n - y_{n-1} = h\left[\frac{5}{12}f(t_n, y_n) + \frac{8}{12}f(t_{n-1}, y_{n-1}) - \frac{1}{12}f(t_{n-2}, y_{n-2})\right] + g(t_{n-1}, y_{n-1})\Delta W_{n-1} \qquad \dots(2.27)$$

where $t \in J;\ h = \frac{T - t_0}{N};\ \Delta W_{n-1} = W(t_n) - W(t_{n-1})$.

2.3 Variable Step Size Method for Solving SODE's, [42]

The numerical solution of SODE's will be found using variable step size method, which may be considered as a new approach in this topic, where the considered SODE's has the form:

$$dy_t = f(t, y_t)\,dt + g(t, y_t)\,dW_t;\ y_t(t_0) = y_{t_0} \qquad \dots(2.28)$$

In all fixed step-size methods, the local truncation error will depends on the step size h and on the used numerical method. But, in variable step-size method, we shall find the numerical solution y_t for the SODE given in eq.(2.28), that is accurate to within a specified tolerance ε.

Therefore, it turns out for reasonable effective estimates of the step-size, it is required to attain a specified local truncation error (tolerance) ε. The variable step-size method, which will be considered here is based upon the comparison between the estimates of the one and two steps of the numerical value of y_t at some time obtained by the numerical method with local truncation error term that is of the form Ch^p, where C is unknown constant and p is the order of the method. Suppose that we started with the initial condition y_{t_0} with step-size h using certain SLMM and to find the solutions $y_{t_0+h}^{(1)}$ and $y_{t_0+h}^{(2)}$ using the step-size h and $\frac{h}{2}$, respectively. Let:

$$E_{est.} = \| y_{t_0+h}^{(1)} - y_{t_0+h}^{(2)} \| \qquad \ldots(2.29)$$

and here if $E_{est.} \leq \varepsilon$, then there is no problem and one may consider $y_{t_0+h}^{(2)}$ as the solution at $t_0 + h$. Otherwise, if $E_{est.} > \varepsilon$, then one can to find another estimation of the step-size say h_{new}. If this approximation was accepted then this value of h_{new} will be used as the new value of h in the next step; if not, then it will be used as an old h and repeat similarly as above.

Now, a common question may arise, which is how to find h_{new} ?. In this work, a new criterion has been developed for estimating the local truncation error, which control the step-size. The problem of error estimation is the most important problem that impact the user while using variable step-size method.

Theorem (2.1):

Suppose the $y^{(1)}_{t_0+h}$ and $y^{(2)}_{t_0+h}$ are the numerical solution the SODE given in eq.(2.28) using certain SLMM with step sizes h and $\frac{h}{2}$, respectively. If ε is the tolerance and $E_{est.} = \| y^{(1)}_{t_0+h} - y^{(2)}_{t_0+h} \|$, then (the new value of the step size) h_{new} is given by:

$$h_{new} = \frac{\sqrt{h_{old}}\,(\sqrt{2}+1)\,\varepsilon}{\sqrt{2}\,E_{est.}} \qquad \ldots(2.30)$$

where h_{old} refers to the old value of the step size h.

Proof:

Suppose y_t is the actual solution at $t_0 + h$, by taking expectation to the both sides of eq.(2.29) yields:

$$\begin{aligned} E(E_{est.}) &= E(\| y^{(1)}_{t_0+h} - y^{(2)}_{t_0+h} \|) \\ &= E(\| y^{(1)}_{t_0+h} - y^{(2)}_{t_0+h} + y_t - y_t \|) \\ &\le E(\| y^{(1)}_{t_0+h} - y_t \| + \| y^{(2)}_{t_0+h} - y_t \|) \\ &= E(\| y^{(1)}_{t_0+h} - y_t \|) + E(\| y^{(2)}_{t_0+h} - y_t \|) \\ &\le Ch^{1/2} + C\left(\frac{h}{2}\right)^{1/2} \end{aligned}$$

Hence:

$$E(E_{est.}) \le C\frac{\sqrt{h}(\sqrt{2}+1)}{\sqrt{2}}$$

also yields to:

$$C \leq \frac{\sqrt{2}\,E_{est.}}{\sqrt{h}\,(\sqrt{2}+1)} \qquad \ldots(2.31)$$

since, $\varepsilon = Ch_{new} = \dfrac{\sqrt{2}\,E_{est.}}{\sqrt{h_{old}}\,(\sqrt{2}+1)} h_{new}$

and so:

$$h_{new} = \frac{\sqrt{h_{old}}\,(\sqrt{2}+1)\,\varepsilon}{\sqrt{2}\,E_{est.}} \quad \blacksquare \qquad \ldots(2.32)$$

2.4 Solution of SODE's Using Implicit Methods, [1]

When we back to subsection (2.1.1), one can see the difficulty in solving the nonlinear SODE's (2.6) using implicit methods, therefore the predictor-corrector approach may be used to get an improved the results as much as it is required.

Therefore, when using an implicit method, the following two cases may be arised:

(a) If the functions f and g are linear functions, then using an implicit scheme will give no difficulty since the resulting finite difference equation may be simplified to an explicit formula. As an example consider the SODE:

$$dy_t = dt + dW_t,\ y_{t_0} = 0, \qquad \ldots(2.33)$$

where the functions f and g are $f(y_t) = 1$ and $g(y_t) = 1$, which are linear and using the AM_2 method for n = 2, 3,…, N, we get:

$$y_n - y_{n-1} = h\left[\frac{5}{12}f(y_n) + \frac{8}{12}f(y_{n-1}) - \frac{1}{12}f(y_{n-2})\right] + g(y_{n-1})\Delta W_{n-1}$$

Now, apply the functions f and g

$$y_n - y_{n-1} = h\left[\frac{5}{12} + \frac{8}{12} - \frac{1}{12}\right] + \Delta W_{n-1}$$

and if h = 0.1, then upon carrying some simplifications will get:

$$y_n = y_{n-1} + 0.1 + \Delta W_{n-1}$$

Hence, the evaluation of y_n may be achieved without any difficulty.

(b)If the functions f and g are nonlinear functions, then using implicit methods may give a difficulty in solving the resulting nonlinear finite difference equation in terms of y_n.

Therefore, two approaches may be used to solve such equations, which are by using either Newton-Raphson method or predictor-corrector method.

