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Numerical Solutions for Solving Delay Differential Equations of Fractional Order

A Thesis

Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ نَرْفَعُ دَرَجَاتٍ مَّن نَّشَاءُ ۗ وَفَوْقَ كُلِّ ذِي عِلْمٍ عَلِيمٌ ﴾

صدق الله العظيم

سورة يوسف

ألآية (76)

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Qutaiba Wadi Ibrahim
2016.

SUPERVISOR CERTIFICATION

I certify that this thesis entitled **" Numerical Solutions for Solving Delay Differential Equations of Fractional Order "** was prepared **" Qutaiba Wadi Ibrahim "** under my supervision at Department of Mathematics and Computer Application, College of Science, Al-Nahrain University, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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SUMMERY

The main theme of this thesis is to study and find the numerical solution of fractional order delay differential equations, and may be divided into two sub objectives, as follows:

The first objective is to prove the existences, uniqueness and the stability of the solutions of fractional order delay differential equations.

The second objective is to find the numerical solutions of fractional order delay differential equations by using the operational matrices of the generalized Hat functions.

In this thesis, a modified technique by combining the method of steps and generalized Hat functions for solving fractional order delay differential equations will be proposed.

This technique converts the fractional order delay differential equations on a given interval to a fractional order non-delay differential equations over that interval, by using the function depend on previous interval. Then apply the operational matrix for generalized Hat function on the obtained fractional order non-delay differential equations to transform linear and nonlinear the fractional order non-delay differential equations into a system of algebraic equations and find the solution.

Some illustrative examples are presented and the results of these examples are compared with the existing methods such as Chebyshev wavelet method and the exact solution in order to illustrate the accuracy and efficiency of the proposed method.

CONTENTS

CHAPTER THREE: A Modified Approach for Solving Fractional Order Delay Differential Equations Using Generalized Hat Functions of Second Kind.

List of Symbols and Abbreviations

DDEs	Delay Differential Equations.
FDEs	Fractional Differential Equations.
FDDEs	Fractional order Delay Differential Equations.
PDEs	Partial Differential Equations.
ODEs	Ordinary Differential Equations.
eq.'s	equations.
${}_aI_t^{\alpha}$	the left Riemann – Liouville fractional integrals of order α.
${}_bI_t^{\alpha}$	the right Riemann – Liouville fractional integrals of order α.
I^{α}	The Abel-Riemann (A-R) fractional integral of any order $\alpha > 0$.
D_t^{α}	Riemann–Liouville fractional derivative of order α.
${}^cD_t^{\alpha}$	Caputo fractional derivative of order α.
$[\alpha]$	the greatest integer part of α.
ψ_i's	Hat functions.
y_{exact}	exact solution.
y_{hat}	the solution by generalized Hat functions.
y_{Ch}	the solution by Chebyshev wavelets method.
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Introduction

Delay differential equations (DDEs) play an important role in the research field of various applied sciences such as control theory, electrical networks, population dynamics, environment science, biology and life science. Mathematical models employing delay differential equations turn out to be useful especially in the situation, where the description of investigated systems depends not only on the position of a system in the current time, but also in the past. In such cases the use of ordinary differential equations turns out to be insufficient. The presence of a delayed time argument in the investigated system may frequently influence properties of solutions. The survey of the theory related to delay differential equations can be found e.g. in books [Balachandran,2009], [Bellen,2003], [Erneux,2009], [Kolmanovskii,1999].

Delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. DDEs are also called time-delay systems, systems with aftereffect or dead-time, hereditary systems, equations with deviating argument, or differential-difference equations.

They belong to the class of systems with the functional state, i.e. partial differential equations (PDEs) which are infinite dimensional, as opposed to ordinary differential equations (ODEs) having a finite dimensional state vector.

Four points may give a possible [Richard,2003] explanation of the popularity of DDEs:

(1) After effect is an applied problem: it is well known that, together with the increasing expectations of dynamic performances, engineers need their models to behave more like the real process. Many processes include aftereffect

phenomena in their inner dynamics. In addition, actuators, sensors, communication networks that are now involved in feedback control loops introduce such delays. Finally, besides actual delays, time lags are frequently used to simplify very high order models. Then, the interest for DDEs keeps on growing in all scientific areas and, especially, in control engineering.

