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Approximation Method for Solving Fractional Order Variational Problems using Haar Wavelet

A Thesis
Submitted to the Council of the College of Science / Al-Nahrain University
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Science in Mathematics

By

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June 2014 Shabaan 1435.

بسم الله الرحمن الرحيم

يَٰٓأَيُّهَا ٱلَّذِينَ
ءَامَنُوٓاْ إِذَا قِيلَ لَكُمۡ تَفَسَّحُواْ فِي ٱلۡمَجَٰلِسِ فَٱفۡسَحُواْ يَفۡسَحِ
ٱللَّهُ لَكُمۡۖ وَإِذَا قِيلَ ٱنشُزُواْ فَٱنشُزُواْ يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ
مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ دَرَجَٰتٖۚ وَٱللَّهُ بِمَا تَعۡمَلُونَ خَبِيرٞ ١١

صدق الله العظيم

سورة المجادلة
الآية (11)

Dedication

To …

My Family with Love and Respects

Acknowledgement

Praise is to Allah the Lord of the worlds and peace and blessings be upon the master of humankind Muhammad and his pure Progeny and his relatives and may God curse their enemies until the Day of Judgment.

I would like to express my deepest thanks to my respected supervisors Asst. Prof. Dr. Osama H. Mohammed, and Asst. Prof. Dr. Fadhel S. Fadhel, for their continuous encouragement, advice, discussion and suggestions throughout my study.

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Zaid Amer Mohammed

June, 2014

Supervisor (s) Certifications

We, certify that this thesis entitled ***"Approximation Method for Solving Fractional Order Variational Problems using Haar Wavelet"*** under our supervision at the College of Science / Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Committee Certification

We, the examining committee certify that we have read this thesis entitled ***"Approximation Method for Solving Fractional Order Variational Problems using Haar Wavelet"*** and examined the student ***"Zaid Amer Mohammed"*** in its contents and that in our opinion, it is accepted for the degree of Master of Science in Mathematics.

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Summary

In this thesis, we present a clear procedure of solutions for the fractional variational problems via Haar wavelet technique. The fractional derivative is defined in the Riemann-Liouville sense.

The main theme of this thesis is oriented about two objects:

The first objective is to study the simplest fractional variational problem with two fixed boundary conditions and find its approximate solution by using the direct Haar wavelet method.

The scond objective is about studying the fractional variational problems with one movable condition (undetermined condition) and finding its approximate solution by using the direct Haarwavelet method.

The approximate solution for the considered classes of variational problem can be obtained directly from the functional and there is no need to solve the fractional Euler-Lagrange equation therefore the proposed approach (direct Haar wavelet method) can give us a simplest and accurate solution for such kind of variational problems of fractional order.

List of Contents

Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in on and more variables [Kilbas,2006].

The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by L'Hopital to Leibniz, which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in \mathbb{N}_0 = \{0,1,2,...\}$ when $n = \frac{1}{2}$ (what if $n = \frac{1}{2}$?).in his replay 30 September 1695, Leibnize wrote L' Hopital as follows: "… this is apparentParadox from which, one day, useful consequences will be drawn …", [Kilbas, 2006]. Since that time fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace Fourier, Able, Liouville, Riemann, and Laurent. But it was not until 1884 that the theory of generalized operators achieved such a level in its development so as to make it suitable as a point of departure for the modern mathematician [Caponetto, 2010].

By then the theory had been extended to include D^m operators, where m could be rational or irrational, positive or negative, real or complex.Thus the name fractional calculus become somewhat of misnomer. A better description might be differentiation and integration to an arbitrary order [Caponetto, 2010].Although, the concept of the fractional derivatives was introduced already in the middle of the 19^{th} century by Riemann and Liouville, [Lepik,2007]. The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [Oldham,1974] published in 1974. After that, the number of publications about the fractional calculus has rapidly increased and today there exist at least two international journals which are devoted almost entirely to the subject of fractional calculus (i) journal of fractional calculus and (ii)fractional calculus and applied analysis and for an historical overview on fractional calculus, see [Oldham,1974], [Miller,1993] and [Kilbas,2006].

Recent practical applications of fractional calculus in engineering, physics, and biology can be found in [Podlubny,1999], [Sabatier,2007], [Das,2008]and [Mainardi, 2010].

A fractional calculus of variations problem is a subtopic of fractional calculus and it is a problem in which either the objective functional or the constraint equation or both contain at least one fractional derivative term, [Agrawal,2002].

This occurs naturally in many problems of physics, mechanics and engineering in order to provide more accurate models of physical phenomena (see [El-Nabulsi,2007] and [Mozyrska,2011)], However, the fractional calculus of variations is a new field;Its starting point appear to be the references [Riewe,1996], [Riewe,1997] where Riewe developed the nonconcentrativeLagrangian, Hamiltonian, and other concepts of classical mechanics using fractional derivative, [Agrawal,2008].

Agrawal[Agrawal,2001] presented a heuristic approach to obtain differential equations of fractionally damped system. Later Agrawal[Agrawal,2002] combined the calculus of variations and the concept of fractional derivatives to develop Euler-Lagrange equations for fractional variational problems.Klimek [Klimek,2001] presented a fractional sequential mechanics model with symmetric fractional derivatives. in [Klimek,2002] stationary conservation laws for fractional differential equations with variable coefficients.Dresigmeyer and Yong [Dresigmeyer,2003] presented nonconservativeLagrangian mechanics using generalized function approach. In[Dresigmeyer,2004] the author show that obtaining differential equations for a nonconservative system using fractional derivatives may not be possible.

The fractional Euler-Lagrange equations was used by Baleanu and Coworker to model fractionalLagrangian and Hmiltonian formulations with linear velocities [Baleanu,2004], [Muslih,2005a] and Hamiltonian equations for fractional variational problems [Muslih,2005b]. References[Agrawal,2004] [Agrawal,2005] present formulations for deterministic and stochastic analyses of fractional optimal control problems.

Tarasov and Zaslvasky [Tarasov, 2005] have used variational Euler-Lagrange equations fractional generalization of the Ginzburg-Landou equation for fractal media.Fractional Euler-Lagrange equations are difficult to solve explicitly and consequently it is of interest to develop efficient numerical schemes for such dynamical systems

In this thesis, we present the direct Haar wavelet method to solve fractional variational problems with transversality/ fixed boundary conditions.Haar wavelet theory has been innovated and applied to various fields in engineering ([Strang,1989]-[Hsiao,2000]), and have proved to be a wonderful mathematical tool.The procedure begins by assuming the admissible functions by Haar wavelets with coefficients to be determined, then establishing an operational matrix for performing integration and finding the necessary condition for exterimization, solving the resulting algebraic equation gives the Haar coefficients. This indicates that for the class of problems that will be considered, the numerical solution can be obtained directly from the functional, and there is no need to solve the fractional Euler-Lagrange equations.

This thesis consist of three chapters :

In chapter one which is entitled basic concepts we give some necessary and important definitions of fractional calculus in addition to the definition of the Haar function and its main properties such as multiplication ,function approximation and operational matrix of integration and its operational matrix of fractional integration.

In chapter two we solve the classical fractional variational problems with natural conditions using Haar wavelet method.

Finally In chapter three we solve the fractional variational problems with transversality conditions using Haar wavelet method. It is remarkable that all calculations are made by using computer software Mathcad14.

Chapter One

Basic Concepts

1 Basic Concepts

1.1 Introduction

This chapter consists of five sections, in section 1.2 the gammaand beta functions are given, in section 1.3 we present some definitions of fractional order integration while in section 1.4 some definitions of fractional order derivatives are presented ,finally in section 1.5Haar functions and its main properties are given.

1.2 The Gamma and Beta Functions *[Oldham,1974]:*

The complete gamma function $\Gamma(x)$ plays an important role in the theory of fractional calculus. A comprehensive definition of $\Gamma(x)$ is that provided by Euler limit:

$$\Gamma(x)=\lim_{N\to\infty}\left(\frac{N!N^{x}}{x(x+1)(x+2)...(x+N)}\right),\ x>0 \qquad \ldots(1.1)$$

but the integral transform definition is given by:

$$\Gamma(x)=\int_{0}^{\infty} y^{x-1}e^{-y}dy\,,\ x>0 \qquad \ldots(1.2)$$

is often more useful, although it is restricted to positive value of x. An integration by parts applied to equation (1.2) leads to the recurrence relationship:

$$\Gamma(x+1)=x\Gamma(x) \qquad \ldots(1.3)$$

This is the most important property of gamma function. The same result is a simple consequence of equation(1.1), since $\Gamma(1) = 1$, this recurrence shows that for positive integer n:

$$\Gamma(n + 1) = n\Gamma(n)$$

$$= n! \qquad \ldots(1.4)$$

The following are the most important properties of the gamma function:

1. $\Gamma\left(\frac{1}{2}-n\right)=\frac{(-4)^n n!\sqrt{\pi}}{(2n)!}$

2. $\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!\sqrt{\pi}}{4^n n!}$

3. $\Gamma(-x)=\frac{-\pi\csc(\pi x)}{\Gamma(x+1)}$

4. $\Gamma(nx)=\sqrt{\frac{2\pi}{n}}\left[\frac{nx}{\sqrt{2\pi}}\right]^n\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right), n \in \mathbb{N}^+$

A function that is closely related to the gamma function is the complete beta function $\beta(p,q)$. For positive value of the two parameters p and q; the function is defined by the beta integral:

$$\beta(p,q)=\int_0^1 y^{p-1}(1-y)^{q-1}\,dy,\ p, q > 0 \qquad \ldots(1.5)$$

which is also known as the Euler's integral of the second kind. If either p or q is nonpositive, the integral diverges otherwise $\beta(p,q)$ is defined by the relationship:

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad \ldots(1.6)$$

wherep and q> 0.

Both beta and gamma functions have "incomplete" analogues. The incomplete beta function of argument x is defined by the integral:

$$\beta_x(p,q)=\int_0^x y^{p-1}(1-y)^{q-1}\,dy \qquad \dots(1.7)$$

and the incomplete gamma function of argument x is defined by:

$$\gamma^*(c,x)=\frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}e^{-y}\,dy$$

$$=e^{-x}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(j+c+1)} \qquad \dots(1.8)$$

$\gamma^*(c,x)$ is a finite single-valued analytic function of x and c.

1.3 Fractional Integration

There are many literatures introduce different definitions of fractional integrations, such as:

1. ***Riemann-Liouville integral, [Oldham,1974]:***

<u>Definition (1.1), (Riemann–Liouville Fractional Integrals):</u>

Let $f \in L^1[a, b]$ and $0 < \alpha < 1$. The left and Right Riemann–Liouville Fractional integrals of order α of a function f is defined respectively by:

$${}_aI_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \qquad \dots(1.9)$$

$${}_xI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt \qquad \dots(1.10)$$

for all $x\in[a,b]$.

2. Weyl fractional integral, [Oldham,1974]:

The left hand fractional order integral of order $\alpha > 0$ of a given function f is defined as:

$$ {}_{-\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}} dy, \ x > -\infty \qquad \ldots(1.11)$$

and the right fractional order integral of order $\alpha > 0$ of a given function f is given by:

$$ {}_{\infty}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}} dy, \ x < \infty$$

3. Abel-Riemann fractional integral, [Mittal,2008]:

The Abel-Riemann (A-R) fractional integral of any order $\alpha > 0$, for a function f(x) with $x \in \mathbb{R}^+$ is defined as:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\tau)^{\alpha-1} f(\tau) d\tau, \ x > 0, \ \alpha > 0 \qquad \ldots(1.12)$$

$I^0 = I$ (identity operator)

The A-R integral posses the semigroup property:

$$I^{\alpha} I^{\beta} = I^{\alpha+\beta}, \text{ for all } \alpha, \beta \geq 0 \qquad \ldots(1.13)$$

1.4 Fractional Derivatives

Many literatures discussed and presented fractional derivatives of certain functions, therefore in this section, some definitions of fractional derivatives are presented:

1. Riemann-Liouville formula of fractional derivatives, [Oldham,1974], [Nishimoto,1983]:

Among the most important formula used in fractional calculus is the Riemann-Liouville formula. For a given function f(x), $\forall\ x \in [a,b]$; the left and right hand Riemann-Liouville fractional derivatives of order $\alpha > 0$ and m is a natural number, are given by:

$$ {}_{X}D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}} dt \qquad \ldots(1.14)$$

$$ {}_{X}D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(x-t)^{\alpha-m+1}} dt \qquad \ldots(1.15)$$

Wherem– $1 < \alpha \leq m$, $m \in \mathbb{N}$.