2.4.1 Newton-Raphson Method:

The Newton-Raphson method will be used to solve the resulting nonlinear equation in terms of y_n at each step of the discretization points of the time interval and it is known that the Newton-Raphson method require an initial value for each step of the scheme, which may be found approximately using any explicit one-step method. As an example, consider the solution of the SODE:

$$dy_t = -(1 + 0.01\,y_t^2)(1 - y_t^2)\,dt + 0.1(1 - y_t^2)\,dW_t,\ y_{t_0} = 0$$

The functions f and g are:

$$f(y_t) = -(1 + 0.01\,y_t^2)(1 - y_t^2) \quad \text{and} \quad g(y_t) = 0.1(1 - y_t^2)$$

which are nonlinear, and upon using AM_2 for n = 2, 3,…, N; which has the form:

$$y_n - y_{n-1} = h\left[\frac{5}{12}f(y_n) + \frac{8}{12}f(y_{n-1}) - \frac{1}{12}f(y_{n-2})\right] + g(y_{n-1})\Delta W_{n-1}$$

Now, apply the functions f and g to get:

$$y_n - y_{n-1} = h\left[\frac{5}{12}(y_n^2 - 1)(1 + 0.01y_n^2) + \frac{8}{12}(y_{n-1}^2 - 1)(1 + 0.01y_{n-1}^2) - \right.$$
$$\left.\frac{1}{12}(y_{n-2}^2 - 1)(1 + 0.01y_{n-2}^2)\right] + 0.1(1 - y_{n-1}^2)\Delta W_{n-1}$$

for n = 2,3, …, N, and if h = 0.1, we get:

$$y_n - y_{n-1} = \frac{5}{120}(y_n^2 - 1)(1 + 0.01y_n^2) + \frac{8}{120}(y_{n-1}^2 - 1)(1 + 0.01y_{n-1}^2) -$$
$$\frac{1}{120}(y_{n-2}^2 - 1)(1 + 0.01y_{n-2}^2) + 0.1(1 - y_{n-1}^2)\Delta W_{n-1}$$

for n = 2,3, …, N; where y_{n-1} and y_{n-2} are given in prior, but y_n is unknown and hence a nonlinear equation for y_n is obtained, which is simplified and equated to zero, which will yields to:

$$F(y_n) = y_n - y_{n-1} - \frac{5}{120}(y_n^2 - 1)(1 + 0.01y_n^2) - \frac{8}{120}(y_{n-1}^2 - 1)$$

$$(1 + 0.01y_{n-1}^2) + \frac{1}{120}(y_{n-2}^2 - 1)(1 + 0.01y_{n-2}^2) -$$

$$0.1(1 - y_{n-1}^2)\Delta W_{n-1}$$

Hence, $F(y_n) = 0$ and y_{n-1}; y_{n-2} are given. Also, in order to use Newton-Raphson method, we need:

$$F'(y_n) = 1 - \frac{5}{120}(1.98y_n + 0.04y_n^3)$$

Therefore, one can get the solution at each point of the mesh by solving a nonlinear algebraic equation resulting from the finite difference equation by using Newton-Raphson method given by:

$$y_n^{m+1} = y_n^m - \frac{F^m(y_n)}{F'^m(y_n)}$$

$m = 0,1,2,\ldots$

2.4.2 Predictor-Corrector Methods for Solving SODE's, [20]:

The Adam's-Bashforth and Adam's -Moulton methods having been derived in the nineteenth century [4], their fixed weighting was customarily used to reduce the computational overhead of each step.

The Adam's -Bashforth family of predictor-corrector methods [4] are explicit, linear, multistep techniques. Each successive member of the family has a higher order of convergence, and the family can be extended indefinitely. The Adam's -Moulton family of predictor-corrector methods [35] are, similarly, implicit, linear, multistep techniques, and can be similarly extended to an arbitrarily high order of convergence. This predictor-corrector combined method will be termed as Adam's -Bashforth-Moulton. For clarity, we will refer to the order of convergence of both the Adam's-Bashforth predictor phase “Adam's -Bashforth-Moulton” fixed-grid method of order 3-4.

Now, the Adam's-Bashforth-Moulton predictor-corrector method can be constructed from the Adam's -Bashforth method (an explicit method) and the Moulton rule (an implicit method).

First, the predictor step; starting from the correct value y_{n-1}, calculate an initial value $\tilde{y}_n$ via the Adam's-Bashforth (AB_2) method:

$$\tilde{y}_n = y_{n-1} + h\left[\frac{3}{2} f(t_{n-1}, y_{n-1}) - \frac{1}{2} f(t_{n-2}, y_{n-2})\right] + g(t_{n-1}, y_{n-1})\Delta W_{n-1} \qquad \ldots(2.34)$$

Next, the corrector step; improve the initial guess through iteration of Moulton rule:

$$y_n = y_{n-1} + h\left[\frac{5}{12} f(t_n, \tilde{y}_n) + \frac{8}{12} f(t_{n-1}, y_{n-1}) - \frac{1}{12} f(t_{n-2}, y_{n-2})\right] + g(t_{n-1}, y_{n-1})\Delta W_{n-1} \qquad \ldots\ldots(2.35)$$

This iteration is repeated for some fixed n-times or until the guesses converge to within some error tolerance ε:

$$|\tilde{y}_n - \tilde{y}_{n-1}| \le \varepsilon \qquad \ldots(2.36)$$

2.5 Numerical Results

In this section, some illustrative examples will be considered, which have for comparison purpose an exact solution. These examples will be solved using the considered approaches given and discussed previously in this chapter.

Remarks (2.1):

1. The argument of the considered examples is $t \in [0, 1]$ and the step size used for discretizing this interval is with $h = 0.1$.
2. The obtained results for these examples are represented at average of 10000 simulted solution by using N(0, h) random number generations for the Wiener process W_t.