(2) Delay systems are still resistant to many classical controllers: one could think that the simplest approach would consist in replacing them by some finite-dimensional approximations.

Unfortunately, ignoring effects which are adequately represented by DDEs is not a general alternative: in the best situation (constant and known delays), it leads to the same degree of complexity in the control design. In worst cases (time-varying delays, for instance), it is potentially disastrous in terms of stability and oscillations.

(3) Delay properties are also surprising since several studies have shown that voluntary introduction of delays can also benefit the control.

(4) In spite of their complexity, DDEs however often appear as simple infinite-dimensional models in the very complex area of partial differential equations (PDEs).

Delay differential equations were initially introduced in the 18th century by Laplace and Condorect, [Ulsoy,2003]. However, the rapid development of the theory and applications of those equations did not come until after the Second World War, and continues till today. The basic theory concerning the stability of systems described by equations of this type was developed by Pontryagin in 1942. Important works have been written by Smith in 1957, Pinney in 1958, Bellman and Cooke in 1963, Halanay in 1966, Myshkis in 1972, Hale 1977, Yanusherski in 1978 and Marshal in 1979. On the other hand, many complicated physical problems described in terms of partial differential equations can be approximated by much simpler problems described in terms of delay differential equations, [Pinney,1958]. The impetus has mainly been due to the developments in many fields, such as the control theory, mathematical biology, and mathematical economics, etc. Minorsky, [Hale, 1977] was one of the first investigators of modern times to study the delay differential equation:

$$y'(t) = f(t, y(t), y(t-\tau)),$$

and its effect on simple feed-back control systems in which the communication time cannot be neglected.

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals).

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject is as old as the differential calculus, and goes back to times when Leibnitz and Newton invented differential calculus [Sabatier,2015].

Fractional calculus is a topic being more than 300 years old. The idea of fractional calculus has been known since the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695 where half-order derivative was mentioned. In a correspondence between Johann Bernoulli and Leibniz in 1695, Leibniz mentioned the derivative of general order.

In 1730 the subject of fractional calculus did not escape Euler's attention. J. L. Lagrange in 1772 contributed to fractional calculus indirectly, when he developed the law of exponents for differential operators. In 1812, P. S. Laplace defined the fractional derivative by means of integral and in 1819 S. F. Lacroix mentioned a derivative of arbitrary order in his 700-page long text, followed by J. B. J. Fourier in 1822, who mentioned the derivative of arbitrary order.

The first use of fractional operations was made by N. H. Abel in 1823 in the solution of tautochrome problem. J. Liouville made the first major study of fractional calculus in 1832, where he applied his definitions to problems in theory.

In 1867, A. K. Grünwald worked on the fractional operations. G. F. Riemann developed the theory of fractional integration during his school days and published his paper in 1892. A. V. Letnikov wrote several papers on this topic from 1868 to 1872. Oliver Heaviside published a collection of papers in 1892, where he showed the so-called Heaviside operational calculus concerned with linear generalized operators. In the period of 1900 to 1970 the principal contributors to the subject of fractional calculus were, for example, H. H. Hardy, S. Samko, H. Weyl, M. Riesz, S. Blair, etc. The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [Oldham,1974] published in1974. After that, a number of publications about the fractional calculus has rapidly increased and today there exist at least two international journals (i) journal of fractional calculus and (ii) fractional calculus and applied analysis and for a historical overview on fractional calculus, see [Oldham,1974], [Miller,1993] and [Kilbas,2006].

Fractional differential equations are applied to model wide range of physical problems including nonlinear oscillation of earth quakes [He,1998], fluid-dynamic traffic [He,1999], frequency dependent damping behavior of many viscoelastic materials, signal processing [Panda,2006] and control theory [Bohannan,2008]. Moreover, in several areas of applied mathematics [Asl,2003] fractional differential equations are often used. These are also used in the study of epidemics, age-structured population growth [Kuang,1993], automation, traffic flow and in many engineering problems.

The fractional order delay differential equation is a generalization of the delay differential equation to arbitrary noninteger order. For most of fractional order delay differential equations, exact solutions are not known. Therefore, different numerical methods [Wang,2013], [Moghaddam,2013], [Morgado,2013] have been developed and applied for providing approximate solutions.

As we mention above that the exact solution of delay differential equations of fractional order is not known therefore in this thesis a generalized Hat basis functions [Tripathi, 2013] together with the method of steps will be used in order to find the numerical solution of delay differential equations of fractional order.