2. The A-R fractional derivative, [Nishimoto,1983]:

The A-R fractional derivative of order $\alpha > 0$ is defined as the inverse of the corresponding A-R fractional integral, i.e.,

$$D^{\alpha} I^{\alpha} = I \qquad \ldots(1.16)$$

for positive integer m, such that $m - 1 < \alpha \leq m$,

$$(D^m I^{m-\alpha}) I^{\alpha} = D^m (I^{m-\alpha} I^{\alpha}) = D^m I^m = I$$

i.e.,

$$D^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dx^m} \displaystyle\int_a^x \dfrac{f(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{cases} \qquad \ldots(1.17)$$

3. Caputo fractional derivative, [Caputo ,1967]:

In the late sixties, an alternative definition of fractional derivatives was introduced by Caputo. Caputo and Mirandi used this definition in their work on the theory of viscoelasticity. According to Caputo’s definition:

$${}^{c}D_{x}^{\alpha} = I^{m-\alpha}D^{m}, \text{ for } m-1 < \alpha \leq m$$

which means that:

$${}^{c}D_{x}^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \dfrac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m \end{cases}$$

The basic properties of the Caputo fractional derivative are:

1. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

2. $${}_{a}I_{x}^{\alpha}\, {}^{c}D_{x}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+}) \frac{x^k}{k!}.$$

3. Caputo’s fractional differentiation is linear operator, similar to integer order differentiation:

$${}^{c}D_{x}^{\alpha}[\lambda f(x) + \mu g(x)] = \lambda\, {}^{c}D_{x}^{\alpha} f(x) + \mu\, {}^{c}D_{x}^{\alpha} g(x)$$

4. Grünwald fractional derivatives, [Oldham,1974]:

The Grünwald derivatives of any integer order to any fractional order derivatives, can take the form:

$$D^{\alpha} f(x)=\lim_{N\to\infty}\left\{\frac{\left(\frac{x}{N}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(x-j\left(\frac{x}{N}\right)\right)\right\} \quad \ldots(1.18)$$

1.5 Haar Wavelets

Haar functions have been used since 1910, when they were introduced by Hungarian mathematician AlferdHaar, [Haar, 1910].

The orthogonal set of Haar function is defined as shown in Figs.(1.1-1.8) that is a square waves with magnitude of ±1 in some interval and zero elsewhere. The first curve of Fig.(1.1) is that $h_0(x) = 1$ during the whole interval $0 \leq x \leq 1$. It is called the scaling function. The second curve $h_1(x)$ is the fundamental square wave, or mother wavelet which also spans the whole interval [0,1]. All the other subsequent curve are generated from $h_1(x)$ with two operation translation and dilation, $h_2(x)$ is obtained from $h_1(x)$ with dilation, i.e., $h_1(x)$ is compressed from the whole interval [0,1] to half interval [0,1/2] to generate $h_2(x)$, $h_3(x)$ is the same as $h_2(x)$ but shifted (translated) to the right by 1/2. Similarly, $h_2(x)$ is compressed from the half interval to a quarter interval to generate $h_4(x)$. The function $h_4(x)$ is translated to the right by 1/4, 2/4, 3/4 to generate $h_5(x)$, $h_6(x)$ and $h_7(x)$; respectively.

In general:

$h_n(x) = h_1(2^j x - k/2^j)$, $n = 2^j + k$, $j \geq 0$, $0 < k \leq 2^j$

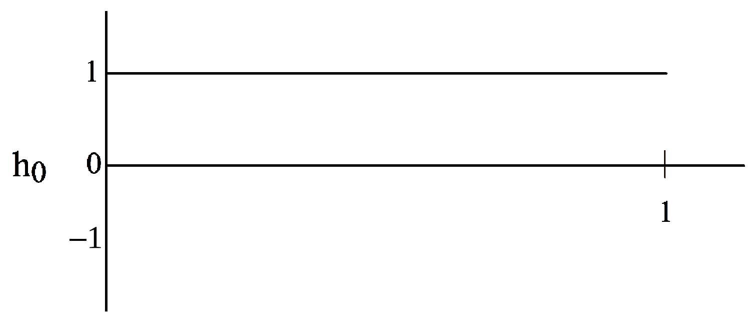

Fig.(1.1) First Haar function.

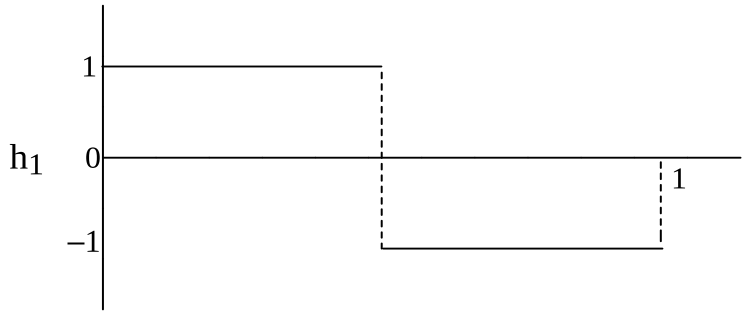

Fig.(1.2) Second Haar function.

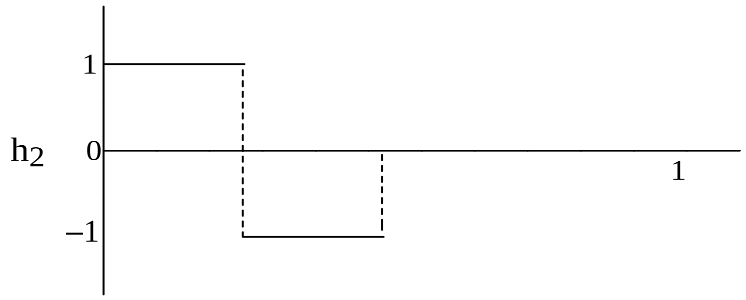

Fig.(1.3) Third Haar function.

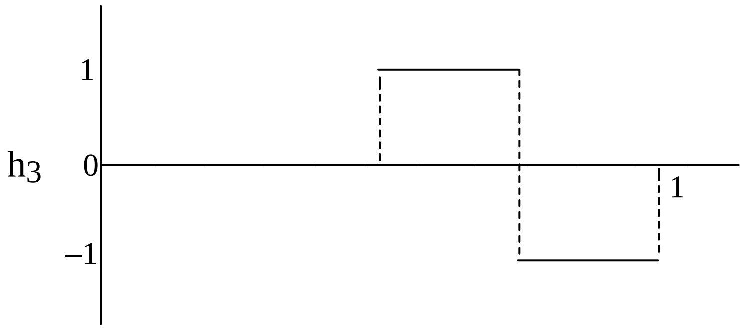

Fig.(1.4) Fourth Haar function.

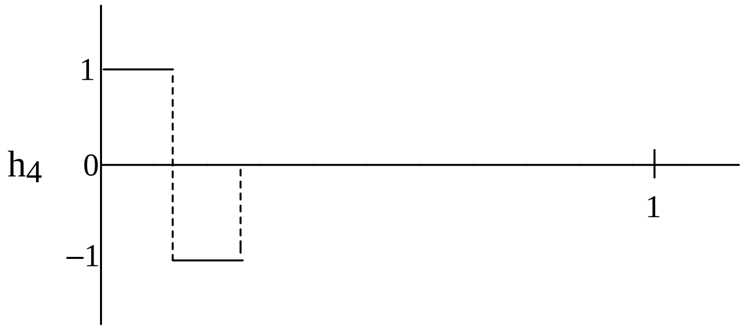

Fig.(1.5) Fifth Haar function.

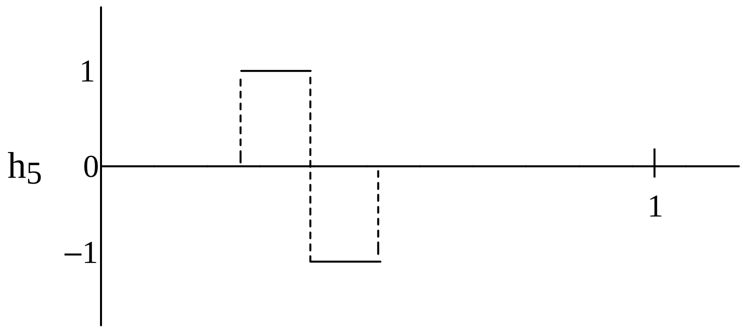

Fig.(1.6) Sixth Haar function.

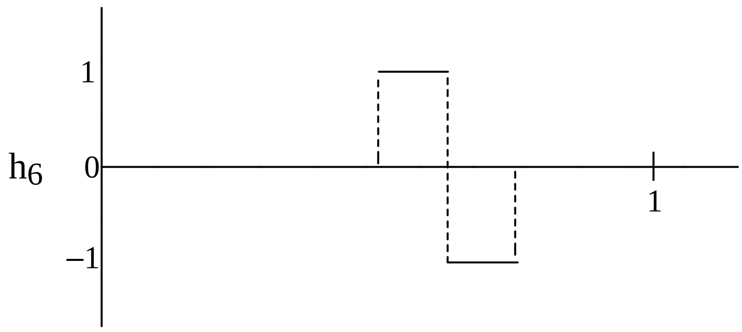

Fig.(1.7) Seventh Haar function.

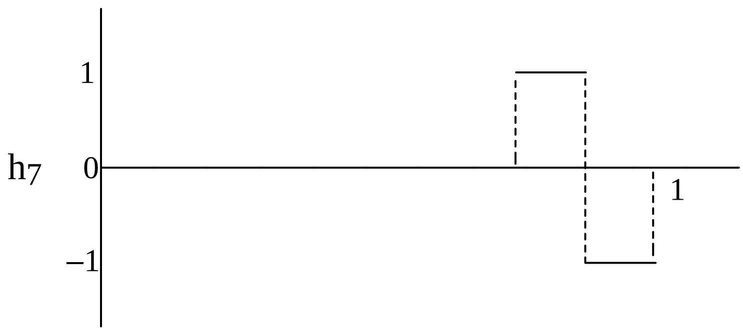

Fig.(1.8) Eighth Haar function.

This orthogonal basis is a reminiscent of the Walsh basis, in which each Walsh function contains many wavelets to fill the interval [0,1] completely, and to form a global basis. While each Haar function contains just one wavelet during some subinterval of time, and remains zero elsewhere the Haar set form a local basis.

All the Haar wavelets are orthogonal to each other:

$$\int_0^1 h_i(x)h_\ell(x)\,dx = 2^{-j}\delta_{i\ell}$$

$$= \begin{cases} 2^{-j}, & i = \ell = 2^{j+k} \\ 0, & i \neq \ell \end{cases}$$

Therefore, they form a very good transform basis.

1.5.1 Multiplication of HaarWavelets

Two basic multiplication properties of Haar wavelets are as follows:

(i) For any two Haar wavelets $h_n(t)$ and $h_l(t)$ with $n < l$.

$$h_n(t)h_l(t) = \rho h_l(t) \qquad \ldots(1.19)$$

$$\rho = h_n(2^{-i}(q + \frac{1}{2}))$$

$$= \begin{cases} 1, & 2^{i-j}k \leq q < 2^{i-j}(k + \frac{1}{2}), \\ -1, & 2^{i-j}(k + \frac{1}{2}) \leq q < 2^{i-j}(k+1), \\ 0, & \text{otherwise}, \end{cases}$$

where:

$$\begin{cases} n = 2^j + k, & j \geq 0, \quad 0 \leq k \leq 2^j \\ l = 2^i + q, & i \geq 0, \quad 0 \leq q \leq 2^i \end{cases} \qquad \ldots(1.20)$$

(ii) The square of any Haarwavelet is a block pulse with magnitude of 1 during both positive and negativehalf waves.

In the study of variational problems via Haar wavelets, it usually needs to evaluate the integration of $H_{(m)}(x)H^{T}_{(m)}(x)$ where

$$H_{(m)}(x) \triangleq [h_0(x),...,h_{m-1}(x)]^T$$

Let usdefined

$$H_{(m)}(x)H^T_{(m)}(x) \triangleq M_{(m\times m)}(x) \qquad \ldots(1.21)$$

$$H_a \triangleq \left[h_0(x), h_1(x),..., h_{\frac{m}{2}-1}(x)\right]^T = H_{(\frac{m}{2})}, H_b \triangleq \left[h_{\frac{m}{2}}(x), h_{\frac{m}{2}+1}(x),..., h_{m-1}(x)\right]^T \qquad \ldots(1.22)$$

M(x) is the Haar product matrix, which satisfies the following recursive formula equation (1.23) and the integration relation Equation (1.24).

$$M_{(m\times m)}(x) = \begin{bmatrix} M_{(\frac{m}{2}\times\frac{m}{2})}(x) & \Phi_{(\frac{m}{2}\times\frac{m}{2})}\text{diag}[H_b] \\ \text{diag}[H_b]\Phi^T_{(\frac{m}{2}\times\frac{m}{2})} & \text{daig}[\Phi^{-1}_{(\frac{m}{2}\times\frac{m}{2})}H_a] \end{bmatrix}, M_{(1\times1)}(x) = h_0(x) \qquad \ldots(1.23)$$

where

$$\Phi_{m\times m} \triangleq [H_m(x_0)H_m(x_1)...H_m(x_{m-1})], \tfrac{i}{m} \le x_i \le \tfrac{i+1}{m}$$

$$\int_0^1 H_{(m)}(\tau)H^T_{(m)}(\tau)d\tau = \int_0^1 M_{(m\times m)}(\tau)d\tau = \frac{1}{m}\Phi_{m\times m}\Phi^T_{m\times m}$$

$$= \begin{bmatrix} I_{(2\times2)} & & & & 0 \\ & \frac{1}{2}I_{2\times2} & & & \\ & & \frac{1}{4}I_{4\times4} & & \\ & & & \ddots & \\ 0 & & & & \frac{2}{m}I_{(\frac{m}{2}\times\frac{m}{2})} \end{bmatrix}$$

$$\square\ K_{(m\times m),}\ \text{for } m > 2 \qquad \ldots(1.24)$$

Equation (1.24) is very important for solving variational problems.

1.5.2Haar Wavelets Operational Matrix

In this subsection we shall begin with the more convenient way for representing Haar wavelets in computer and for $x\in[A,B]$ which was given by [Lepik, 2007] and for this purpose we define the quantity $M=2^J$, where J is the maximal level of resolution and divide the interval [A,B] into 2M subintervals of equal length; each subinterval has the length.