Example (2.1),[30]:

Consider the SODE:

$$dy_t = -(1 + 0.01\, y_t^2)(1 - y_t^2)\, dt + 0.1(1 - y_t^2)\, dW_t$$

with the initial condition $y_{t_0} = 0$, and the exact solution is given by

$$y_t = \frac{(1+y_{t_0})e^{-2t+0.2W_t} + y_{t_0} - 1}{(1+y_{t_0})e^{-2t+0.2W_t} - y_{t_0} + 1}$$

The results of this example and its comparison with the exact solution are given in tables (2.1)-(2.3) using explicit variable step size method, implicit method using Newton-Raphson and predictor-corrector methods, respectively:

Table (2.1)

The exact and numerical results of example (2.1) using explicit variable step size method.

t_i	Exact solution	Numerical solution	Absolute error
0.1	–0.09966	–0.09894	0.00072
0.2	–0.1973	–0.19592	0.00138
0.3	–0.29105	–0.28796	0.00309
0.4	–0.3820	–0.37554	0.00646
0.5	–0.46166	–0.45595	0.00571
0.6	–0.53484	–0.52992	0.00484
0.7	–0.60242	–0.59251	0.00991
0.8	–0.66290	–0.65152	0.0011
0.9	–0.72625	–0.70381	0.022
1	–0.76158	–0.74899	0.013

Table (2.2)

The exact and numerical results of example (2.1) using explicit and implicit (Newton-Raphson) methods.

t_i	Exact solution	Explicit method	Absolute error	Implicit method	Absolute error
0.1	–0.09966	–0.09871	0.00095	–0.09771	0.00019
0.2	–0.1973	–0.19667	0.00063	–0.19271	0.00459
0.3	–0.29105	–0.29118	0.00013	–0.28626	0.00479
0.4	–0.3820	–0.3808	0.0012	–0.37614	0.00586
0.5	–0.46166	–0.46023	0.00143	–0.45489	0.00677
0.6	–0.53484	–0.53217	0.00267	–0.52739	0.00745
0.7	–0.60242	–0.60021	0.00221	–0.61006	0.00764
0.8	–0.66290	–0.65901	0.00389	–0.65560	0.0073
0.9	–0.72625	–0.72223	0.00402	–0.73573	0.00948
1	–0.76158	–0.75643	0.0515	–0.7520	0.00958

Table (2.3)

The exact and numerical results of example (2.1) using implicit (predictor-corrector) method.

t_i	Exact solution	Numerical solution	Absolute error
0.2	–0.1973	–0.1971	0.0002
0.3	–0.29105	–0.29109	0.0004
0.4	–0.3820	–0.3825	0.0005
0.5	–0.46166	–0.46766	0.0006
0.6	–0.53484	–0.534	0.0008
0.7	–0.60242	–0.60241	0.0009
0.8	–0.66290	–0.66290	0.0000
0.9	–0.72625	–0.72625	0.0000
1	–0.76158	–0.76158	0.0000

Example (2.2),[30]:

Consider the linear SODE:

$$dy_t = y_t\,dt + 0.5\,y_t\,dW_t$$

with the initial condition $y_{t_0} = 1$, and the exact solution is given by

$$y_t = y_{t_0} \exp(0.875t + 0.5W_t)$$

The results of this example and its comparison with the exact solution are given in table (2.4) using explicit variable step size method:

Table (2.4)

The exact and numerical results of example (2.2) using explicit variable step size method.

t_i	Exact solution	Numerical solution	Absolute error
0.2	1.024	1.023	0.001
0.3	1.085	1.065	0.02
0.4	1.135	1.098	0.037
0.5	1.210	1.165	0.045
0.6	1.323	1.274	0.049
0.7	1.298	1.24	0.058
0.8	1.576	1.513	0.063
0.9	1.565	1.499	0.066
1	1.851	1.764	0.087

Chapter Three

Richardson and Variable Order Methods for Solving Stochastic Ordinary Differential Equations

Numerical methods for solving ODE's constructed by translating a deterministic numerical method (like the Euler's method or LMM's or Runge-Kutta methods, etc.), and modifying such methods to solve SODE's. However, merely translating and applying certain deterministic numerical methods to SODE's will generally not provide accurate results, [13].Suitably appropriate numerical methods for SODE's should take into account a detailed analysis of the order of convergence, as well as, stability of the numerical scheme and the behavior of the error . In contrast to strong approximations which require that the simulated paths are close to the solution y of the SDE, weak approximations need not necessarily approximate these paths. If one aim is to compute, for instance, a moment of the solution, the expectation of a terminal pay-off or a general functional of the form E(g(y(T))), where E stands for the expectation and g is a certain polynomial; then the weak approximations are the method of choice. Instead of approximating the path, it is sufficient to approximate adequately the probability distribution that corresponds to the exact solution y, [39].

This chapter consists of four sections. In section (3.1) an illustration to the strong and weak convergence criteria are given. In sections (3.2) and (3.3), we study and introduce the higher-order Richardson extrapolation method and variable order method for approximating the solution of functionals diffusion of Itô kind. Under appropriate regularity conditions, it is shown that those methods allow considerable increase in the weak order of convergence of a discrete time one step approximation methods. Numerical method experiments indicate the efficiency of Richardson extrapolation method and variable order method based on higher-order weak schemes for solving SODE's with additive noise.

Finally, in section (3.4), some examples are solved using those methods discussed in sections (3.2) and (3.3) and then comparing the results with the exact solution, which are given here for comparison propose.

3.1 Convergence Criteria

Since many SODE's cannot be solved explicitly, numerical schemes are employed. There are various numerical schemes (for instance see [28]) and in order to access their usefulness and practicality, certain criteria are required in which to access the various schemes. The convergence criterion is just one of many other criterions, like mean square stability and asymptotic stability in which the cost of computation that can be used when assessing the usefulness of different numerical schemes.

Convergence of random sequences may be classified into two classes, namely, strong and weak convergence. Convergence with probability one, mean square convergence and convergence in probability are the most commonly used convergence criterion in the strong class while convergence in distribution and weak convergence are used with the weak class. For the weak class, only the distribution function is required and not the actual random variables of the underlying probability space.

3.1.1 Strong Convergence Criterion:

In many practical areas, like direct simulations, filtering or testing statistical estimators, a good path wise approximation is usually required and for these instances, the absolute error criterion is appropriate. The criterion gives a measure of path wise closeness at the end of the time interval [0, T], [28].