This thesis consists of three chapters, as well as, this introduction.

In chapter one, fundamental concepts of delay differential equations and fractional calculus are given, while in chapter two, the existence and stability of the solutions of fractional order delay differential equations are presented. Finally, in chapter three a modified approach for solving fractional order delay differential equation using generalized Hat functions operational matrices together with the method of steps, with illustrative examples have been given. It is remarkable that all the calculations have been done using Mathcad 14.

CHAPTER ONE
Fundamental Concepts

1.1 Introduction

In this chapter we shall introduce the basic concepts of delay differential equations and fractional calculus which are necessary for the construction of this thesis.

This chapter consists of three sections. In section (1.2) the basic concepts of delay differential equations were given. In section (1.3) we shall give a brief introduction to the subject of fractional calculus including the beta and gamma functions, the fractional integration and fractional derivatives.

1.2 Delay Differential Equations [Bellman, 1963]

Delay differential equation "DDE" is defined as an unknown function $y(t)$ and some of its derivatives, evaluated at arguments that differ by any of fixed number of values $\tau_1, \tau_2, \tau_3, \dots, \tau_k$. The general form of the n-th order DDE is given by

$$F(t, y(t), y(t-\tau_1), \dots, y(t-\tau_k), y'(t), y'(t-\tau_1), \dots, y'(t-\tau_k),$$

$$y^{(n)}(t), y^{(n)}(t-\tau_1), \dots, y^{(n)}(t-\tau_k)) = 0, \qquad (1.1)$$

where F is a given functional and $\tau_i, \forall\ i = 1, 2, \dots, k$; are given fixed positive number called the "time delay".

In some literature equation (1.1) is called a difference differential equation or functional differential equation, [Bellman,1963], or an equation with time lag [Halanay,1966], or a differential equation with deviating arguments, [Driver,1977].

The emphasis will be, in general, on the linear equations with constant coefficients of the first order and with one delay

$$a_0 y'(t) + a_1 y'(t-\tau) + b_0 y(t) + b_1 y(t-\tau) = f(t), \qquad (1.2)$$

where $f(t)$ is a given continuous function and τ is a positive constant and a_0, a_1, b_0, and b_1 are constants (if $f(t) = 0$, then equation (1.2) is said to be homogenous; otherwise it is nonhomogeneous).

The kind of initial conditions that should be used in DDE's differ from ODE's so that one should specify in DDE's an initial function on some interval of length τ, say $[t_0 - \tau, t_0]$ and then try to find the solution of equation (1.2) for all $t \geq t_0$. Thus, we set $y(t) = p_o(t)$, for $t_0 - \tau \leq t \leq t_0$ where $p_0(t)$ is some given continuous function. Therefore, the solution of DDE consists of finding a continuous extension of $p_0(t)$ into a function $y(t)$ which satisfies (1.2) for all $t \geq t_0$, [Halanay, 1966].

Delay differential equation given by equation (1.2) can be classified into three types which are retarded, neutral and mixed. The first type means an equation where the rate of change of state variable y is determined by the present and past states of the equation (1.2) where the coefficient of $y'(t-\tau)$ is zero, i.e., $(\mathrm{a}_0 \neq 0, \mathrm{a}_1 = 0)$. If the rate of change of state depends on its own past values as well on its derivatives, the equation is then of neutral type, equation (1.2) where the coefficient of $y(t-\tau)$ is zero, i.e., $(\mathrm{a}_0 \neq 0, \mathrm{a}_1 \neq 0 \text{ and } \mathrm{b}_1 = 0)$, while the third type is a combination of the previous two types, i.e., $(\mathrm{a}_0 \neq 0, \mathrm{a}_1 \neq 0,\ \mathrm{b}_0 = 0 \text{ and } \mathrm{b}_1 \neq 0)$.

1.2.1 Solution of the First Order Delay Differential Equations [Driver, 1977]:

In this section, we shall introduce two analytical methods used to solve the delay differential equations.

1. Method of steps (method of successive integrations), [Smith, 2011]:

The method of steps or the method of successive integrations is an elementary method that can be used to solve some DDEs analytically. It is much more intuitive and can be used to solve DDEs with variable coefficients

This method converts the DDE on a given interval to an ODE over that interval, by using the known history function for that interval. The resulting equation is solved, and the process is repeated in the next interval with the newly found solution serving as the history function for the next interval. we shall consider some illustrative examples for all types of DDE.