$\Delta x= (B - A)/2M$.

Two parameters are introduced the dilation parameter j for which $j = 0, 1,\ldots, J$ and the translation parameter $k = 0,1,\ldots,m-1$ where $m = 2^j$. The wavelets number i is identify as $i = m + k + 1$ the ithHaar wavelet is defined as:

$$h_i(x) = \begin{cases} 1, & \text{for } x \in [\xi_1(i),\xi_2(i)], \\ -1, & \text{for } x \in [\xi_2(i),\xi_3(i)], \\ 0, & \text{elsewhere.} \end{cases} \qquad \ldots(1.25)$$

where:

$$\xi_1(i) = A + 2k\mu\ ,\quad \xi_2(i) = A + (2k+1)\mu\Delta x$$

$$\xi_3(i) = A + 2(k+1)\mu\Delta x\ ,\quad \mu = {}^{M}\!/_{m}$$

The case $i = 1$ corresponding to the scaling function

$$h_1(x) = \begin{cases} 1, \text{for } x \in [A,B] \\ 0,\ \text{elsewhere} \end{cases} \qquad \ldots(1.26)$$

The following notations are introduced:

$$p_{i,1}(x) = \int_0^x h_i(\tau)d\tau$$

$$p_{i,v+1}(x) = \int_0^x p_{i,v}(\tau)d\tau, \ v = 1, 2, \ldots$$

These integrals can be evaluated by using equation (1.25) and the first two of them are given by:

$$p_{i.1}(x) = \begin{cases} x - \xi_1(i), & x \in [\xi_1(i), \xi_2(i)) \\ \xi_3(i) - x, & x \in [\xi_2(i), \xi_3(i)) \\ 0, & \text{Otherwise.} \end{cases} \qquad \ldots(1.27)$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \xi_1(i))^2, & x \in [\xi_1(i), \xi_2(i)) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3(i) - x)^2, & x \in [\xi_2(i), \xi_3(i)) \\ \frac{1}{4m^2}, & x \in [\xi_3(i), 1) \\ 0, & \text{Otherwise.} \end{cases} \qquad \ldots(1.28)$$

In general:

$$p_{i,n}(x) = \begin{cases} 0, & x < \xi_1(i) \\ \frac{1}{n!}(x - \xi_1(i))^n, & x \in [\xi_1(i), \xi_2(i)] \\ \frac{1}{n!}\left[(x - \xi_1(i))^n - 2(x - \xi_2(i))^n\right], & x \in [\xi_2(i), \xi_3(i)] \\ \frac{1}{n!}\left[(x - \xi_1(i))^n - 2(x - \xi_2(i))^n + 2(x - \xi_3(i))^n\right], & x > \xi_3(i) \end{cases}$$
$$\ldots(1.29)$$

For example, if J = 2, then:

$$P_{4,1} = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

and if J = 3, then:

$$P_{8,1} = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 & -4 & 4 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Following Figs. (1.9-1.16) represents the first integral of $h_i(x)$, for all i= 0, 2, …, 7.

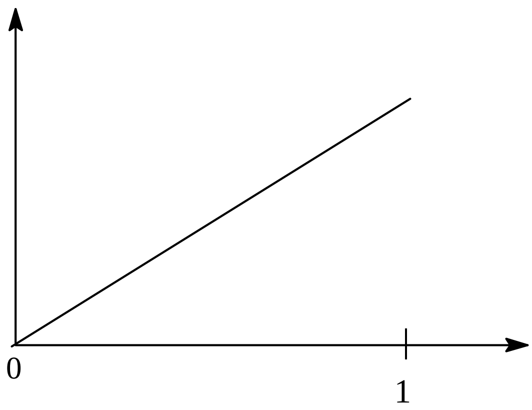

Fig.(1.9) Integration of the first Haar wavelet.

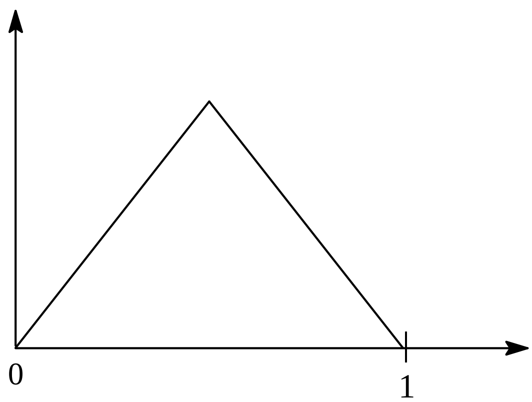

Fig.(1.10) Integration of the second Haar wavelet.

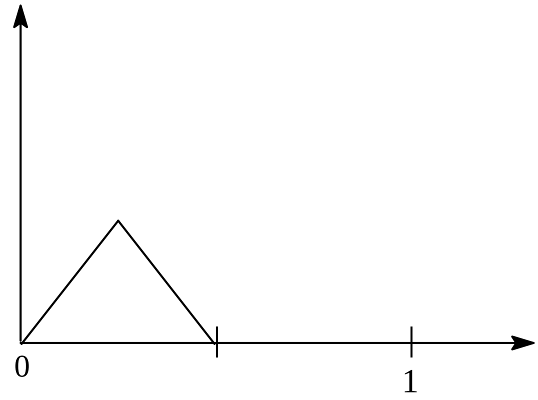

Fig.(1.11) Integration of the third Haar wavelet.

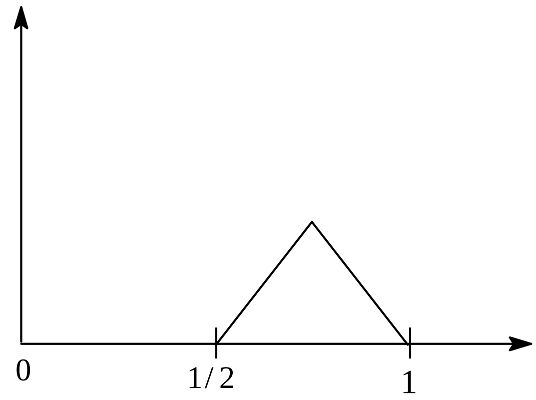

Fig.(1.12) Integration of the forthHaar wavelet.

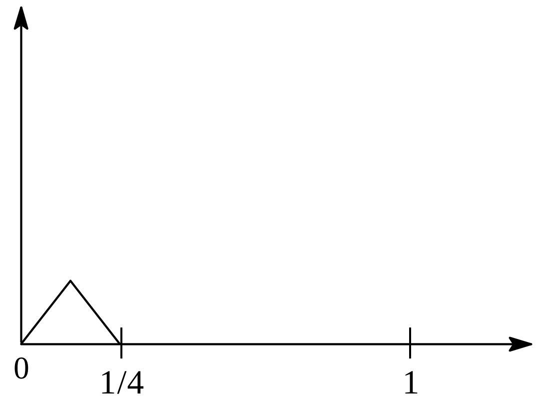

Fig.(1.13) Integration of the fifth Haar wavelet.

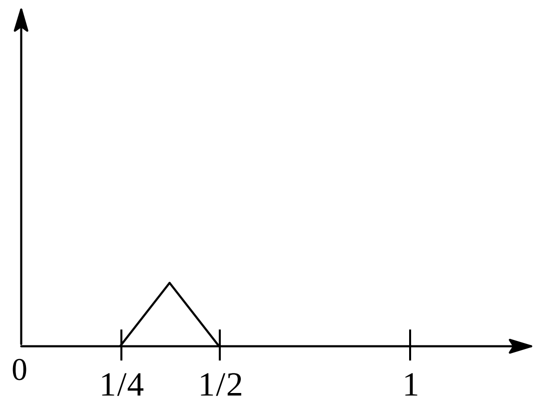

Fig.(1.14) Integration of the sixth Haar wavelet.

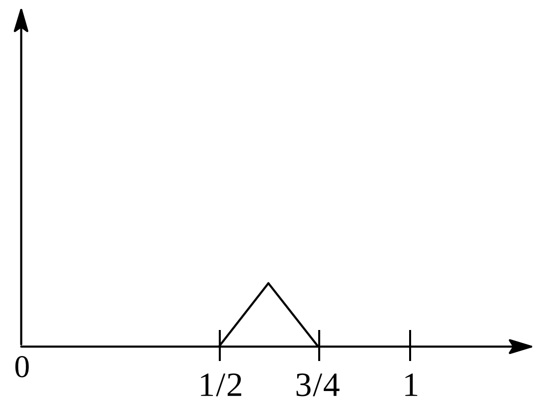

Fig.(1.15) Integration of the seventh Haar wavelet.

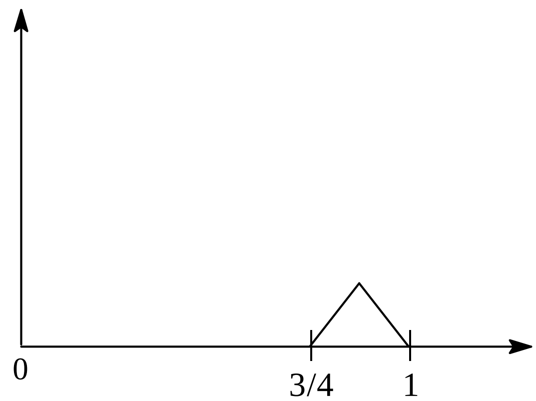

Fig.(1.16) Integration of the eighth Haar wavelet.

1.5.3 Function Approximation and operational Matrix

Any function $f(x) \in L^2([0,1])$ can be expanded in term of Haar series as:

$$f(x) = \sum_{i=1}^{\infty} c_i h_i(x) \qquad \ldots(1.30)$$

where the coefficients c_i are determined by:

$$c_i = 2^j \int_0^1 f(x) h_i(x)$$

The series in equation (1.30) contains an infinite number of terms. If f(x) is piecewise constant or may be approximated as piecewise constant, then the sum in equation(1.30) may be terminated after m terms, that is:

$$f(x) = \sum_{i=0}^{m-1} c_i h_i(x) = C_m^T H_m(x) = \hat{f}(x) \qquad \ldots(1.31)$$

$\hat{f}$denotes the truncated sum, the Haar coefficients vector C_m and Haar vector $H_m(x)$ are defined as:

$$C_m = [c_0, c_1, \ldots, c_{m-1}]^T$$

$$H_m(x) = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^T \qquad \ldots(1.32)$$

Taking the collocation points as following

$$x_i = A + (i - 0.5)\Delta x, i = 1,2, \ldots 2M \qquad \ldots(1.33)$$

By letting A = 0, B = 1 and hence $\Delta x = \frac{1}{2M}$ in equation(1.33). We define the m-square Haar matrix $\Phi_{m\times m}$ as:

$$\Phi_{m\times m} = \left[H_m\left(\frac{1}{4M}\right) H_m\left(\frac{3}{4M}\right) \ldots H_m\left(\frac{4M-1}{4M}\right)\right] \qquad \ldots(1.34)$$

Correspondingly, we have:

$$\hat{f}_m = \left[\hat{f}\left(\frac{1}{4M}\right) \hat{f}\left(\frac{3}{4M}\right) \ldots \hat{f}\left(\frac{4M-1}{4M}\right)\right] = C_m^T \Phi_{m\times m} \qquad \ldots(1.35)$$

Because the m-square Haar wavelets matrix $\Phi_{m\times m}$ is an invertible matrix, the Haar coefficients vector C_m^T can be given by:

$$C_m^T = \hat{f}_m \Phi_{m\times m}^{-1} \qquad \dots(1.36)$$

1.5.3.1 Block Pulse Function (BPF)

Defines a m – Set of Block Pulse Function (BPF) as:

$$b_i(x) = \begin{cases} 1, {}^{i}/_{m} \le x < {}^{(i+1)}/_{m} \\ 0, \quad \text{Otherwise} \end{cases} \qquad \dots(1.37)$$

where $i = 0,1,2 \dots, m-1$.

The functions $b_i(x)$are disjoint and orthogonal, that is:

$$b_i(x)b_l(x) = \begin{cases} 0 & , i \ne l \\ b_i(x) & , i = l \end{cases} \qquad \dots(1.38)$$

Kilicman and Zhour [Kilicman, 2007] have given the block pulse operational matrix of fractional order integration F^{α} as following:

$${}_{a}I_x^{\alpha} B_m(x) = F^{\alpha} B_m(x) \qquad \dots(1.39)$$

where:

$$F^{\alpha} = \frac{1}{(2M)^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 \cdots & \xi_{2M-1} \\ 0 & 1 & \xi_1 \cdots & \xi_{2M-2} \\ 0 & 0 & 1 \dots & \xi_{2M-3} \\ \vdots & \vdots & \vdots \ \ddots & \vdots \\ 0 & 0 & 0\ 0 & \vdots \\ 0 & 0 & 0\ 0 & 1 \end{bmatrix} \qquad \dots(1.40)$$

where:

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$$

1.5.3.2 Operational matrix of the fractional order integration of Haar Wavelet Functions

The integration of $H_m(x)$ defined in equation (1.32) can be approximated by Haar series with Haar coefficient matrix P as:

$$\int_0^x H_m(\tau)d\,\tau \approx P_{m\times m}H_m(x) \qquad \dots(1.41)$$

where a m-square matrix P is called the Haar wavelets operational matrix of integration [Chen, 1997].

Zhao, [Zhao, 2010] derive the Haar wavelets operational matrix of the fractional order integration.