Consider a practical sample path of the Wiener process, i.e., W_T is given (and hence known) therefore there is no randomness in the SODE and hence no randomness in X_T [15]. The increments in the given Wiener process are then used to obtain the numerical approximation Y(T). The absolute error criterion is defined as:

$$\varepsilon = E(|X_T - Y(T)|)$$

Here, the Euclidean norm is used, X_T is the Itô process at time T while Y(T) is the approximation obtained by approximately integrating the SODE in a sequence of time steps, i.e., from the numerical scheme. Therefore, the error is the expectation of the absolute value of the difference between the approximation Y(T) and the Itô process X_T at time T.

The numerical scheme is consistent if the approximation Y(T) converges to X_T as h tends to zero. Therefore, a discrete time approximation Y(T) with maximum step size δ converges strongly to X at time T if [28]:

$$\lim_{\delta\to 0} E(|X_T - Y(T)|) = 0 \qquad \ldots(3.1)$$

A discrete time approximation Y^h ***converges strongly*** with order p > 0 at time T if there exists a positive constant C, which does not depend on the step size h, and δ > 0, such that:

$$E\left(\left\|X_T - Y_T^h\right\|\right) \le C h^p$$

holds for each $h = \dfrac{T - t_0}{N} \in (0, \delta)$; where N is the number of subintervals of the interval $J = [t_0,T]$,[15].

3.1.2 Weak Convergence Criterion:

In some cases, approximating some functional of the Itô process is of interest, such as the mean and variance of the probability distribution. Thus, the weak convergence criterion is used since the requirements for their simulation are not as demanding as for path wise approximations, [28]. Here the sample path W_T is not known but is drawn from the distribution of Wiener processes.

Since W_T and X_T are a random variables. The numerical approximation Y(T) is also a random variable, because Y(T) is obtained using samples increments of Wiener-process.

A general time discrete approximation Y with maximum time step size δ converges weakly to X at time T as $\delta \longrightarrow 0$ with respect to a class C of test functions g: $\mathbb{R}^d \longrightarrow \mathbb{R}$, if we have:

$$\lim_{\delta \to 0} |E(g(X_T)) - E(g(y(T)))| = 0, \text{ for } g \in C$$

A discrete time approximation Y_T^h with step size h is said to be ***converges weakly*** with order $p > 0$ to X at time T as $h \longrightarrow 0$, if for each $g \in C_p^{2(p+1)}(\mathbb{R}^d, \mathbb{R})$ there exists a positive constant C, which does not depend on h and a finite number $\delta > 0$, such that:

$$|E(g(X_T)) - E(g(Y_T^h))| \le Ch^p$$

holds for each $h \in (0, \delta)$,[16].

Remark (3.1):

We shall discuss first the Richardson extrapolation method which is considered as a special case a general scheme of variable order method.

3.2 Richardson Extrapolation Method for Solving SODE's

Consider the Itô process y_t satisfying the SODE with one Wiener process:

$$dy_t = f(t, y_t)\, dt + g(t, y_t)\, dW_t;\ y_{t_0} = y_0 \qquad \ldots(3.2)$$

for $t \in J$, where $J = [t_0, T]$, $t_0 \in [0, \infty)$, $y_{t_0} \in \mathbb{R}$ and where f and g are the drift and diffusion functions respectively.

We shall suppose that f and g are at least Lipschitz functions and satisfy the linear growth bound and that all of the following initial moments are exists:

$$E(|y|^r) < \infty,\ r = 1, 2, \ldots \qquad \ldots(3.3)$$

so that we have a unique solution of (3.2) for which all moments exist.

To define an appropriate measure for the rate of convergence, we shall say that a discrete - time approximation y_t converges weakly with order $p \in \{1, 2, \ldots\}$ if for each $g \in C_p^{\infty}(\mathbb{R}^d, \mathbb{R})$ there exist a constant C_g, which does not depend on h, such that:

$$M_h = |E(g(y(h))) - E(g(y(T)))| < C_g h^p \qquad \ldots(3.4)$$

for all $h \in (0, 1)$; [43], [34].

A first-order weak approximation (see [43] and [34]) is provided by Euler's scheme:

$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1}) + g(t_{n-1}, y_{n-1})\Delta W_{n-1} \qquad \ldots(3.5)$$

Here, ΔW_{n-1} represent an independent N(0,1) distributed Gaussian random variables.

We turn now to Richardson extrapolation methods for the simulation of functionals of Itô diffusion based on discrete-time weak approximation, assuming in what follows that the function for $g \in C_p^{\infty}(\mathbb{R}^d, \mathbb{R})$ is given.

The series weak error expansion for some $N \geq 1$ has the form:

$$E(g(y(T)) - g(y(h))) = \sum_{j=1}^{N} a_{2j} h^{2j} + O(h^{2N+1}) \qquad \text{….(3.6)}$$

where a_2, $a_{4,}$ … are constants independent of h, then the process of Richardson extrapolation method consists of successively eliminating terms in the error expansion to produce approximations of higher order.

Form (3.6), we have for the step size h and $\frac{h}{2}$, respectively:

$$\left.\begin{aligned} E(g(y(T))) &= E(g(y(h))) + \sum_{j=1}^{N} a_{2j} h^{2j} + O(h^{2N+1}) \\ E(g(y(T))) &= E\left(g\left(y\left(\frac{h}{2}\right)\right)\right) + \sum_{j=1}^{N} a_{2j}\left(\frac{h}{2}\right)^{2j} + O(h^{2N+1}) \end{aligned}\right\} \qquad \text{…(3.7)}$$

Multiplying the second equation in (3.7) by 4 and subtracting the first equation, yields to:

$$3E(g(y(T))) = 4E\left(g\left(y\left(\frac{h}{2}\right)\right)\right) - E(g(y(h))) + \sum_{j=2}^{N}\left(\frac{1}{2^{2j-2}} - 1\right)a_{2j}h^{2j} + O(h^{2N+1}) \quad \ldots(3.8)$$

The multiplicative factor 4 was chosen to cancel the h^2 terms. Therefore, eq. (3.8), shows that:

$$E_1(g(y(h))) = \frac{4E\left(g\left(y\left(\frac{h}{2}\right)\right)\right) - E(g(y(h)))}{3} \quad \ldots(3.9)$$

which is an $O(h^4)$ approximation to $E(g(y(T)))$.