The best well known method for solving DDE's is the method of steps or the method of successive integrations which is used to solve a DDE of the form:

$$y'(t) = f\big(t, y(t), y(t-\tau), y'(t-\tau)\big), \quad t \geq t_0 \tag{1.3}$$

with initial condition $y(t) = p_o(t)$, for $t_0 - \tau \leq t \leq t_0$. For such equations the solution is constructed step by step as follows:

Step 1: On the interval $[t_0 - \tau, t_0]$, the function $y(t)$ is the given function $p_o(t)$, so one can obtain the solution in the next step interval $[t_0, t_0 + \tau]$ by solving the following equation:

$$y'(t) = f\big(t, y(t), p_o(t-\tau), {p_o}'(t-\tau)\big), \quad t_0 \leq t \leq t_0 + \tau,$$

with initial condition $y(t_0) = p_o(t_0)$. If we consider that $p_1(t)$ is the desired first step solution, which exists by virtue of continuity hypotheses.

Step 2: On the interval $[t_0, t_0 + \tau]$, the function $y(t)$ is the given function $p_1(t)$, therefore one can find the solution $p_2(t)$ to the equation:

$$y'(t) = f\big(t, y(t), p_1(t-\tau), {p_1}'(t-\tau)\big), \quad t_0 + \tau \leq t \leq t_0 + 2\tau$$

with initial condition $y(t_0 + \tau) = p_1(t_0 + \tau)$.

These steps may be continued for subsequent intervals.

In general, by assuming that $p_k(t)$, $\forall(k = 1,2\,,\dots)$ is defined on the interval $[t_0 + (k-2)\tau, t_0 + (k-1)\tau]$, then, one can find the solution $p_k(t)$ to the equation:

$$y'(t) = f\big(t, y(t), p_{k-1}(t-\tau), p_{k-1}{}'(t-\tau)\big), t_0 + (k-1)\tau \le t \le t_0 + k\tau$$

with the initial condition:

$$y(t_0 + (k-1)\tau) = p_{k-1}(t_0 + (k-1)\tau).$$

We illustrate the method in the following example.

Example (1.1):

Consider the neutral first order DDE:

$$y\,'(t) = y\,'(t-1) + t\,, t\ \ge\ 0$$

with initial condition

$$p_0(t) = t\ +\ 1, \text{for } -1 \le t \le 0.$$

Solution:

To find the solution in the first interval [0, 1]. We solve the following:

$$\begin{aligned} y\,'(t) &= p_0'(t-1) + t \\ &= t\ +\ 1, \text{for } 0 \le t \le 1. \end{aligned} \tag{1.4}$$

Integrating both sides of eq.(1.4) from 0 to t where $0 \le t \le 1$, we have:

$$\int_0^t y'(s)ds = \int_0^t (s+1)ds$$

and hence:

$$y_1(t) = \frac{t^2}{2} + t + \ 1, \text{for } 0 \le t \le 1$$

In order to find the solution in the second step interval suppose that:

$$p_1(t) = y_1(t) = \frac{t^2}{2} + t + \ 1$$

is the initial condition. Since $p_1(t)$is defined on the whole segment $[0, 1]$.

Hence by forming the new equation:

$$y'(t) = p_1'(t-1) + t \text{ , for } 1 \leq t \leq 2, \tag{1.5}$$

where

$$p_1(t) = \frac{t^2}{2} + t + 1, \text{for } 0 \leq t \leq 1 \text{ .}$$

One can find the solution in the next step interval $[1, 2]$, and solving eq. (1.5) for $y\ (t)$, we have:

$$y'(t) = p_1'(t-1) + t$$

$$= 2t \text{ , for } 1 \leq t \leq 2 \text{ .} \tag{1.6}$$

Integrating both sides of eq. (1.6) from 1 to t where $1 \leq t \leq 2$, we get:

$$y(t) = t^2 + \frac{3}{2} \text{ ,for } 1 \leq t \leq 2.$$

Therefore, $y(t)$ is the desired second step solution which is denoted by:

$$y(t) = p_2(t) = t^2 + \frac{3}{2} \text{ ,for } 1 \leq t \leq 2$$

Similarly, we proceed to the next intervals.