He introduced the Riemann-Liouville fractional order integration, as given in chapter one as:

$$\left({}_aI_x^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_o^x (x-\tau)^{\alpha-1}f(\tau)d\tau = \frac{1}{\Gamma(\alpha)}x^{\alpha-1} * f(x) \qquad \dots(1.42)$$

Where$\alpha \in \mathbb{R}$ is the order of integration, $\Gamma(\alpha)$ is the Gamma function and $x^{\alpha-1} * f(x)$ is the convolution product of $x^{\alpha-1}$ and $f(x)$.

Now if f(x) is expanded in Haar function, the Riemann- liouville fractional order integration is solved via the Haar function, because the Haar functions are piecewise constant, it may be expanded into m- term Block Pulse Function (BPF) as:

$$H_m(x) = \Phi_{m\times m}B_m(x)^T \qquad \dots(1.43)$$

Where$B_m(x) \triangleq [b_0(x)b_1(x) \dots b_i(x) \dots b_{m-1}(x)]$

Next, we shall derive the Haar wavelets operational matrix of the fractional order integration by letting

$$({}_aI_x^{\alpha}H_m)(x) = P_{m\times m}^{\alpha}H_m(x) \qquad \dots(1.44)$$

where the m-square matrix$P_{m\times m}^{\alpha}$ is called the Haar wavelets operational matrix of the fractional integration.

Using equations(1.39) and (1.43) we have

$$({}_{a}\mathrm{I}_{\mathrm{x}}^{\alpha}\mathrm{H}_{\mathrm{m}})(\mathrm{x}) \approx \left({}_{a}\mathrm{I}_{\mathrm{x}}^{\alpha}\Phi_{\mathrm{m}\times\mathrm{m}}\mathrm{B}_{\mathrm{m}}\right)(\mathrm{x}) = \Phi_{\mathrm{m}\times\mathrm{m}}({}_{a}\mathrm{I}_{\mathrm{x}}^{\alpha}\mathrm{B}_{\mathrm{m}})(\mathrm{x})$$

$$\approx \Phi_{\mathrm{m}\times\mathrm{m}}\mathrm{F}^{\alpha}\mathrm{B}_{\mathrm{m}}(\mathrm{x}) \qquad \ldots(1.45)$$

From equations(1.44) and (1.45), we get:

$$\mathrm{P}_{\mathrm{m}\times\mathrm{m}}^{\alpha}\mathrm{H}_{\mathrm{m}}(\mathrm{x}) = \mathrm{P}_{\mathrm{m}\times\mathrm{m}}^{\alpha}\Phi_{\mathrm{m}\times\mathrm{m}}\ \mathrm{B}_{\mathrm{m}}(\mathrm{x})$$

$$= \Phi_{\mathrm{m}\times\mathrm{m}}\ \mathrm{F}^{\alpha}\ \mathrm{B}_{\mathrm{m}}(\mathrm{x}) \qquad \ldots(1.46)$$

Then the Haar wavelet operational matrix of the fractional order of integration $\mathrm{P}_{\mathrm{mxm}}^{\alpha}$ is given by:

$$\mathrm{P}_{\mathrm{m}\times\mathrm{m}}^{\alpha} = \Phi_{\mathrm{m}\times\mathrm{m}}\ \mathrm{F}^{\alpha}\ \Phi_{\mathrm{m}\times\mathrm{m}}^{-1} \qquad \ldots(1.47)$$

For example, let $\alpha = 0.5$, J = 2 hence m = 8, the operational matrix $\mathrm{P}_{\mathrm{m}\times\mathrm{m}}^{\alpha}$ is computed below as:

$$P_{8\times8}^{0.5} = \begin{bmatrix} 0.7523 & -0.2203 & -0.1558 & -0.0820 & -0.1102 & -0.0580 & -0.0447 & -0.0377 \\ 0.2203 & 0.3116 & -0.1558 & 0.2296 & -0.1102 & -0.0580 & 0.1756 & 0.0782 \\ 0.0410 & 0.1148 & 0.2203 & -0.0350 & -0.1102 & 0.1623 & -0.0389 & -0.0063 \\ 0.0779 & -0.0779 & 0 & 0.2203 & 0 & 0 & -0.1102 & 0.1623 \\ 0.0094 & 0.0196 & 0.0812 & -0.0032 & 0.1558 & -0.0247 & -0.0026 & -0.0009 \\ 0.0112 & 0.0439 & -0.0551 & -0.0194 & 0 & 0.1558 & -0.0247 & -0.0026 \\ 0.0145 & -0.0145 & 0 & 0.0812 & 0 & 0 & 0.1558 & -0.0247 \\ 0.0275 & -0.0275 & 0 & -0.0551 & 0 & 0 & 0 & 0.1558 \end{bmatrix}$$

Chapter Two

Haar Wavelet Method for Solving Simplest Fractional Variational Problems

2 Haar Wavelet Method for Solving Simplest Fractional Variational Problems

2.1 Introduction

This chapter consists of seven sections , in section 2.2 classical calculus of variational is presented , in section 2.3 the simplest fractional variational problem was given, while in section 2.4. The case of $\alpha, \beta \in \mathbb{R}^+$ and Several Functions are discussed and we present in section 2.5 The problem of Lagrange and the multiplier rule.

In section 2.6 the Haar wavelet direct method was used to solve the simplest fractional variational problems. Finally two numerical examples are given in section 2.7 .

2.2 Classical Calculus of Variation Problem

Let us examine for extreama, by considering a funetional of the simplest form:

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\,dx \qquad \ldots(2.1)$$

where the end points of the admissible curves are fixed, i.e., $y(x_0) = y_0$ and $y(x_1) = y_1$.

We can define the first variation of functional J(y) as the part with linear increment which is linear if δy is defined by:

$$\delta(J) = J(y + \delta y) - J(y)\,\big|_{\text{linear part in } \delta y}$$

which equals zero at the optimum solution y(x).

Now, suppose an extremum occurs at the curve y = y(x) along with all admissible solutions y = y*(x) and hence we can define the variation in the solution y to be $\delta y = y(x) - y^*(x)$ and since the first variation is a function of x, then it can be differentiated with the property that:

$$(\delta y)' = (y(x) - y^*(x))'$$

$$= y'(x) - y^{*\prime}(x)$$

$$= \delta y'$$

Therefore, if:

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

Then:

$$J[y + \delta y] = \int_{x_0}^{x_1} F(x, y + \delta y, y' + \delta y')\,dx$$

Hence:

$$J[y + \delta y] = J[y + \delta y] - J[y]$$

$$= \int_{x_0}^{x_1} F(x, y + \delta y, y' + \delta y')\,dx - \int_{x_0}^{x_1} F(x, y, y')\,dx$$

$$= \int_{x_0}^{x_1} \left[F(x, y + \delta y, y' + \delta y') - F(x, y, y')\right]dx$$

and upon using Taylor series expansion to the first degree or linear part of the increment function, we have:

$$\delta J = \int_{x_0}^{x_1} \left(F_y \delta y + F_{y'} \delta y' \right) dx$$

and upon using the method of integrations to the second part of the integral therefore we have:

$$\delta J = \int_{x_0}^{x_1} \left(F_y \delta y - \frac{d}{dx} F_{y'} \delta y \right) dx$$

$$= \int_{x_0}^{x_1} \left(F_y - \frac{d}{dx} F_{y'} \right) \delta y \, dx = 0$$

Now, since δy is an arbitrary function, hence by using the fundamental lemma of calculus of variation, we have:

$$F_y - \frac{d}{dx} F_{y'} = 0$$

which is the required necessary condition to be satisfied on the optimum solution y(x). This condition is called the Euler-Lagrange equation (for simplicity Euler equation).

2.3 The Simplest Fractional Variational Problem

Several definitions of a fractional derivative have been proposed in chapter one. These definitions include Riemann-Liouville, Grunwald-Letnikov, Weyl and Caputo, fractional derivatives. Here, we formulate the variational problem in terms of the left and the right Riemann-Liouville fractional derivatives, which are defined as:

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\, d\tau $$

and

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{-d}{dx}\right)^n \int_x^b (x-\tau)^{n-\alpha-1} f(\tau)\, d\tau $$

where α is the order of the derivative, such that $n - 1 < \alpha \le n$. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$ {}_aD_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^{\alpha},\ {}_xD_b^{\alpha} f(x) = \left(\frac{-d}{dx}\right)^{\alpha},\ \alpha = 1, 2, \ldots \qquad \ldots(2.2) $$

Note that, in the literature of Riemann-Liouville fractional derivative generally means the LRLFD. From physical point of view, if x is considered as a time scale, the RRLFD represent, an operation performed on the future state of the process f(x). This derivative has generally been neglected with the assumption that the present state of a process dose not depend on the results of its future development, [Agrawal,2002].

However, the derivation to follow will show that both derivatives naturally occur in a problem of fractional calculus of variations.

The first simplest fractional calculus of variation problem can be defined as follows; let F(x,y,u,v) be a function with continuous first and second (partial) derivatives with respect to all of its arguments. Then, among all functions y(x), which have continuous LRLFD of order α and RRLFD of order β for $a \le x \le b$ and satisfy the boundary conditions:

$$ y(a) = y_a,\ y(b) = y_b \qquad \ldots(2.3) $$

Find the function for which the functional:

$$ J[y] = \int_a^b F(x, y, {}_aD_x^{\alpha} y, {}_xD_b^{\beta} y)\, dx \qquad \ldots(2.4) $$

is an extremum, where $0 < \alpha, \beta \le 1$. The continuity requirement on F can be given more precisely. However, these assumptions are made for simplicity.

Note that:

(1) We have included the LRLFD and RRLFD generality.

(2) We first consider $0 < \alpha, \beta \le 1$. The case of $\alpha, \beta \in \mathbb{R}^+$ will be considered shortly in the next section.

(3) When $\alpha = \beta = 1$, the above problem reduces to the simplest variational problem.

To develop the necessary conditions for the extremum, assume that $y^*(x)$ is the desired function, let $\varepsilon \in \mathbb{R}^+$, and define a family of curve:

$$y(x) = y^*(x) + \varepsilon\eta(x) \qquad \dots(2.5)$$

which satisfy the boundary conditions, i.e., we require that:

$$\eta(a) = \eta(b) = 0 \qquad \dots(2.6)$$

since ${}_aD_x^{\alpha}$ and ${}_xD_b^{\beta}$ are linear operators, it follows that:

$${}_aD_x^{\alpha} y(x) = {}_aD_x^{\alpha} y^*(x) + \varepsilon\, {}_aD_x^{\alpha} \eta(x) \qquad \dots(2.7)$$

$${}_xD_b^{\beta} y(x) = {}_xD_b^{\beta} y^*(x) + \varepsilon\, {}_xD_b^{\beta} \eta(x) \qquad \dots(2.8)$$

Substituting Equations (2.5), (2.7) and (2.8) into Equation (2.4) we find that for each $\eta(x)$

$$J = J[\varepsilon] = \int_a^b F(x, y^* + \varepsilon\eta, {}_aD_x^{\alpha} y^* + \varepsilon\, {}_aD_x^{\alpha}\eta, {}_xD_b^{\beta} y^* + \varepsilon\, {}_xD_b^{\beta}\eta)\,dx \qquad \dots(2.9)$$

is a function of ε only. Note that $J[\varepsilon]$ is extremum at $\varepsilon = 0$.

Differentiating equation (2.9) with respect to ε, we obtain:

$$\frac{dJ}{d\varepsilon} = \int_a^b \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial\, {}_aD_x^{\alpha} y}\, {}_aD_x^{\alpha}\eta + \frac{\partial F}{\partial\, {}_xD_b^{\beta} y}\, {}_xD_b^{\beta}\eta \right] dx \qquad \text{…(2.10)}$$

Equation(2.10) is also called the variations of J[y] at y(x) along η(x).

A necessary condition for J[ε] to have an extremum is that $\frac{dJ}{d\varepsilon}$ must be zero and this should be true for all admissible η(x). This leads to the condition that for J[y] to have an extremum for y = y*(x) is that:

$$\int_a^b \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial\, {}_aD_x^{\alpha} y}\, {}_aD_x^{\alpha}\eta + \frac{\partial F}{\partial\, {}_xD_b^{\beta} y}\, {}_xD_b^{\beta}\eta \right] dx = 0 \qquad \text{…(2.11)}$$

for all admissible η(x). Using the formula for fractional integration by parts, the second integral in equation (2.11) can be written as[Riewe,1996] ,[Samko,1993]:

$$\int_a^b \frac{\partial F}{\partial\, {}_aD_x^{\alpha} y}\, {}_aD_x^{\alpha}\eta\, dx = \int_a^b {}_xD_b^{\alpha}\left(\frac{\partial F}{\partial\, {}_aD_x^{\alpha} y} \right)\eta\, dx \qquad \text{…(2.12)}$$

provided that $\frac{\partial F}{\partial\, {}_aD_x^{\alpha} y}$ or η is zero at x = a and x = b. Using equation (2.6) this condition is satisfied and it follows that equation (2.12) is valid.