Observe that we did not actually need to know the value of the coefficient a_2 but only that, the error expansion had the form (3.6). The process can be continued from (3.8) in this direction, when:

$$E(g(y(T))) = E_1(g(y(h))) - \frac{3}{4}a_4h^4 + \ldots$$

and

$$\left.\begin{aligned} E(g(y(T))) &= E_1(g(y(h))) - \frac{3}{4}a_4h^4 + \ldots \\ E(g(y(T))) &= E_1\left(g\left(y\left(\frac{h}{2}\right)\right)\right) - \frac{3}{4}a_4\left(\frac{h}{2}\right)^4 + \ldots \end{aligned}\right\} \quad \ldots(3.10)$$

Similarly, multiplying the second equation in (3.10) by 16 and subtracting the first equation and eliminating the h^4 term, yields to:

$$15E(g(y(T))) = 16E_1\left(g\left(y\left(\frac{h}{2}\right)\right)\right) - E_1(g(y(h))) \quad \ldots(3.11)$$

and obtaining the order-six approximation:

$$E_2(g(y(h))) = \frac{16E_1\left(g\left(y\left(\frac{h}{2}\right)\right)\right) - E_1(g(y(h)))}{15} \qquad \ldots(3.12)$$

In general, using the mathematical induction, one obtain recursively the $O(h^{2n+2})$ approximation, of the general form as:

$$E_0(g(y(h))) = E(g(y(h))) \qquad \ldots(3.13)$$

$$E_n(g(y(h))) = \frac{4^n E_{n-1}\left(g\left(y\left(\frac{h}{2}\right)\right)\right) - E_{n-1}(g(y(h)))}{4^n - 1} \qquad \ldots(3.14)$$

for all n = 1, 2, …;

Note that, to find $E_2(g(y(h)))$,one must calculate $E_1(g(y(h)))$ which in turn requires the computation of $E_1(g(y(h/4)))$. For simplicity, the following diagram illustrates such decencies.

Level	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	
0	$E_0(g(y(h)))$				
1	$E_0(g(y(h/2)))$	$E_1(g(y(h)))$			
2	$E_0(g(y(h/4)))$	$E_1(g(y(h/2)))$	$E_2(g(y(h)))$		
3	$E_0(g(y(h/8)))$	$E_1(g(y(h/4)))$	$E_2(g(y(h/2)))$	$E_3(g(y(h)))$	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

3.3 Variable Order Methods for Solving SODE's

Using the SLMM's in connection with variable order methods used for solving ODE's to derived a new approach for solving SODE's with more accurate results will give more accurate result. This method will be referred to as the variable order method for solving SODE's:

Consider the SODE:

$$dy_t = f(t, y_t)\, dt + g(t, y_t)\, dW_t;\ y_{t_0} = y_0 \qquad \ldots(3.15)$$

In this investigation, approximation are studied for expectations of functions of the solution, i.e., E(g(y(T))), where *g* is a real-valued smooth function, that is, weak approximation. The weak error is defined as:

$$E(g(y(T)) - g(y(h))) \qquad \ldots(3.16)$$

The primary goal of this investigation is to prove that the variable order method has a weak error expansion of the form:

$$E(g(y(T)) - g(y(h))) = a_1h + a_2h^2 + \ldots \qquad \ldots(3.17)$$

where a_1, a_2, … are some constants independent of h and by using several approximations $E(g(y(h_0)))$, $E(g(y(h_1)))$, $E(g(y(h_2)))$, …; with $h_0 > h_1 > h_2 > \ldots$; where h_0, h_1, h_2, … are the step sizes.

Now, to successively eliminate the terms in the error expansion, thereby producing approximations using methods of higher and higher order. The sequence of step sizes used was $h_j = h/2^j$; j = 0, 1, 2, …; where h is some starting step size. If a_1 in eq.(3.17) is not zero, then the approximation scheme E(g(y(T))) is only of order h. To obtain approximations of order h^2, and we proceed as follows:

Find the weak error expansion using two different step sizes h_0 and h_1, such that $h_1 < h_0$, as follows:

$$\begin{aligned}E(g(y(T)) - g(y(h_0))) &= a_1h_0 + a_2h_0^2 + a_3h_0^3 + \ldots \\ E(g(y(T)) - g(y(h_1))) &= a_1h_1 + a_2h_1^2 + a_3h_1^3 + \ldots\end{aligned} \quad \ldots(3.18)$$

and upon subtracting h_0 times the second equation from h_1 times the first equation and solving for E(g(y(T))), one may get:

$$E(g(y(T))) = \frac{h_1E(g(y(h_0))) - h_0E(g(y(h_1)))}{h_1 - h_0} - a_2h_0h_1 - a_3h_0h_1(h_0 + h_1) - a_4(h_0^2 + h_0h_1 + h_1^2) - \ldots$$

$$= E(g(y(h_1))) + \frac{E(g(y(h_1))) - E(g(y(h_0)))}{\frac{h_0}{h_1} - 1} - a_2h_0h_1 - a_3h_0h_1(h_0 + h_1) - a_4(h_0^2 + h_1h_2 + h_1^2) - \ldots$$

Thus, letting:

$$E_1(g(y(h_0))) = E(g(y(h_1))) + \frac{E(g(y(h_1))) - E(g(y(h_0)))}{\frac{h_0}{h_1} - 1} \quad \ldots(3.19)$$

which is an $O(h_0^2)$ approximation to E(g(y(T))). Since $h_1 < h_0$ and any two pair h_j and h_{j+1} may be used in the above elimination process, one may see that in general:

$$E_1(g(y(h_j))) = E(g(y(h_{j+1})))+\frac{E(g(y(h_{j+1})))-E(g(y(h_j)))}{\frac{h_j}{h_{j+1}}-1} \quad \ldots(3.20)$$

which is also an $O(h_j^2)$ approximation to E(g(y(T))). Now, we have:

$$\left.\begin{aligned} &E(g(y(T))) = E_1(g(y(h_0)))-a_2h_0h_1-a_3h_0h_1(h_0+h_1) - \\ &\qquad a_4h_0h_1(h_0^2 + h_0h_1 + h_1^2) - \ldots \\ &\text{and} \\ &E(g(y(T))) = E_1(g(y(h_1)))-a_2h_1h_2-a_3h_1h_2(h_1+h_2) - \\ &\qquad a_4h_1h_2(h_1^2 + h_1h_2 + h_2^2) - \ldots \end{aligned}\right\} \quad \ldots(3.21)$$

and upon eliminating the terms involving a_2, we obtain:

$$E(g(y(T))) = E_2(g(y(h_0)))+a_3h_0h_1h_2+a_4h_0h_1h_2(h_0+h_1+h_2)+\ldots$$

where:

$$E_2(g(y(h_0))) = E_1(g(y(h_1)))+\frac{E_1(g(y(h_1)))-E_1(g(y(h_0)))}{\frac{h_0}{h_2}-1} \quad \ldots(3.22)$$

which is an $O(h_0^3)$ approximation to E(g(y(T))). More generally:

$$E_2(g(y(h_j)))=E_1(g(y(h_{j+1})))+\frac{E_1(g(y(h_{j+1})))-E_1(g(y(h_j)))}{\frac{h_j}{h_{j+2}}-1} \quad \ldots(3.23)$$

which is also an $O(h_j^3)$ approximation to E(g(y(T))). Similarly, continuing in this manner, the following recursively sequence may be defind:

$$E_0(g(y(h_j))) = E(g(y(h_j))) \qquad \ldots(3.24)$$

$$E_n(g(y(h_j))) = E_{n-1}(g(y(h_{j+1}))) + \frac{E_{n-1}(g(y(h_{j+1}))) - E_{n-1}(g(y(h_j)))}{\dfrac{h_j}{h_{j+n}} - 1} \qquad \ldots(3.25)$$

for all n = 1, 2, …; j = 0, 1, …

On the basis of the results for $E(g(y(h_j)))$ and $E_2(g(y(h_j)))$, it seems that $E_n(g(y(h_j)))$ provides an $O(h_j^{n+1})$ approximation to $E(g(y(T)))$. This may be verified directly by following the evolution of the general term $a_n\, h^n$ in the error expansion, but is perhaps obtained more easily by the following alternative approach obtained from equations (3.24) and (3.25), which is given in the following diagram:

Level	$O(h_j)$	$O(h_j^2)$	$O(h_j^3)$	$O(h_j^4)$	
0	$E_0(g(y(h_0)))$				
1	$E_0(g(y(h_1))) \to$	$E_1(g(y(h_0)))$			
2	$E_0(g(y(h_2))) \to$	$E_1(g(y(h_1))) \to$	$E_2(g(y(h_0)))$		
3	$E_0(g(y(h_3))) \to$	$E_1(g(y(h_2))) \to$	$E_2(g(y(h_1))) \to$	$E_3(g(y(h_0)))$	…
⋮	⋮	⋮	⋮	⋮	⋱

3.4 Numerical Results

As an illustration and for comparison purpose, we consider in this section, some illustrative examples, which are for comparison between the numerical schemes used in this wotk, the same examples considered in chapter two. But, first consider the following remarks:

Example (3.1):

Resolving example (2.1) using Richardson extrapolation method with explicit Euler's method and variable order method with explicit Euler's method we get the results present in tables (3.1)-(3.6).

Table (3.1)
The approximate results for the weak solution using Richardson extrapolation method.

Level	**O(h^2)**	**O(h^4)**	**O(h^6)**	**O(h^8)**	
0	9.989×10^{-9}				
1	9.986×10^{-9}	9.986×10^{-9}			
2	9.992×10^{-9}	9.994×10^{-9}	9.986×10^{-9}		
3	9.986×10^{-9}	9.985×10^{-9}	9.993×10^{-9}	9.986×10^{-9}	…
⋮	⋮	⋮	⋮	⋮	⋱

Table (3.2)
The exact results for the weak solution using Richardson extrapolation method.

Level	**O(h^2)**	**O(h^4)**	**O(h^6)**	**O(h^8)**	
0	9.988×10^{-9}				
1	9.986×10^{-9}	9.985×10^{-9}			
2	9.992×10^{-9}	9.994×10^{-9}	9.986×10^{-9}		
3	9.986×10^{-9}	9.984×10^{-9}	9.992×10^{-9}	9.986×10^{-9}	…
⋮	⋮	⋮	⋮	⋮	⋱

Table (3.3)
The absolute error between the approximate and exact results for the weak solution using Richardson extrapolation method.

Level	**O(h^2)**	**O(h^4)**	**O(h^6)**	**O(h^8)**	
0	1.372×10^{-12}				
1	0	1×10^{-12}			
2	0	0	0		
3	0	10×10^{-13}	10×10^{-13}	0	…
⋮	⋮	⋮	⋮	⋮	⋱

Table (3.4)

The approximate results for the weak solution using variable order method.

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	1.003×10^{-4}				
1	9.975×10^{-5}	9.915×10^{-5}			
2	9.924×10^{-4}	9.872×10^{-5}	9.857×10^{-5}		
3	1.157×10^{-4}	1.335×10^{-4}	1.458×10^{-4}	1.676×10^{-4}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table (3.5)

The exact results for the weak solution using variable order method.

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	9.894×10^{-5}				
1	9.89×10^{-5}	9.887×10^{-5}			
2	9.877×10^{-5}	9.863×10^{-5}	9.855×10^{-5}		
3	1.154×10^{-4}	1.334×10^{-4}	1.456×10^{-4}	1.526×10^{-4}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table (3.6)

The absolute error between the approximate and exact result for the weak solution using variable order method.

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	1.367×10^{-6}				
1	8.455×10^{-7}	2.8×10^{-7}			
2	8.936×10^{-7}	9×10^{-8}	2×10^{-8}		
3	3.163×10^{-7}	10×10^{-8}	2×10^{-7}	1.5×10^{-5}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example (3.2):

Resolving example (2.2) using Richardson extrapolation method with explicit Euler's method and variable order method with explicit Euler's method we get the results present in tables (3.7)-(3.12).

Table (3.7)
The approximate results for the weak solution using Richardson extrapolation method.

Level	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	
0	3.832×10^{-5}				
1	2.991×10^{-5}	2.711×10^{-5}			
2	1.164×10^{-5}	5.551×10^{-6}	2.935×10^{-5}		
3	7.891×10^{-6}	6.641×10^{-6}	1.042×10^{-5}	2.978×10^{-5}	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table (3.8)
The exact results for the weak solution using Richardson extrapolation method.

Level	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$	
0	2.832×10^{-5}				
1	2.991×10^{-5}	2.721×10^{-5}			
2	1.164×10^{-5}	5.548×10^{-6}	2.934×10^{-5}		
3	7.892×10^{-6}	6.644×10^{-6}	1.042×10^{-5}	2.977×10^{-5}	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table (3.9)

The absolute error between the approximate and exact results for the weak solution using Richardson extrapolation method.