Example (1.2):

Consider the retarded first order DDE:

$$y'(t) = y(t-1) \text{ , } t \geq 0$$

with the initial condition:

$$y(t) = p_o(t) = t, \text{ for } -1 \leq t \leq 0.$$

Solution:

To find the solution in the first step interval $[0, 1]$ we have to solve the following equation:

$$y'(t) = p_o(t-1)$$

$$= t - 1, \text{ for } 0 \leq t \leq 1. \quad (1.7)$$

Integrating both sides of eq.(1.7) from 0 to t where $0 \leq t \leq 1$, we have:

$$\int_0^t y'(s)ds = \int_0^t (s-1)ds$$

and hence after carrying some calculations we get the first time step solution:

$$y_1(t) = \frac{t^2}{2} - t, \qquad \text{for } 0 \leq t \leq 1.$$

In order to find the solution in the second step interval, suppose that:

$$p_1(t) = y_1(t) = \frac{t^2}{2} - t, 0 \leq t \leq 1$$

Since $p_1(t)$ is defined on the whole segment $[0, 1]$.

Hence by forming the new equation:

$$y'(t) = p_1(t-1), \text{ for } 1 \leq t \leq 2 \quad (1.8)$$

with the initial condition

$$p_1(t) = \frac{t^2}{2} - t, \text{for } 0 \leq t \leq 1$$

On the next step interval $[1, 2]$, One can find the solution of eq. (1.8)

$$y'(t) = p_1(t-1), \qquad 1 \leq t \leq 2$$

$$= \frac{(t-1)^2}{2} - (t-1)$$

$$= \frac{t^2}{2} - t + \frac{1}{2} - t + 1$$

$$= \frac{t^2}{2} - 2t + \frac{3}{2}, 1 \leq t \leq 2. \quad (1.9)$$

Integrating both sides of eq. (1.9) from 1 to t, where $t \in [1, 2]$, we get:

$$y(t) = \frac{t^3}{6} - t^2 + \frac{3}{2}t - \frac{7}{6}, \text{for } 1 \leq t \leq 2$$

Similarly, let:

$$y_2(t) = \frac{t^3}{6} - t^2 + \frac{3}{2}t - \frac{7}{6}$$

and suppose $p_2(t)$ is the desired second step solution, i.e.,

$$p_2(t) = y_2(t) = \frac{t^3}{6} - t^2 + \frac{3}{2}t - \frac{7}{6}$$

Since $p_2(t)$ is defined on the whole segment $[1,2]$ hence by forming the new equation:

$$y'(t) = p_2(\mathrm{t} - 1), \text{ for } 2 \leq t \leq 3 \qquad (1.10)$$

with the initial condition:

$$p_2(t) = \frac{t^3}{6} - t^2 + \frac{3}{2}t - \frac{7}{6}.$$

Similarly, one can find $y_3(t)$, $y_4(t)$ and so on.

Example (1.3):

Consider the mixed DDE:

$$y'(t) = y(t-1) + 2y'(t-1), \quad t \geq 0$$

with initial condition:

$$p_0(t) = 1, \text{ for } 0 \leq t \leq 1 .$$

Solution:

To find the solution in the first step interval [1,2], we will solve the following equation:

$$y'(t) = p_0(t-1) + 2{p_0}'(t-1), \text{ for } 1 \leq t \leq 2$$

Hence

$$y'(t) = 1, \text{ for } 1 \leq t \leq 2 . \qquad (1.11)$$

By integrating both sides of eq. (1.11) from 1 to t, where $1 \leq t \leq 2$, we have:

$$\int_1^t y'(\mathrm{s})ds = \int_1^t ds$$

and hence:

$$y(t) = t, \text{ for } 1 \le t \le 2$$

and suppose that $p_1(t)$ is the desired first step solution

$$y_1(t) = p_1(t) = t, \text{ for } 1 \le t \le 2$$

Since $p_1(t)$ is defined on the whole segment [1,2], hence by forming the new equation:

$$y'(\mathrm{t}) = p_1(\mathrm{t}-1) + 2p_1{}'(\mathrm{t}-1) \text{ , for } 2 \le t \le 3$$

with initial condition:

$$y_1(t) = p_1(t) = t, \text{ for } 1 \le t \le 2$$

and so on, we proceed to the next intervals.