Similarly, the third integral in equation (2.11) is:

$$\int_a^b \frac{\partial F}{\partial\, {}_xD_b^{\beta} y}\, {}_xD_b^{\beta}\eta\, dx = \int_a^b {}_aD_x^{\beta}\left(\frac{\partial F}{\partial\, {}_xD_b^{\beta} y} \right)\eta\, dx \qquad \text{…(2.13)}$$

substitute equations (2.12) and (2.13) into equation (2.11), we get:

$$\int_a^b \left[\frac{\partial F}{\partial y} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha} y} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta} y} \right]\eta\, dx = 0 \qquad \text{…(2.14)}$$

since $\eta(x)$ is arbitrary, it follows from a well established result in calculus of variations that:

$$\frac{\partial F}{\partial y} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta}y} = 0 \qquad \ldots(2.15)$$

Equation(2.15) is the Euler-Lagrange equation for the fractional calculus of variations problem. Thus, we have:

Theorem (1.1),[Agrawal,2002]:

Let J[y] be a functional of the form:

$$\int_a^b F(x, y, {}_aD_x^{\alpha}y, {}_xD_b^{\beta}y)\,dx$$

Defined on the set of functions y(x), which have continuous LRLFD of order α and RRLFD of order β in [a,b] and satisfy the boundary conditions $y(a) = y_a$ and $y(b) = y_b$. Then a necessary condition for J[y] to have an extremum for a given function y(x) is that y(x) satisfy following Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta}y} = 0$$

Note that for fractional calculus of variation problems, the resulting Euler-Lagrange equation contains both the LRLFD and the RRLFD. This is expected since the optimum function must satisfy both terminal conditions. Further, for $\alpha = \beta = 1$, we have ${}_aD_x^{\alpha} = \frac{d}{dx}$ and ${}_xD_b^{\beta} = \frac{-d}{dx}$ and equation (2.15) reduces to the standard Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad \ldots(2.16)$$

where $y' = \frac{dy}{dx}$.

2.4 The Case of $\alpha, \beta \in \mathbb{R}^+$ and Several Functions

We now consider further generalization of the above problem. Specifically, we consider two different cases first, in which $\alpha_j, \beta_j \in \mathbb{R}^+$ (j = 1, 2, …), i.e., one can have multiple positive α and β, and second, in which one more than one function. In both cases, we consider the end points fixed.

Case 1: Fixed end points and $\alpha_j, \beta_j \in \mathbb{R}^+$ (j = 1, 2, …):

Assume that α_j (j = 1, 2, …, n) and β_k (k = 1, 2, …, m) are two sets of real numbers all greater than zero.

$$\alpha_{max} = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, .., \beta_m\} \qquad \ldots(2.17)$$

is the maximum of all these numbers, and M is an integer such that $M - 1 \leq \alpha_{max} < M$. Assume that $F(x,y,z_1, \ldots, z_{m+n})$ is a function with continuous first and second partial derivatives with respect to all its arguments, and consider a functional of the form:

$$J[y] = \int_a^b F(x, y, {}_aD_x^{\alpha_1} y, \ldots, {}_aD_x^{\alpha_n} y, {}_xD_b^{\beta_1} y, \ldots, {}_xD_b^{\beta_m} y)\, dx \qquad \ldots(2.18)$$

The problem can now be defined as follows: Among all functions y(x) satisfying the conditions:

$$y(a) = y_{a0},\ y'(a) = y_{a1},\ \ldots,\ y^{(M-1)}(a) = y_{a(M-1)} \qquad \ldots(2.19a)$$

$$y(b) = y_{b0},\ y'(b) = y_{b1},\ \ldots,\ y^{(M-1)}(b) = y_{b(M-1)} \qquad \ldots(2.19b)$$

Find the function for which equation (2.18) has an extremum. Here it is implicitly assumed that y(x) meets all the differentiability requirements.

The necessary condition for this problem can be found following the approach presented above. This leads to:

Theorem (2.2),[Agrawal,2002]:

Let J[y] be a functional of the form given by equation (2.18) defined on the set of functions satisfying the boundary conditions given by equation (2.19). Then a necessary condition for J[y] to have an extremum for a given function y(x) is that y(x) satisfy the Euler-Lagrange equation:

$$\frac{\partial F}{\partial y}+\sum_{j=1}^{n} {}_{x}D_{b}^{\alpha_j}\frac{\partial F}{\partial\, {}_{a}D_{x}^{\alpha_j}y}+\sum_{k=1}^{m} {}_{a}D_{x}^{\beta_k}\frac{\partial F}{\partial\, {}_{x}D_{b}^{\beta_k}y}=0 \qquad \dots(2.20)$$

As a special case, consider that $\alpha_j = j$ (j = 1, 2, …, n) and that F does not contain the ${}_{x}D_{b}^{\beta_k}y$ (k = 1, 2, …, m) terms. In this case,using equation (2.2) we have :

$$\frac{\partial F}{\partial y}+\sum_{j=1}^{n}\left(\frac{-d}{dx}\right)^{j}\frac{\partial F}{\partial y^{(j)}}=0 \qquad \dots(2.21)$$

Thus, for integral order derivatives, the necessary conditions obtained using fractional calculus of variations approach reduces to that obtained using standard calculus of variations approach.

Case 2: Fixed end points and several functions:

The simplest fractional variational problem discussed in section (2.3) can be generalized in a straight forward manner to problems containing several unknown functions.

This problem can be defined as follows:

Let $F(x,y_1,y_2,\dots,y_n,z_1,z_2,\dots,z_{2n})$ be a function with continuous first ad second (partial) derivatives with respect to all its arguments. For $0 < \alpha, \beta \le 1$, consider the problem of finding necessary conditions for an extremum of a functional of the form:

$$J[y_1,\dots,y_n]=\int_a^b F(x,y_1,\dots,y_n,{}_{a}D_{x}^{\alpha}y_1,\dots,{}_{a}D_{x}^{\alpha}y_n,{}_{x}D_{b}^{\beta}y_1,\dots,{}_{x}D_{b}^{\beta}y_n)\,dx \qquad \dots(2.22)$$

which depends on n continuously differentiable functions $y_1(x)$, $y_2(x)$, …, $y_n(x)$ satisfying the boundary conditions:

$$y_j(a) = y_{ja},\ y_j(b) = y_{jb},\ j = 1, 2, \ldots, n \qquad \ldots(2.23)$$

Note that, no relationship exists among the functions $y_j(x)$ ($j = 1, 2, \ldots, n$). Therefore, the necessary condition for the functional in equation (2.22) to have an extremum can be found by considering the variations of each function one at a time. Thus, we have:

Theorem (2.3), [Agrawal,2002]:

A necessary condition for the curve

$$y_j = y_j(x)\ (j = 1, 2, \ldots, n) \qquad \ldots(2.24)$$

which satisfies the boundary conditions given by equation (2.23) to be an extreamal of the functional given by equation(2.22) is that the functions $y_j(x)$ satisfy the following Euler-Lagrange equation:

$$\frac{\partial F}{\partial y_j} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y_j} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta}y_j} = 0,\ j = 1, 2, \ldots, n \qquad \ldots(2.25)$$

In vector notation, the above condition can be written as:

$$\frac{\partial F}{\partial y} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta}y} = 0 \qquad \ldots(2.26)$$

where $y \in \mathbb{R}^n$.

The above problem considers several functions, but only one LRLFD of order $\alpha \le 1$ and one RRLFD of order $\beta \le 1$. The problem of finding extremum of a functional consisting of multiple functions and multiple LRLFD and RRLFD of order greater than zero can be developed using the discussion presented in cases 1 and 2 above.

2.5 The Problem of Lagrange and the Multiplier Rule

In this section we consider the following problem: Find the extremum of the functional:

$$J[y] = \int_a^b F(x, y, {}_aD_x^{\alpha} y, {}_xD_b^{\beta} y)\, dx \qquad \ldots(2.27)$$

such that:

$$\Phi(x,y) = 0 \qquad \ldots(2.28)$$

and

$$y_{s1(j)}(a) = y_{s1(j)a},\ y_{s2(j)}(b) = y_{s2(j)b}\ (\ j = 1, 2, \ldots, n - m) \qquad \ldots(2.29)$$

where $y \in \mathbb{R}^n$, $\Phi \in \mathbb{R}^m$, $m < n$, and s1 and s2 are two sets of n numbers obtained by reordering the numbers 1 to n. It is assumed that the constrained functions $\phi_j(x,y) = 0$ ($j = 1, 2, \ldots, m$) are all independent. This problem is essentially the same as that of Lagrange except that in this case the functional contains the LRLFD and the RRLFD. For this reason, we will call this problem as the problem of Lagrange containing fractional derivatives or simply a fractional Lagrange problem. This is a special case, and in a general fractional Lagrange problem, Φ may also contain the left and the right fractional derivatives.

To develop the necessary conditions for the problem, note that y at the two ends are completely known. This follows from the fact that the constraints $\phi_j(x,y) = 0$ ($j = 1, 2, \ldots, m$) are all independent and the values of $n - m$ functions $y_j(x)$ ($j = 1, 2, \ldots, n$) are specified at both ends. Therefore, the values of the rest of the functions at the two ends can be determined using a technique such as Newton-Raphson.

Suppose y*(x) is the solution to the above problem, and define:

$$y(x) = y^*(x) + \varepsilon\eta(x) \qquad \ldots(2.30)$$

where ε is a sufficiently small number, and $\eta(x) \in \mathbb{R}^n$ is a variation of y(x) consistent with the constraints, i.e., y(x) satisfies equation (2.28). from the above discussion, it follows that:

$$\eta(a) = \eta(b) = 0 \qquad \ldots(2.31)$$

Substituting equation (2.31) into equation (2.28), expanding the resulting vector into Taylor series, and neglecting second and higher order terms in ε, we get:

$$\frac{\partial \Phi}{\partial y}\eta(x) = 0 \qquad \ldots(2.32)$$

Equation (2.32) clearly indicates that not all functions $\eta_j(x)$ (j = 1, 2, …, n) can be independent. Substituting equation (2.30) into equation (2.27), we get a function that is only dependent on ε. Extremum of this function requires that its derivative with respect to ε must be zero. This leads to:

$$\int_a^b \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial\, {}_aD_x^{\alpha} y}\, {}_aD_x^{\alpha}\eta + \frac{\partial F}{\partial\, {}_xD_b^{\beta} y}\, {}_xD_b^{\beta}\eta \right] dx = 0 \qquad \ldots(2.33)$$

The left-hand side of equation (2.33) is the directional derivative of J at y(x) in the direction η(x). Using the formula for fractional integration by parts and equation (2.31), it follows that:

$$\int_a^b \left[\frac{\partial F}{\partial y} + {}_xD_b^{\alpha} \frac{\partial F}{\partial\, {}_aD_x^{\alpha} y} + {}_aD_x^{\beta} \frac{\partial F}{\partial\, {}_xD_b^{\beta} y} \right] \eta \, dx = 0 \qquad \ldots(2.34)$$

Here the elements of η(x) are not all independent, and therefore its coefficients cannot be set to zero. Equation (2.15) motivates the following:

Definition(2.1),[Agrawal,2002]:

An admissible arc y*(x) is said to satisfy the multiplier rule if there exists a vector of multipliers $l(x) \in \mathbb{R}^m$ continuous on [a,b], and a function:

$$\bar{F}(x,y,{}_aD_x^{\alpha}y,{}_xD_b^{\beta}y,l) = F(x,y,{}_aD_x^{\alpha}y,{}_xD_b^{\beta}y) + l^T(x)\Phi(x,y) \qquad \ldots(2.35)$$

such that:

$$\frac{\partial \bar{F}}{\partial y} + {}_xD_b^{\alpha}\frac{\partial \bar{F}}{\partial\, {}_aD_x^{\alpha}y} + {}_aD_x^{\beta}\frac{\partial \bar{F}}{\partial\, {}_xD_b^{\beta}y} = 0 \qquad \ldots(2.36)$$

is satisfied along y*(x).

Theorem (2.4),[Agrawal,2002]:

Every minimizing arc y*(x) must satisfy the multiplier rule.

Proof:

To prove this, multiply equation (2.32) with $l^T(x)$ and add the results to equation (2.34), to get:

$$\int_a^b \left[\frac{\partial F}{\partial y} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta}y} + l^T(x)\frac{\partial \Phi}{\partial y} \right] \eta \, dx = 0 \qquad \ldots(2.37)$$

It can now be shown that:

$$\frac{\partial F}{\partial y} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta}y} + l^T(x)\frac{\partial \Phi}{\partial y} = 0 \qquad \ldots(2.38)$$

This follows from the fact that l(x) may be selected such that m of the n equations in equation (2.38) are zero. This is true since $\partial\Phi/\partial y$ has a full rank. Rest of the η's can be selected as independent and therefore the other $n - m$ equations in (2.38) follows by using equation (2.37) and applying a theorem in calculus of variations. Note that equation (2.36) can now be obtained using equation (2.35) and (2.38). Equation (2.38) will be called the Euler-Lagrange equation for constrained fractional variational problems. ■

Follows we obtain the Euler-Lagrange equations for an unconstrained and a constrained fractional variational problems.