Level	**O(h^2)**	**O(h^4)**	**O(h^6)**	**O(h^8)**	
0	0				
1	8×10^{-7}	1×10^{-7}			
2	0	3×10^{-9}	10×10^{-9}		
3	0	3×10^{-9}	0	1×10^{-8}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table (3.10)

The approximate results for the weak solution using variable order method.

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	1.462×10^{-4}				
1	1.472×10^{-5}	1.483×10^{-5}			
2	1.481×10^{-4}	1.49×10^{-5}	1.492×10^{-5}		
3	1.51×10^{-4}	1.54×10^{-4}	1.558×10^{-4}	1.791×10^{-4}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table (3.11)

The exact results for the weak solution using variable order method.

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	1.424×10^{-5}				
1	1.419×10^{-5}	1.414×10^{-5}			
2	1.419×10^{-5}	1.419×10^{-5}	1.42×10^{-5}		
3	1.439×10^{-4}	1.46×10^{-4}	1.474×10^{-4}	1.482×10^{-4}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table (3.12)
The absolute error between the approximate and exacts result for the weak solution using variable order method.

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	1.32×10^{-4}				
1	5.3×10^{-7}	6.9×10^{-7}			
2	1.339×10^{-4}	7.1×10^{-7}	7.2×10^{-7}		
3	7.1×10^{-4}	8×10^{-6}	8.4×10^{-6}	3.09×10^{-5}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Conclusions and Recommendations

The following conclusions may be drown from the present study

1. Variable step size methods improve the accuracy of the results, but it requires more calculation which will increases the consuming time.
2. Richardson extrapolation method and Variable order method give a high accurate results in comparison with SLMM's, respectively.

Also from the present study the following conclusions may be drown as an open problems for the future work:

1. Deriving higher order models of the SLMM's to solve SODE's.
2. Applying Richardson extrapolation method and variable order method for solving SODE's based on explicit stochastic Runge-Kutta methods.
3. Using the proposed methods given in this thesis to solve SODE's with multi-Wiener process.

References

[1] Al-Tememy N. Z., "Numerical Solutions of Stochastic Ordinary Differential Equations ", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Mustansiriya University, Baghdad, Iraq., 2011.

[2] Arnold L., "Stochastic Differential Equations; Theory and Applications", John Wiley and Sons, Inc., 1974.

[3] Artemiev S. S. and Averina T. A., "Numerical Analysis of System of Ordinary and Stochastic Differential Equations "Applications of Mathematics , Springer-Verlag , Berlin, 1997.

[4] Bashforth F. and Adams J. C., "An Attempt to test the Theories of Capillary Action by comparing the theoretical and measured forms of drops of fluid. With an explanation of the method of integration employed in constructing the tables which give the theoretical forms of such drops", Cambridge University, 1983.

[5] Black F. and Scholes M., "The Pricing of Options and Corporate Liabilities", J. Political Economy, 81, 637-659, 1973.

[6] Buckwar E. and Winkler R., "Improved linear Multi-Step Methods for Stochastic Ordinary Differential Equations" J. Computer Applications of Mathematics, 205 (2), 912-922, 2007.

[7] Buckwar E., Bokor R. H. and Winkler R., "Asymptotic Mean-Squre Stability of Two-Step Methods For Stochastic Ordinary Differential Equations" BIT, Numerical Mathematics, 46(2), 261-282, 2006.

[8] Buckwar E., and Winkler R., "Multi-step methods for Stochastic Differential Equations and their application to problems with small noise" SIAM J. Numerical Analysis 44 (2), 779-803, 2006.

[9] Burrage K. and Burrage P. M., "Order Conditions of Stochastic Rung-Kutta Methods By B-series", SIAM. J. Applied Numerical Mathematics, 38(5), 1626-1646, 2000.

[10] Burrage P. M., "Runge-Kutta Methods for Stochastic Differential Equations", Ph.D. Thesis, Department of Mathematics, Queensland University, Brisbane, Queensland, Australia, 1999.

[11] Burrage K. and Burrage P. M., "General Order Conditions for Stochastic Runge-Kutta Methods for Both Commuting and Non-Commuting Stochastic Ordinary Differential Equation Systems", Applied Numerical Mathematics, 28, 161-177, 1998.

[12] Burrage K., Burrage P. M. and Belward J., "A bound on the maximum strong order of stochastic Runge-Kutta methods for stochastic ordinary differential equations", BIT, 37 (4), 771–780, 1997.

[13] Burrage K. and Burrage P. M., "High Strong Order Explicit Runge-Kutta Methods for Stochastic Ordinary Differential Equations", Applied Numerical Mathematics, 22, 81-101, 1996.

[14] Butcher J. C., "The Numerical Analysis of Ordinary Differential Equations", John Wiley, U.K., 1987.

[15] Cao R. and Pope S. B., "Numerical Integration of Stochastic Differential Equations: Weak Second-Order Mid-Point Scheme for Application in the Composition PDF Method", Journal of Computational Physics, Vol.185 (1), 194-212, 2003.

[16] Carletti M., "Numerical Solution of Stochastic Differential Problems in the Bioscience", Journal of Computational and Applied Mathematics, Vol.185, 422-450, 2006.

[17] Dahlquist G., "23 years of Numerical Instability", BIT, Numerical Mathematics, 25, 188-204, 1985.

[18] Evans L. C., "An Introduction to Stochastic Differential Equations", Version 12, Lecture Notes, Short Course at SIAM Meeting, July, 2006.

[19] Han S., "Numerical Solution of Stochastic Differential Equations", M.Sc. Thesis, University of Edinburgh and Heriot-Watt University, Scotland, UK, 2005.

[20] Hayes A. P., "The Adams-Bashforth-Moulton Integration Method Generalized To An Adaptive Grid" University of

Maine,120 Benntt Hall, Orono , Maine 4469-5709, USA and rew .Hayes@umit.maine.edu.

[21] Hussain A. S., " On Numerical Solutions of Some Stochastic Ordinary Differential Equations ", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq., 2009.

[22] Ikeda N. and Watanabe S., "Stochastic Differential Equations and Diffusion Process" North-Holland, Amsterdam (Kodansha LTD., Tokyo), 1989.