The next example considers the solution of DDE with variable delay which can be solved by successive integration method.

Example (1.4):

Consider the retarded first order DDE:

$$y'(t) = -y(t-e^t)\ , \quad 0 \le t \le 1$$

with initial condition:

$$p_0(t) = 1, \text{ for } -1 \le t \le 0.$$

Solution:

To find the solution in the first step interval [0,1] we have to solve the following equation:

$$y'(t) = -p_0(t-e^t)\ , \quad 0 \le t \le 1$$

$$= -1\ , \quad 0 \le t \le 1 \qquad (1.12)$$

Integrating both sides of eq. (1.12) from 0 to t where $0 \leq t \leq 1$, we have:

$$\int_0^t y'(s)ds = -\int_0^t ds$$

and hence after carrying some calculations we get the first time step solution:

$$y(t) = 1 - t, \text{for } 0 \leq t \leq 1$$

In order to find the solution in the second step interval suppose that:

$$y_1(t) = p_1(t) = 1 - t, \text{ for } 0 \leq t \leq 1$$

Therefore:

$$y_1(t) = 1 - t, \text{ for } 0 \leq t \leq 1$$

Since $p_1(t)$ is defined on the whole segment $[0, 1]$.

Hence by forming the new equation:

$$y'(t) = -p_1(t - e^t)\,, \;\; 0 \leq t \leq 1$$

$$= -1 + (t - e^t), \;\; 0 \leq t \leq 1. \qquad (1.13)$$

By integrating both sides of eq. (1.13) from 1 to t, where $1 \leq t \leq 2$, we have:

$$y(t) = \frac{t^2}{2} - t - e^t + 3.2, \text{for } 1 \leq t \leq 2$$

Similarly, let:

$$y_2(t) = \frac{t^2}{2} - t - e^t + 3.2, \text{ for } 1 \leq t \leq 2\,,$$

and suppose $p_2(t)$ is the desired second step solution, i.e.,

$$p_2(t) = y_2(t)$$

$$= \frac{t^2}{2} - t - e^t + 3.2,, \text{for } 1 \leq t \leq 2\,.$$

Since $p_2(t)$ is defined on the whole segment $[1, 2]$, hence by forming the new equation:

$$y'(t) = -p_2(\mathrm{t} - \mathrm{e}^{\mathrm{t}})\,, \;\; 2 \leq t \leq 3$$

with initial condition

$$p_2(t) = \frac{t^2}{2} - t - \mathrm{e}^{\mathrm{t}} + 3.2,, \text{for } 1 \le t \le 2$$

similarly, one can find $y_3(t)$, $y_4(t)$and so on.

2. Laplace transform method, [Ross, 1984], [Gupta, 2014]:

Laplace transformation method is also, one of the most widely used methods for solving DDE's

Suppose that y is a real-valued function of the real variable defined for $t > 0$. Let s be a parameter that we shall assume to be real, and consider the function Y defined by

$$L[y(t)] = Y(s) = \int_0^\infty e^{-st} y(t)dt, \tag{1.14}$$

for all values of s for which this integral exists. The function $L[y]$ defined by the integral (1.14) is called the Laplace transformation of the function y and we shall denote the Laplace transform $L[y]$ of y by $Y(s)$.

We say that $y(t) = L^{-1}[Y(s)]$ is the (unique) inverse Laplace transform of $Y(s)$.

We also recall that the Laplace transform is a linear operator. In particular, if $L[y(t)]$ and $L\{x(t)\}$ exist, then

$$L[y(t) + x(t)] = L[y(t)] + L[x(t)]$$

and

$$L[cy(t)] = cL[y(t)] \text{ , } c \text{ is constant.}$$

This method can be used in two different approaches for solving delay differential equations.

The first approach is by mixing between the method of steps and the Laplace transform method and the other approach is by applying the Laplace transform method directly to the original DDE.

The First Approach, [Brauer, 1973]:

This approach depends mainly on applying first the method of steps to transform the DDE into ODE and then applying Laplace transformation method to solve the resulting equation. To this end consider the following example:

Example (1.5):

Consider the neutral DDE:

$$y'(t) = y'(t-1) + t\,, \quad 0 \le t \le 1$$

with initial condition:

$p_0(t) = 1 + t$, for $-1 \le t \le 0$.