<u>Example (2.1)[Agrawal,2002]:</u>

As the first example, consider the following unconstrained fractional variational problem:

$$\text{Minimize } J[y] = \frac{1}{2}\int_0^1 ({}_0D_x^{\alpha} y)^2 \, dx \qquad \ldots(2.39)$$

such that:

$$y(0) = 0 \text{ and } y(1) = 1 \qquad \ldots(2.40)$$

This example with α = 1, for which the solution is y(x) = x, is often considered in textbooks on variational calculus. It can be shown that for this problem, the Euler-Lagrange equation is:

$$ {}_xD_1^{\alpha}({}_0D_x^{\alpha} y) = 0 \qquad \ldots(2.41)$$

It can be shown that for $\alpha > 1/2$, the solution is given as:

$$y(x) = (2\alpha - 1)\int_0^x \frac{dt}{[(1-t)(x-t)]^{1-\alpha}} \qquad \ldots(2.42)$$

Example (2.2) [Agrawal,2002]:

As the second example, consider the following constrained fractional variational problem:

$$\text{Minimize } J[y] = \frac{1}{2}\int_0^1 \left[y_1^2 + y_2^2 \right] dx \qquad \ldots(2.43)$$

such that:

$$ {}_0D_x^{\alpha} y_1 = y_1 + y_2 \qquad \ldots(2.44)$$

$$y(0) = 1 \qquad \ldots(2.45)$$

This example with integral order derivative is often considered in textbooks on optimal control. It can be shown that for this problem, the Euler-Lagrange equation is:

$$y_1 + l + {}_xD_l^{\alpha} l = 0 \qquad \ldots(2.46)$$

$$y_2 - l = 0 \qquad \ldots(2.47)$$

2.6 Direct Haar Wavelet Method for Solving Simplest Fractional Variational Problems:

In this section we shall consider the problem of exterimization of the functional J of the form:

$$J[y(t)] = \int_a^b F(x, y, {}_aD_x^{\alpha} y, {}_xD_b^{\beta} y)dx \qquad \ldots(2.48)$$

Satisfying the boundary conditions :

$$y\,(a) = y_a \quad \text{and} \quad y(b) = y_b \qquad \ldots(2.49)$$

where ${}_aD_x^{\alpha}y$ and ${}_xD_b^{\beta}y$ are considered to be the LRLFD and RRLFD of ordered α and β respectively.

The regular method for solving problem (2.48)-(2.49) as given in section (2.3) through the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} + {}_xD_b^{\alpha}\frac{\partial F}{\partial\, {}_aD_x^{\alpha}y} + {}_aD_x^{\beta}\frac{\partial F}{\partial\, {}_xD_b^{\beta}y} = 0$$

In this section we shall use Haar wavelet functions to establish the direct method for fractional variational problems.

Unlike other direct methods, beginning with the assumption of the variable itself, the method we have started here is state by assuming ${}_aD_x^{\alpha}y$ as Haar wavelet whose coefficients are to be determined.

$${}_aD_x^{\alpha}y = \sum_{i=0}^{\infty} c_i h_i(x)$$

Taking finite terms as approximation, we have

$${}_aD_x^{\alpha}y \simeq \sum_{i=0}^{m-1} c_i h_i(x) = c_m^T H_m(x) \qquad \ldots(2.50)$$

Applying the Riemann-Liouville fractional integration of order α to the both sides of equation (2.50) yields [killbas, 2006]:

$$y(x) - \sum_{j=1}^{n} \frac{y^{(n-j)}(0)}{\Gamma(\alpha - j + 1)}(y-a)^{\alpha - j} \simeq C_m^T P_{m\times m}^{\alpha} H_m(x)$$

Thus y(x) can be expressed as:

$$y(x) \simeq C_m^T P_{m\times m}^{\alpha} H_m(x) + \sum_{j=1}^{n} \frac{y^{(n-j)}}{\Gamma(\alpha - j + 1)}(y-a)^{\alpha - j} \qquad \ldots(2.51)$$

The other terms in the functional (2.48) are expanded also in terms of Haar wavelets and therefore through substitution we have

$$J = J(c_0, c_1, ..., c_{m-1}) \qquad \ldots(2.52)$$

The original extremization of fractional problem shown in (2.48) becomes the extremization of functional of finite set of variable in equation (2.52)

Taking partial derivatives of J with respect to c_i and setting them equal to zero, we obtain

$$\frac{\partial J}{\partial c_i} = 0, i = 0, 1, ..., m-1$$

Solving for c_i and hence we have the desired solution. After substituting these values into equation(2.51).

2.7 Numerical Examples

In order to illustrate the efficiency and applicability of the numerical procedure which was given in the above section following two numerical examples are considered in this section.

Example (2.3):

Consider the functional:

$$J[y] = \frac{1}{2}\int_0^1 ({}_0D_x^{\alpha} y)^2 dx \qquad \ldots(2.53)$$

$$y(0) = 0 \text{ , } y(1) = 1 \qquad \ldots(2.54)$$

Let:

$${}_0D_x^{\alpha} y(x) = C_m^T H_m(x) \qquad \ldots(2.55)$$

Taking ${}_0I_x^{\alpha}$ to the both sides of equation (2.55), we get:

$$y(x) = C_m^T P_{m\times m}^{\alpha} H_m(x) \qquad \ldots(2.56)$$

Also the other boundary condition that we have is :

$$y(1) = 1$$

which implies that

$$C_m^T P_{m\times m}^{\alpha} H_m(1) = 1 \qquad \ldots(2.57)$$

Substituting equation (2.55) into (2.53), we have:

$$J = \frac{1}{2}\int_0^1 C_m^T H_m(x) H_m^T C_m dx$$

$$J = \frac{1}{2} C_m^T \int_0^1 H_m(x) H_m^T(x) dx C_m$$

Hence

$$J = \frac{1}{2} C_m^T K_{m\times m} C_m \qquad \ldots(2.58)$$

Case 1: if $\alpha = 1$, in this case the exact solution was given [Agrawal,2002] as $y(x)=x$ then equation (2.55) becomes:

$$y' = C_m^T H_m(x) \qquad \ldots(2.59)$$

and hence integrating equation (2.59) from 0 to x, thus we get:

$$y(x) = \int_0^x C_m^T H_m(x) dx + 0 = C_m^T P_{m\times m} H_m(x) \qquad \ldots(2.60)$$

For the final boundary condition $y(1) = 1$ equation (2.60), yields:

$$y(1) = C_m^T \int_0^1 H_m(x) dx = 1 \qquad \ldots(2.61)$$

Note that the definite integral of $h_0(x)$ from zero to one is equal to one, while the definite integrals of h_1, h_2, …, h_7 are all equal to zero for $m = 8$, or:

$$\int_0^1 h_0(\tau)d\tau = 1, \int_0^1 h_i(\tau)d\tau = 0 \qquad ,i=1,2,\ldots,7 \qquad \ldots(2.62)$$

Substituting (2.62) into (2.61), we have

$$C_8^T\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T = 1 = c_0$$

This information should be substituted into equation (2.58), we then have:

$$J \cong \frac{c_1^2}{2} + \frac{c_2^2}{4} + \frac{c_3^2}{4} + \frac{c_4^2}{8} + \frac{c_5^2}{8} + \frac{c_6^2}{8} + \frac{c_7^2}{8} + \frac{1}{2}$$

where:

$$K_{8\times 8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

For extermization, we take the partial derivatives of J with respect to c_i, $i = 1,2,\ldots,7$, and set it equal to zero

$$\frac{\partial J}{\partial c_1} = 0, \frac{\partial J}{\partial c_2} = 0, \cdots, \frac{\partial J}{\partial c_7} = 0$$

Therefore, we get $c_1=c_2=\ldots=c_7=0$ and hence we have

$$y' \cong \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} H_8(x)$$

and y(x) is obtained from equation (2.60) as

$$y(x) = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} P_{8\times 8} H_8(x)$$

where

$$P_{8\times8} = \frac{1}{16}.\begin{bmatrix} 8 & -4 & -2 & -2 & -1 & -1 & -1 & -1 \\ 4 & 0 & -2 & 2 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{-1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{-1}{4} & 0 & \frac{-1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$y(x) = 0.5h_0(x) - 0.25h_1(x) - 0.125h_2(x) - 0.125h_3(x) - \\ 0.063h_4(x) - 0.063h_5(x) - 0.063h_6(x) - 0.063h_7(x)$$

Case 2: if $\alpha = 0.6$ then

$$y^{(0.6)} = C_m^T H_m(x) \qquad \ldots(2.63)$$

And hence taking ${}_0I_x^{0.6}$ to the both sides of (2.63) thus we get

$$y(x) = C_m^T P_{m\times m}^{(0.6)} H_m(x) \qquad \ldots(2.64)$$

Also, for the final boundary condition y(1) = 1, and for m=8 equation (2.64) yields:

$$c_7 = \frac{1}{0.007}[0.022c_6 + 0.031c_2 + 0.004128c_4 + 0.134c_3 - \\ 1.075c_0 + 0.008891c_5 + 0.287c_1 + 1] \qquad \ldots(2.65)$$

substituted equation (2.65) into (2.58) thus and after making extermization to J by taking partial derivative with respect to $c_i = i$, $i = 0,1,\ldots,6$ and set them equal to zero

$$\frac{\partial J}{\partial c_i} = 0\,,\ \ i = 0,1,\ldots,6$$

Therefore:

$$y^{(0.6)}(x) \square [0.841 - 0.225 - 0.049 - 0.21 - 0.031 - 0.028 - 0.069 - 0.091]H_8(x)$$

and y(x) is obtained from equation (2.60) as:

$$y(x) = [0.841 - 0.225 - 0.049 - 0.21 - 0.031 - 0.028 - 0.069 - 0.091]P^{(0.6)}_{8\times 8}H_m(x)$$

where

$$P^{(0.6)}_{8\times 8} = \begin{bmatrix} 0.699 & -0.238 & -0.157 & -0.094 & -0.104 & -0.062 & -0.051 & -0.044 \\ 0.238 & 0.224 & -0.157 & 0.22 & -0.104 & -0.062 & 0.156 & 0.081 \\ 0.047 & 0.11 & 0.147 & -0.027 & -0.104 & 0.145 & -0.029 & -5.323\times 10^{-3} \\ 0.079 & -0.076 & 0 & 0.147 & 0 & 0 & -0.104 & 0.145 \\ 0.011 & 0.02 & 0.072 & -2.662\times 10^{-3} & 0.097 & -0.018 & -2.109\times 10^{-3} & -2.21\times 10^{-3} \\ 0.013 & 0.039 & -0.052 & -0.015 & 0 & 0.097 & -0.018 & 2.109\times 10^{-3} \\ 0.016 & -0.016 & 0 & 0.072 & 0 & 0 & 0.097 & -0.018 \\ 0.026 & -0.026 & 0 & -0.052 & 0 & 0 & 0 & 0.097 \end{bmatrix}$$

Hence:

$$y(x) = 0.513h_0(x) - 0.239h_1(x) - 0.103h_2(x) - 0.162h_3(x) - 0.06h_4(x) - 0.048h_5(x) - 0.061h_6(x) - 0.086h_7(x)$$

Following table (2.1) gives the approximate solution of example (2.3) for different values of α and compares the result for α = 1 with exact solution, which was given in [Agrawal,2002].

Table (2.1) **The approximate solution of example 2.3 for different values of α with comparison with the exact solution when α=1**

x \ α	***1***	***0.5***	***0.6***	***0.8***	***Exact for $\alpha=1$***
0	0.062	−0.108	0.111	−0.148	0.000
0.125	0.188	−0.208	0.231	−0.182	0.125
0.250	0.375	−0.335	0.389	−0.381	0.250
0.375	0.438	−0.363	0.425	−0.481	0.375
0.5	0.75	−0.573	0.68	−0.802	0.5
0.625	0.688	−0.563	0.651	−0.741	0.625
0.750	0.875	−0.762	0.889	1.12	0.750
0.875	0.938	−1.06	1	0.011	0.875

More accurate results can be obtained for larger values of m.

Example (2.4):

Find the extremal of the following functional:

$$J=\int_0^1[({}_0D_x^{\alpha}y)^2+x({}_0D_x^{\alpha}y)]dx \qquad \ldots(2.66)$$

$$y(0)=0,\ y(1)=\frac{1}{4} \qquad \ldots(2.67)$$

For solving this problem by the Haar direct method, we assume that ${}_0D_x^{\alpha}y(x)$ can be expanded in terms of Haar wavelet as:

$${}_0D_x^{\alpha}y(x)\simeq\sum_{i=0}^{m-1}c_ih_i(x)$$

or:

$${}_0D_x^{\alpha}y(x)\simeq C_m^TH_m(x) \qquad \ldots(2.68)$$

Here also we let m = 8.