[23] Jassim H. A., "Solution of Stochastic Linear Ordinary Delay Differential Equations", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq, 2009.

[24] Karatzas I. and Shreve S. E., "Brownian Motion and Stochastic Calcuus", 2^{nd} Edition, Springer-Verlag, Berlin, 1999.

[25] Kloeden P. E. and Platen E., “The Numerical Solution of Stochastic Differential Equations”, Springer-Verlag, New York, 1999.

[26] Kloeden P. E, Platen E. and Schurz H., "Numerical Solution of Stochastic Differential Equations through Computer Experiments", Springer-Verlag, Berlin, 1997.

[27] Kloeden P. E, Platen E. and Hofmann N., "Extrapolation Methods for the Weak Approximation of ITO Diffusions"

SIAM J. Numerical Analysis, Vol.32. No5, 1519-1534, October, 1995.

[28] Kloeden P. E. and Platen E., "The Numerical Solution of Stochastic Differential Equations", 2nd Edition, V.23, Application of Mathematics, New York, Springer-Verlag, Berlin, 1995.

[29] Lax P.D. and Richtmyer R. D., "Survey of the Stability of linear finite difference equations", Comm. Pure Application of Mathematics , 9: 267-293, 1956 .

[30] Mauthner S., "Step Size Control in the Numerical Solution of Stochastic Differential Equations", Journal of Computational and Applied Mathematics, 100, 93-109, 1998.

[31] Merton R., "The Theory of Rational Option Pricing", Bell Journal of Economics and Management Science, 4, 141-183, 1973.

[32] Milsten G. N., Platen E. and Schure H., "Blanced Implicit Methods for Stiff Stochastic Systems”, SIAM J. Numerical Analysis, 35 ,1010-1019.1999.

[33] Milstein G. N., "Numerical Integration of Stochastic Differential Equations", Kluwer Academic Publishers, Dordrecht, 1995.

[34] Milsten G. N., "A Method of Second Order Accuracy Integration of Stochastic Differential Equations", Theorem Prob. Application, 23, 396-401, 1978.

[35] Moulton, F. R. "New Methods in Exterior Ballistics", University of Chicago, 1926.

[36] Platen E., "An Introduction to Numerical Methods for Stochastic Differential Equations", Acta Numerical, 8, 197-246, 1999.

[37] Platen E. and Wagner W., "On Taylor Formula for a Class of Itô Processes", Prob. Math. Statist. 3, 37-51, 1982.

[38] Protter P., "Stochastic Integration and Differential Equations" Springer-Verlag ,Berlin,1992.

[39] Rößler A., "Runge-Kutta Methods for Numerical Solution of Stochastic Differential Equations", Ph.D. Thesis, Department of Mathematics, Technology University, Germany, 2003.

[40] Sickenberger Th.,"Efficient Transient Noise Analysis in Circuit Simulation", Ph.D. Thesis, Humboldt-Universita Berlin , 2008.

[41] Sobczyk K., "Stochastic Differential Equations with Applications to Physics and Engineering", Kluwer Academic Publishers, Dordrecht, 1991.

[42] Subhi M. M., "Solutions of Stochastic Ordinary Differential Equations Using Variable Step Size Rung-Kutta Methods", M.Sc. Thesis, Department of Mathematics, College of Science, Al-Nahrain University, Baghdad, Iraq., 2012.

[43] Talay D., " Efficient Numerical Schemes for The Approximation of Expectations of Functionals of the Solution of an Stochastic Differential Equation and Applications, Lecture Notes in control and in form.sci, 61, Springer, New York, 294-313, 1984.

[44] Φksendal B., "Stochastic Differential Equations", Applications of Mathematics 21, Springer-Verlag, Berlin, 1998.

المستخلص

أهداف هذه الرسالة يمكن أن ينقاد في ثلاث إتجاهات رئيسية:

الهدف الاول هو في دراسة، وبالتفصيل، الجانب النظري والأساسي لموضوع حسبان التفاضل والتكامل التصادفي Stochastic Calculus ومن ثم دراسة الطرائق متعددة الخطوات Linear Multistep Methods لحل المعادلات التفاضلية التصادفية الاعتيادية Stochastic Ordinary Differential Equations وبرهان بعض النتائج المتعلقة بهذا الموضوع، بالإضافة إلى، دراسة متسلسلات نشر تايلور وتطبيقاتها ومن صيغة Itô التصادفي.

الهدف الثاني يتمثل بدراسة طريقة مارياما ذات الخطوتين Two Steps Maruyama Method وأيضاً دراسة الحلول العددية للمعادلات التفاضلية التصادفية بإستحدام الطرائق الضمنية Implicit Methods والتي عولجت بإستخدام الطرائق المتبعة لحل المعادلة الجبرية الغير خطية الناتجة من إستخدام الصيغة الضمنية لطرائق متعددة الخطوات، حيث تمثلت هذه بطريقتين ألا وهما طريقة نيوتن-رافسون Newton-Raphson Method وطريقة التنبؤ والتصحيحPredictor Corrector Method، ومن ثم إقتراح أسلوب جديد لحل المعادلات التفاضلية التصادفية الاعتيادية باستخدام طرائق متغيرة الخطوة Variable Step Size Method.

الهدف الثالث تمثل بإستحداث طريقة ريجاردسون للاستكمال Richardson Extrapolation Method ومن الرتبة العليا وطرائق متغيرة الرتبة Variable Order Method لحل المعادلات التفاضلية التصادفية الاعتيادية، والتي أدت إلى زيادة ملحوظة في دقة النتائج المستحصل عليها من إستخدام الطرائق الاعتيادية.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جــامعة النهــــريــن
كليــة العلوم
قسم الريـاضيات وتطبيقات الحاسوب

تحسين دقة الطرائق متعددة الخطوات التصادفية لحل المعادلات التفاضلية التصادفية الاعتيادية

رسالة
مقدمة إلى كلية العلوم – جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل

نبأ رحيم كريم
(بكالوريوس رياضيات/ كلية علوم / جامعة النهرين ٢٠٠٩)

اشراف

أ.م.د.فاضل صبحي فاضل   أ.د.علاء الدين نوري احمد

ربيع الثاني ١٤٣٤   شباط ٢٠١٣