Solution:

To find the solution in the first step interval $[0, 1]$, we apply the method of steps, to get:

$$y'(t) = p_0{}'(t-1) + t\,, \quad 0 \le t \le 1$$

Therefore

$$y'(t) = 1 + t\,, \text{for } 0 \le t \le 1 \tag{1.15}$$

Now, taking the Laplace transform to the both sides of eq. (1.15), we have:

$$L\{y'(t)\} = L\{1\} + L\{t\}$$

$$sY(s) - y(0) = \frac{1}{s} + \frac{1}{s^2}$$

and so the Laplace transform of the solution $y(t)$ into $Y(s)$ is given by:

$$Y(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} \tag{1.16}$$

Taking inverse Laplace transform to both sides of eq. (1.16), we have:

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2!}L^{-1}\left[\frac{2!}{s^3}\right]$$

$$y(t) = 1 + t + \frac{t^2}{2}, \text{for } 0 \le t \le 1 .$$

Hence, the solution in the first step interval is given by:

$$p_1(t) = y(t) = 1 + t + \frac{t^2}{2}, \text{for } 0 \le t \le 1$$

In order to find the solution in the second step interval [1,2], we proceed similarly as in the first step with initial condition:

$$p_1(t) = 1 + t + \frac{t^2}{2}, \text{for } 0 \le t \le 1$$

and hence:

$$y'(t) = p_1{}'(\mathrm{t} - 1) + t \text{, for } 0 \le t \le 1$$

with the equivalent ODE

$$y'(t) = 2t \text{, for } 1 \le t \le 2$$

with initial condition: $y(1) = \frac{5}{2}$.

By making changing for the independent variable, we set $v = t - 1$ then $v \in [0,1]$, so that

$$y'(v + 1) = 2(v + 1) \text{, with } y(0) = \frac{5}{2},$$

and by considering:

$$z(v) = y(v + 1),$$

which implies that:

$$z'(v) = 2(v + 1) , z(0) = \frac{5}{2} , v \in [0,1] \tag{1.17}$$

Taking the Laplace transform to the both sides of eq. (1.17), we have:

$$sZ(s) - z(0) = \frac{2}{s} + \frac{2}{s^2} \ ,$$

where $Z(s)$ is the Laplace transform of $z(v)$ hence:

$$Z(s) = \frac{5}{2s} + \frac{2}{s^2} + \frac{2}{s^3}. \qquad (1.18)$$

Taking the inverse Laplace transform to the both sides of eq. (1.18), we have:

$$z(v) = v^2 + 2v + \frac{5}{2}.$$

Hence the solution in the second step interval $[1, 2]$ is given by:

$$z(v) = y(t) = (t-1)^2 + 2(t-1) + \frac{5}{2}$$

Similarly, we proceed to the next intervals.

Second Approach, [Brauer, 1973]:

This approach is to solve DDE's by using Laplace transform method directly without using the method of steps. Laplace transformation method is extremely useful in obtaining the solution of the linear DDE's with constant coefficients. Let us illustrate this method by considering the following example:

Example (1.6):

Consider the retarded first order DDE:

$$y'(t) = y(t-1), \ \ t \geq 0 \,. \qquad (1.19)$$

with the initial condition:

$$y(t) = p_o(t) = t, \text{ for } -1 \leq t \leq 0 \,,$$

such that $y(0) = 0, y'(0) = 1$.

Solution:

Applying the Laplace transform method to both sides of the equation (1.19), we get:

$$sY(s) = \int_{0}^{\infty} y(t-1)e^{-st}dt$$

Using the transform $v = t - 1$, yields:

$$\int_{0}^{\infty} y(t-1)e^{-st}dt = \int_{-1}^{\infty} y(v)e^{-s(v+1)}dv$$

$$= e^{-s}\int_{-1}^{0} y(v)e^{-sv}dv + e^{-s}\int_{0}^{\infty} y(v)e^{-sv}dv$$

$$= e^{-s}\int_{-1}^{0} ve^{-sv}dv + e^{-s}\int_{0}^{\infty} y(v)e^{-sv}dv$$

Since $y(v) = v$, for $-1 \leq z \leq 0$.