There is a variable x involved in equation (2.66) explicitly it can be expanded into Haar series in the time interval [0,1).

$$X \approx d_m^T H_m(x) \qquad \ldots(2.69)$$

Substituting (2.68) and (2.69) into (2.66), we have:

$$J \approx \int_0^1 \left[C_m^T H_m(x) H_m^T(x) C_m + C_m^T H_m(x) H_m^T(x) d_m \right] dx$$

Hence

$$J \approx C_m^T K_{m\times m} C_m + C_m^T K_{m\times m} d_m \qquad \ldots(2.70)$$

Case 1: if α = 1,in this case the exact solution was given in [Hsiao,2006] as $y(x) = \frac{x}{2}\left(1 - \frac{x}{2}\right)$

then equation (2.68) becomes:

$$y' = C_m^T H_m(x)$$

and hence integrating the above equation from 0 to x, thus we get:

$$y(x) = \int_0^x C_m^T H_m(x) dx + 0 = C_m^T P_{m\times m} H_m(x) \qquad \ldots(2.71)$$

For the final boundary condition $y(1) = \frac{1}{4}$ equation (2.71) yields

$$y(1) = C_m^T \int_0^1 H_m(x) dx = \frac{1}{4}$$

Which implies that $c_0 = \frac{1}{4}$ and this information should be substituted into equation (2.70), we then have:

$$J \approx c_1^2 - \frac{c_2}{16} - \frac{c_3}{16} - \frac{c_4}{64} - \frac{c_5}{64} - \frac{c_6}{64} - \frac{c_7}{64} - \frac{c_1}{4} + \frac{c_2^2}{2} + \frac{c_3^2}{2} + \frac{c_4^2}{4} + \frac{c_5^2}{4} + \frac{c_6^2}{4} + \frac{c_7^2}{4} + 0.1875$$

For extermization, we take the partial derivatives of J with respect to c_i, i = 1,2,…,7, and set them equal to zero

$$\frac{\partial J}{\partial c_1} = 0, \frac{\partial J}{\partial c_2} = 0, \cdots, \frac{\partial J}{\partial c_7} = 0$$

Therefore , we get $c_1=c_2=\ldots=c_7=0$ and hence

$$y' \approx \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} H_8(x)$$

and y(x) is obtained from equation (2.71) as:

$$y(x) = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} P_{8\times 8} H_m(x)$$

$$y(x) = 0.166h_0(x) - 0.063h_1(x) - 0.047h_2(x) - 0.016h_3(x) - 0.027h_4(x) - 0.02h_5(x) - 0.012h_6(x) - 3.875\times 10^{-3} h_7(x)$$

Case 2: if $\alpha = 0.6$

$$y^{(0.6)} = C_m^T H_m(x) \qquad \ldots(2.72)$$

and hence upon taking ${}_0I_x^{0.6}$ to the both sides of (2.72), thus we get:

$$y(x) = C_m^T P_{m\times m}^{(0.6)} H_m(x) \qquad \ldots(2.73)$$

Also, for the final boundary condition y(1) = $\frac{1}{4}$ equation (2.74) yields:

$$c_7 = \frac{1}{0.007}[0.022c_6 + 0.031c_2 + 0.004128c_4 + 0.134c_3 - 1.075c_0 + 0.008891c_5 + 0.287c_1]$$

After making extermization to J by taking partial derivative with respect to, i = 0,1,…,7 and set them equal to zero.

Therefore

$$y^{(0.6)} \approx \begin{bmatrix} 0.214 & -0.029 & -0.049 & -0.12 & -0.043 & -0.012 & 7.546\times10^{-4} & 0.01 \end{bmatrix} H_8(x)$$

and y(x) is obtained from (2.73) as :

$$y^{(0.6)} \approx \begin{bmatrix} 0.214 & -0.029 & -0.049 & -0.12 & -0.043 & -0.012 & 7.546\times10^{-4} & 0.01 \end{bmatrix} P_{8\times8}^{0.6} H_8(x)$$

Hence

$$\begin{aligned} y(x) &\approx 0.214h_0(x) - 0.029h_1(x) - 0.049h_2(x) - 0.12h_3(x) \\ &-0.043h_4(x) - 0.012h_5(x) + 7.546\times10^{-4} h_6(x) - 0.01h_7(x) \end{aligned}$$

following table (2.2) represent the approximate solution of example (2.4) for different values of α with comparison with the exact solution at α=1.

Table (2.2) **The approximate solution of example 2.4 for different values of α with comparison with the exact solution when α=1**

x \ *α*	***1***	***0.5***	***0.6***	***0.8***	***Exact for $\alpha = 1$***
0	0.029	0.155	0.093	0.053	0.000
0.125	0.083	0.241	0.179	0.125	0.059
0.250	0.157	0.322	0.265	0.21	0.109
0.375	0.17	0.293	0.246	0.208	0.152
0.5	0.223	0.345	0.317	0.3	0.188
0.625	0.18	0.283	0.254	0.245	0.215
0.750	0.298	0.257	0.24	0.255	0.234
0.875	0.294	0.234	0.221	0.243	0.246

Chapter Three

Haar Wavelet Method for Solving Fractional Variational Problems with Trasversality Conditions

3 Haar Wavelet Method for Solving Fractional Variational Problems with Trasversality Conditions

3.1 Introduction

This chapter consist of four sections. In section 3.2. the generalized Euler-Lagrange equations and the transversility conditions was illustrated, in section 3.3 The direct Haar wavelet method for solving fractional variational problems with transversility conditions in presented, finally two illustrative examples are given in section 3.4.

3.2 The Generalized Euler-Lagrange Equations and the Transversality Conditions

In this section, we present the generalized Euler-Lagrange equation and the transversality conditions for fractional variational problems defined in terms of the Riemann-Liouville and the Caputo derivatives.

We now consider the following fractional variational problem containing the left Riemann-Liouville fractional derivative only. Among all possible functions y(x), find the function y*(x) which minimize the functional:

$$J[y] = \int_0^1 F(x, y, {}_0D_1^{\alpha} y)\, dx \qquad \ldots(3.1)$$

and satisfies the condition

$$y(0) = y_0 \qquad \ldots(3.2)$$

This problem is the same as that considered in [Agrawal, 2002] with two exceptions. First, it does not include the right Riemann-Liouville fractional derivative. This choice made for simplicity. Second, in this problem, the boundary condition is specified only at x = 0, so that we can develop the natural boundary condition. For simplicity, we also assume that $0 < \alpha < 1$ and that all differentiability conditions are met. We further assume that the end points are specified.

Here, the function value is given at one end (say x = 0) but free at the other end (say x = 1). Using the approach presented in [Agrawal,2002], it can be demonstrated that for J[y] to have an extremum, the following conditions must be satisfied:

$$\int_0^1 \left[\frac{\partial F}{\partial y} + {}_x^c D_1^\alpha \frac{\partial F}{\partial\, {}_0 D_x^\alpha y} \right] \delta y \, dx + \left(\frac{\partial F}{\partial\, {}_0 D_x^\alpha y} \right) \delta\, {}_0 D_x^{\alpha-1} y(x) \Bigg|_0^1 = 0 \qquad \ldots(3.3)$$

where $\delta(.)$ is the variation operator and ${}_0 D_x^{\alpha-1} y(x)$ must be interpreted as the fractional integral of order $1 - \alpha$. Since the value the functional of the first term taken only along extermals, consequently , also δy is arbitrary, it follows from a well- established result in calculus of variations that:

$$\frac{\partial F}{\partial y} + {}_x^c D_1^\alpha \frac{\partial F}{\partial\, {}_0 D_x^\alpha y} = 0 \qquad \ldots(3.4)$$

and

$$\left(\frac{\partial F}{\partial\, {}_0 D_x^\alpha y} \right) \delta\, {}_0 D_x^{\alpha-1} y(x) = 0, \; x = 0, 1 \qquad \ldots(3.5)$$

Equations (3.4) and (3.5) are the generalized Euler-Lagrange equation and the transversality conditions for the fractional variational problem defined in terms of the left Riemann-Liouville fractional derivative Equation (3.5) suggests that either:

$$\frac{\partial F}{\partial\, {}_0D_x^{\alpha} y} = 0,\ x = 0,\ 1 \qquad \ldots(3.6)$$

or

$$\delta\, {}_0D_x^{\alpha-1} y(x) = 0,\ x = 0,\ 1 \qquad \ldots(3.7)$$

i.e., ${}_0D_x^{\alpha-1} y(x)$ at the end points should be specified. These boundary conditions are fractional and they are similar to those required when the Laplace transform technique is used. Since y at x = 1 is not specified. it follows that:

$$\left.\left(\frac{\partial F}{\partial\, {}_0D_x^{\alpha} y}\right)\right|_{x=1} = 0 \qquad \ldots(3.8)$$

Equation (3.8) is called the natural boundary conditions, and to obtain the optimum solution, this condition must be satisfied.

Note that equation (3.4) is somewhat different from that presented in [Agrawal, 2002]. It contains a Caputo fractional derivative even when the functional in equation. (3.1) contains no such term. This is because some of the boundary conditions are not specified. Equation (3.1) can be written purely in terms of the Riemann-Liouville fractional derivative. However, in that case, the resulting equations will contain some extra terms.

Now, we consider the following fractional variational problem containing the left Caputo fractional derivative. Among all possible curve y(x), find the curve y*(x), which minimizes the functional:

$$J[y] = \int_0^1 F(x, y, {}_0^cD_1^{\alpha} y)\, dx \qquad \ldots(3.9)$$

and satisfies the initial condition given by (3.2).

Once again, we assume that $0 < \alpha < 1$ and that all differentiability conditions are met. We also assume that the end points are fixed. The approach presented in [Agrawal, 2002], can be used with some minor changes for Caputo derivative to obtain the optimality conditions for this case also. This leads to:

$$\int_0^1 \left[\frac{\partial F}{\partial y} + {}_x D_1^{\alpha} \frac{\partial F}{\partial\, {}_0^c D_x^{\alpha} y} \right] \delta y dx + \left({}_x D_1^{\alpha-1} \frac{\partial F}{\partial\, {}_0^c D_x^{\alpha} y} \right) \delta y(x) \Bigg|_0^1 = 0 \qquad \ldots(3.10)$$

since δy is arbitrary, it follows from a well-established result in calculus of variations that

$$\frac{\partial F}{\partial y} + {}_x D_1^{\alpha} \frac{\partial F}{\partial\, {}_0^c D_x^{\alpha} y} = 0 \qquad \ldots(3.11)$$

and

$$\left({}_x D_1^{\alpha-1} \frac{\partial F}{\partial\, {}_0^c D_x^{\alpha} y} \right) \delta y(x) \Bigg|_0^1 = 0, \ x = 0, 1 \qquad \ldots(3.12)$$

Equations (3.11) and (3.12) are the generalized Euler-Lagrange equation and the transversality conditions for the fractional variational problem defined in terms of the left Caputo fractional derivative.

Note that (3.11) contains a right Riemann-Liouville fractional derivative even when the functional dose not contain any Riemann-Liouville fractional derivative term.

Equation (3.12) suggests that either:

$${}_0 D_x^{\alpha-1} \frac{\partial F}{\partial\, {}_0^c D_x^{\alpha} y} = 0, \ x = 0, 1 \qquad \ldots(3.13)$$

or

$$\delta y(x)\Big|_0^1 = 0\,,\ x = 0,\,1 \qquad \ldots(3.14)$$

i.e., y(x) at the end points should be specified. The boundary conditions resulting from (3.14) are the kinematics boundary conditions. They have no fractional derivative terms, and thus they are consistent with those required by the Laplace transform technique. Since y at x = 1 is not specified, it follows that:

$$\left.\left({}_xD_1^{\alpha-1}\frac{\partial F}{\partial\, {}_0^cD_x^{\alpha} y}\right)\right|_{x=1} = 0 \qquad \ldots(3.15\ )$$

Equation (3.15) is called the natural boundary conditions and the optimum solution must satisfy this condition. Note that this condition, in general, contains fractional derivative terms. Thus fractional variational problems defined in terms of Caputo fractional derivatives may require imposition of fractional boundary conditions.

Follows we shall consider two examples in order to show some applications of the transversality conditions.

Example (3.1):

Consider the following functional:

$$J[y] = \frac{1}{2}\int_0^1 \left[ay^2 + ({}_0D_x^{\alpha} y + y)^2 \right] dx \qquad \ldots(3.16)$$

And the following boundary condition

$$y(0) = 1 \qquad \ldots(3.17)$$

We assume $0 < \alpha < 1$, we will consider two cases:

Case 1: Let a be 0, in this case, the minimum value of J[y] will be 0, if a function could be found that satisfies (3.17) and the differential equation ${}_0D_x^{\alpha}y + y = 0$.

For this poblem the Euler-Lagrange equation and the trasversality condition are:

$$ {}_x^cD_1^{\alpha}({}_0D_x^{\alpha}y + y) + ({}_0D_x^{\alpha}y + y) = 0 \qquad \dots(3.18)$$

and

$$({}_0D_x^{\alpha}y + y) = 0 \quad \text{at} \quad x = 1 \qquad \dots(3.19)$$

Applying the operator ${}_xI_1^{\alpha}$ on both sides of (3.18) and using (3.19) , it can be demonstrated that ${}_0D_x^{\alpha}y + y = 0$ for 0 < x < 1, as expected. Note that the trasversality condition contains a fractional derivative term. Thus, a fractional boundary condition has been used to solve the problem.

Case 2: This time, let a be 1. For this case, the Euler-Lagrange equation is:

$$ {}_x^cD_1^{\alpha}({}_0D_x^{\alpha}y + y) + y + ({}_0D_x^{\alpha}y + y) = 0 \qquad \dots(3.20)$$

and the transversality condition is given by (3.19). Solving (3.20) is not straightforward, and perhaps its closed form solution does not exist. This problem is equivalent to the following fractional optimal control problem [Agrawal, 2004].

Find the optimal control u that minimizes the performance index:

$$J[y] = \frac{1}{2}\int_0^1 \left[y^2 + u^2 \right] dx \qquad \dots(3.21)$$

and satisfies the dynamic constraint:

$$ {}_0D_x^{\alpha}y = -y + u \qquad \dots(3.22)$$

and the initial condition given by (3.17). This problem is solved in [Agrawal, 2004] using a numerical technique. It is demonstrated that u(l) = 0. Using (3.19) and (3.22), it follows that this condition is consistent with the transversality condition.

Example (3.2):

As second example, consider the functional:

$$J[y]=\int_0^1\left[\frac{1}{2}({}_0D_x^{\alpha}y)^2-y\right]dx \qquad \ldots(3.23)$$

and the boundary condition:

$$y(0)=y_0 \qquad \ldots(3.24)$$

Consider that $0<\alpha<1$. In this case, the Euler-Lagrange equation and the natural boundary condition are:

$${}_x^cD_1^{\alpha}({}_0D_x^{\alpha}y)=1 \qquad \ldots(3.25)$$

and

$$({}_0D_x^{\alpha}y)\Big|_{x=1}=0 \qquad \ldots(3.26)$$

respectively. This problem with $\alpha = 1$ and $y_0 = 0$ represents the problem of a uniformly loaded bar fixed at one end and free at the other, and in which case the transversality condition suggests that the strain at the free end should be zero. For linear materials, the stress and the strain are linearly related. Therefore, for $\alpha = 1$, (3.26) also suggests that stress or load at the free end should be zero. If y is the displacement, then dy/dx is known as strain. We may call it first-order strain. Following this, ${}_0D_x^{\alpha}y$ can be called α-order strain. For $\alpha = 1$, it will represent ordinary strain, and for $\alpha = 0$, the displacement.

Note that if a fractional variational problem is defined in terms of Caputo derivatives, then the natural boundary conditions may include Riemann-Liouville derivatives also. Solving such problems analytically may be difficult, so a numerical technique may be necessary.

3.3 The Direct Approach for Solving Fractional Variational Problems with Transversality conditions Using Haar Wavelet Method.

In this section, we shall consider the problem of exterimization of a functional J of the form:

$$J[y(x)] = \int_0^1 F[x, y(x), {}_0D_x^{\alpha} y(x)]dx \qquad \ldots(3.27)$$

satisfying the condition y(0) = y_0,and y(1) is considered to be undetermined where ${}_0D_x^{\alpha}y(x)$ is the Riemann-Liouville fractional derivative. The regular method for solving problem (3.27) is through the Euler-Lagrange equation [Agrawal, 2006]:

$$\frac{\partial F}{\partial y} + {}_x^cD_1^{\alpha} \frac{\partial F}{\partial\, {}_0D_x^{\alpha} y} = 0$$

and

$$\left.\left(\frac{\partial F}{\partial\, {}_0D_x{}^{\alpha} y}\right)\right|_{x=1} = 0$$

where ${}_x^cD_1^{\alpha}$ is the Caputo fractional derivative.

This section mainly uses Haar wavelets to establish the direct method for fractional variational problems.

We start by assuming ${}_0D_x^{\alpha}y(x)$ as Haar wavelets whose coefficients are to be determined:

$$ {}_0D_x^{\alpha}y(x) = \sum_{i=0}^{\infty} c_i h_i(x) \qquad \ldots(3.28)$$

Taking finite terms as an approximation, we have:

$$ {}_0D_x^{\alpha}y(x) \approx \sum_{i=0}^{m-1} c_i h_i(x) = C_m^T H_m(x) \qquad \ldots(3.29)$$

Applying ${}_0I_x^{\alpha}$ to the both sides of equation (3.29), thus y(x) can be expressed as:

$$ y(x) \simeq C_m^T P_{m\times m}^{\alpha} H_m(x) + \sum_{j=1}^{n} \frac{y^{(n-j)}(0)}{\Gamma(\alpha-j+1)}(x-0)^{\alpha-j} \qquad \ldots(3.30)$$

The other terms in the functional of equation (3.27) are known functions of the independent variable x and can be expanded into Haar wavelets through substitution, and finally we have:

$$ J = J(c_0, c_1, \ldots, c_{m-1}) \qquad \ldots(3.31)$$

The original extremiation of a fractional problem shown in equation (3.27) becomes the extremiation of functional of a finite set of variables in equation (3.31).

Taking partial derivatives of J with respect to c_i, and setting them equal to zero, we obtain:

$$ \frac{\partial J}{\partial c_i} = 0,\ i = 0, 1, \ldots, m-1 \qquad \ldots(3.32)$$

solving for c_i, and substituting into equation (3.30), we have the desired result.

3.4 Numerical Examples

In this section, we shall introduce some examples in order to confirm the applicability and reliability of the proposed method which was given in the previous section.

Example (3.3):

Consider the functional:

$$J[y(x)] = \int_0^1 \left[\frac{1}{2} ({}_0D_x^{\alpha} y(x))^2 - y(x) \right] dx \qquad \ldots(3.33)$$

and the boundary condition:

$$y(0) = y_0 \text{ and } y(1) \text{ is unspecified} \qquad \ldots(3.34)$$

Consider that $0 < \alpha < 1$, and for solving this problem by the direct Haar wavelet method, we assume that ${}_0D_x^{\alpha} y(x)$ can be expanded in terms of Haar wavelet, as follows:

$${}_0D_x^{\alpha} y(x) \approx \sum_{i=0}^{m-1} c_i h_i(x) = C_m^T H_m(x) \qquad \ldots(3.35)$$

where:

$$C_m = [c_0 \ \ c_1 \ \ \ldots \ c_{m-1}]^T$$

$$H_m(x) = [h_0(x) \ \ h_1(x) \ \ldots \ h_{m-1}(x)]^T$$

Here, we shall consider y_0=0 and m = 8 and more accurate results may be obtained using larger m.

Now, upon taking the fractional Riemann-Liouville integration to the both sides of equation (3.35), thus we get:

$$y(x) = C_8^T \, P_{8\times 8}^{\alpha} H_8(x) \qquad \ldots(3.36)$$

The other condition according to [Agrawal, 2006], that we have is:

$$\left.\left(\frac{\partial F}{\partial_0 D_x{}^{\alpha} y(x)}\right)\right|_{x=1} = 0$$

and according to our example, we get:

$$\left.{}_0D_x^{\alpha} y(x)\right|_{x=1} = 0$$

which implies that $C_8^T H_8(1) = 0$.

Therefore:

$$c_0 - c_1 - c_3 - c_7 = 0$$

and this gives

$$c_7 = c_0 - c_1 - c_3 \qquad \ldots(3.37)$$

substituting equations (3.35), (3.36) and (3.37) into equation (3.33) yields:

$$J[y(x)] \approx \int_0^1 \left[\frac{1}{2}\left(C_8^T H_8(x) H_8^T(x) C_8\right) - C_8^T P_{8\times 8}^{\alpha} H_8(x) \right] dx$$

$$= \int_0^1 \left[\frac{1}{2}\left(C_8^T H_8(x) H_8^T(x) C_8\right) - C_8^T P_{8\times 8}^{\alpha} H_8(x) \right] dx$$

Therefore:

$$J[y(x)] = \frac{1}{2} C_8^T \int_0^1 H_8(x) H_8^T(x)\, dx\, C_8 - C_8^T P_{8\times 8}^{\alpha} \int_0^1 H_8(x)\, dx \qquad \ldots(3.38)$$

And as we mention in chapter two that the definite integral of $h_0(t)$ from 0 to 1 is equal to 1, while the definite integral of $h_1, h_2, \ldots, h_7$ are equal to zero for m = 8, or:

$$\int_0^1 h_0(x)\, dx = 1, \int_0^1 h_i(x)\, dx = 0, i = 1, 2, \ldots, 7 \qquad \ldots(3.39)$$

Hence, upon using equations (3.37) and (3.39) and substituting into equation (3.38), we get:

$$J \approx \frac{1}{2} C_8^T K_{8\times 8} C_8 - C_8^T P_{8\times 8}^{\alpha} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots(3.40)$$

Following table (3.1) give the approximate solution of example (3.3) for different values of α and compares the results for $\alpha = 1$ with the exact solution which was given in[Agrawal,2002] as:

$$y(x) = x\left(1 - \frac{x}{2}\right)$$

Table (3.1) Comparison of the numerical solution of example (3.3) for different values of α with comparison with the exact solution when $\alpha = 1$

x \ α	***1***	***0.5***	***0.6***	***0.8***	***Exact for $\alpha = 1$***
0	0.059	0.291	0.216	0.115	0.000
0.125	0.169	0.511	0.420	0.273	0.117
0.250	0.318	0.740	0.645	0.466	0.219
0.375	0.341	0.704	0.625	0.473	0.305
0.5	0.536	0.917	0.851	0.695	0.375
0.625	0.449	0.761	0.701	0.574	0.430
0.750	0.503	0.754	0.708	0.593	0.462
0.875	0.492	0.617	0.618	0.591	0.492

Example (3.4):

Consider the functional:

$$J[y(x)] = \int_0^1 \left[{}_0D_x^{\alpha} y(x) \right]^2 + \left[x \left({}_0D_x^{\alpha} y(x) \right) \right] dx \qquad \ldots(3.41)$$

and the boundary conditions:

$$y(0) = 0 \text{ and } y(1) \text{ is unspecified} \qquad \ldots(3.42)$$

and for solving this example also we let:

$${}_0D_x^{\alpha} y(x) = C_m^T H_m(x) \qquad \ldots(3.43)$$

Hence

$$y(x) = C_m^T P_{m\times m} H_m(x) \qquad \ldots(3.44)$$

There is a variable x involved in equation (3.41) explicitly and it can be expanded into Haar series over the interval [0,1]

$$x \simeq d_m^T H_m(x) \qquad \ldots(3.45)$$

Also, the other condition that we have is:

$$2\, {}_0D_x^{\alpha} y(x) + x \Big|_{x=1} = 0$$

which implies that:

$$C_m^T H_m(1) = -\frac{1}{2} \qquad \ldots(3.46)$$

Here we shall take also m=8 and according to equation (3.46) thus we have:

$$c_7 = c_0 - c_1 - c_3 + \frac{1}{2} \qquad \ldots(3.47)$$

and substituting equations (3.43), (3.45) and (3.47) into equation (3.41), we have:

$$J[y(x)] \cong \int_0^1 \left[C_8^T H_8(x) H_8^T(x) C_8 + C_8^T H_8(x) H_8^T(x) d_8 \right] dx$$

$$J[y(x)] \cong C_8^T K_{8\times8} C_8 + C_8^T K_{8\times8} d_8 \qquad \ldots(3.48)$$

Following table (3.2) gives the approximate solution of example (3.4) for different values of α and compares the results for $\alpha = 1$ with the exact solution, which was given in [Hsiao, 2004], as:

$$y(t) = -\frac{x^2}{4}$$

Table (3.2) Comparison of the approximate solution of example (3.4) for different values of α with comparison with the exact solution when $\alpha = 1$

x \ α	**1**	**0.5**	**0.6**	**0.8**	**Exact for $\alpha = 1$**
0	0.001062	–0.008	–0.006441	–0.073	0
0.125	–0.008938	–0.032	–0.026	–0.085	–0.003906
0.250	–0.029	–0.079	–0.065	–0.114	–0.016
0.375	–0.049	–0.109	–0.093	–0.138	–0.035
0.5	–0.108	–0.215	–0.189	–0.214	–0.063
0.625	–0.120	–0.216	–0.193	–0.223	–0.098
0.750	–0.186	–0.437	–0.378	–0.356	–0.141
0.875	–0.222	–0.327	–0.326	–0.367	–0.191

Conclusions and Future Works

Conclusions and Future Works

From the present study one can conclude the following :

1- The numerical solution of the fractional variational problems can be obtained directly from the functional and the there is no need to solve the fractional Euler-Lagrang equation .

2- The procedure considered in this thesis can be considered as a generalization to the results given in [Hsiao,2004].

3- From the Illustrative examples it can be seen that this operational matrix approach can obtain accurate and satisfying results.

4- It in remarkable that by using our approach we do not need to approximate the LRLFD and the RRLFD simulteanously as Haar series in the variational problems which was considered in chapter two and just we approximate the LRLFD as Haar series in order to get the desired (sapproximate) solution and this approach gave us reasonable results if it is compared with the exact solution.

Also , we recommend the following problems as future work :

1- The numerical solution of the fractional variational problems using direct Chebyshev wavelets , direct Legender wavelets methods.

2- The numerical solution of the fractional variational problems using Bernestein operational matrix .

3- The numerical solution of the fractional variational problems with delay using direct Haar wavelet method.

4- The numerical solution of the fuzzy fractional variational problems using direct Haar wavelet method.

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الملخص

في هذه الرسالة ,سوف نستخدم اسلوب واضح لحل مسائل التغاير ذات الرتب الكسرية وذلك باستخدام تقنية مويجات هار. المشتقة الكسرية سوف تكون من نوع ريمان-ليوفيلي.

الغرض الرئيسي لهذه الرسالة يتمحور حول هدفين:

الهدف الاول هو دراسة مسائل التغاير الكسرية البسيطة ذات الشروط الحدودية الثابتة وايجاد الحل التقريبي لها باستخدام طريقة مويجات هار المباشرة.

الهدف الثاني هو دراسة مسائل التغاير الكسرية البسيطة ذات شرط حدودي متحرك (شرط غير محدد) وايجاد الحل التقريبي لها باستخدام طريقة مويجات هار المباشرة.

الحل التقريبي لهذِه الانواع من مسائل التغاير المقترحة ممكن ايجاده مباشرة من الدالي دون الحاجة الى حل معادلة اويلر-لاكرانج الكسرية لذلك الاسلوب المقترح (طريقة مويجات هار المباشرة) أعطتنا حلاً بسيطاً ودقيقاً لهذِه الانواع من مسائل التغاير ذات الرتب الكسرية.

جمهورية العراق
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طريقة تقريبية لحل مسائل التغاير ذات الرتب الكسرية بأستخدام مويجات هار

رسالة

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في الرياضيات

من قبل

زيد عامر محمد
(بكالوريوس علوم، جامعة النهرين، 2008)

إشراف

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