Finally:

$$Y(s) = \left[\frac{-1}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left[\frac{1}{s-e^{-s}}\right] \tag{1.20}$$

From equation (1.20), it follows that:

$$Y(s) = \left[\frac{-1}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]\left[\frac{1}{s-e^{-s}}\right] \tag{1.21}$$

and upon taking the inverse Laplace transform one can find the solution $y(t)$, where it is so difficult to obtain, which in force us to prefer using the numerical methods.

1.3 Fractional Calculus

In this section we shall give some fundamental concepts about fractional calculus, which are needed in this thesis and in order to make this thesis self-contained as soon as possible.

1.3.1 Some Special Functions in Fractional Calculus

In this section, we introduce and discuss gamma and beta functions and their properties. Those functions plays an important role in the theory of fractional derivative and integral.

One of the basic special functions in analysis is $n!$. For non-integer values, or even complex numbers, which is called Euler's gamma function and denoted by $\Gamma(t)$. Gamma function is simply said to be the extension of factorial for real numbers.

1. Gamma Function, [Loverro, 2004]:

The most basic interpretation of the gamma function is simply the generalization of the factorial for all real numbers. The gamma function $\Gamma(t)$ is defined as

$$\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds, \quad t > 0 \qquad (1.22)$$

is often more useful, although it is restricted to positive value of t.

The following are the most important properties of the gamma function:

1. $\Gamma\left(\frac{1}{2} - n\right) = \frac{(-4)^n n! \sqrt{\pi}}{(2n)!}$

2. $\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)! \sqrt{\pi}}{(4)^n n!}$

3. $\Gamma(-t) = \frac{-\pi \csc(\pi t)}{\Gamma(t+1)}$

4. $\Gamma(nt) = \sqrt{\frac{2\pi}{n}} [\frac{nt}{\sqrt{2\pi}}]^n \prod_{k=0}^{n-1} \Gamma\left(t + \frac{k}{n}\right), \ \ n \in N$

5. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2. Beta Function, [Loverro, 2004]:

Another important special function which plays basic role in the theory of fractional calculus is the beta function. For positive value of the two parameters p and q; the function is defined by the beta integral:

$$\beta(p,q) = \int_0^1 s^p (1-s)^q ds \ ,p,q > 0 \tag{1.23}$$

which is also known as the Euler's integral of the second kind. It is well known that, gamma and beta functions are related to each other. If either p or q is non-positive, the integral diverges otherwise $\beta(p,q)$ is defined by the relationship:

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.24}$$

Both beta and gamma functions have "incomplete" analogues. The incomplete beta function of argument t is defined by the integral:

$$\beta_t(p,q) = \int_0^t s^{p-1} (1-s)^{q-1} ds \ ,p,q > 0 \tag{1.25}$$

and the incomplete gamma function of argument t is defined by [1]:

$$\gamma^*(c,t) = \frac{c^{-t}}{\Gamma(t)} \int_0^c s^{t-1} e^{-s} ds$$

$$= e^{-t} \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i+c+1)} \tag{1.26}$$

$\gamma^*(c,t)$ is a finite single-valued analytic function of t and c.

1.3.2 Fractional Order Integration

In this section we shall give some basic definitions and properties of fractional integrals and derivatives.

Definition (1.1): ***(Riemann-Liouville Fractional Order Integral, [Loverro, 2004]:***

Let y: $[a,b] \rightarrow R$ be a function, α a positive real number, n the integer satisfying $n-1 \leq \alpha < n$, and Γ the Euler gamma function. Then, the left and right Riemann–Liouville fractional integrals of order α are defined by:

$${}_aI_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \qquad (1.27)$$

and

$${}_bI_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} y(s) ds, \qquad (1.28)$$

respectively.

For $y(t) \in C^m[a,b],\ \alpha,\beta \geq 0,\ n-1 < \alpha \leq n,\ \alpha+\beta \leq m,\ v \geq -1,$ $t > 0$, we mention the following:

1. $(I_t^{\alpha} I_t^{\beta} y)(t) = \left(I_t^{\beta} I_t^{\alpha} y\right)(t) = (I_t^{\alpha+\beta} y)(t),$
2. $I_t^{\alpha} t^{v} = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} t^{\alpha+v}$,
3. $I_t^{\alpha} e^{av} = t^{\alpha} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+\alpha+1)}$,
4. $I_t^{\alpha} \sin(at) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k (at)^{2k+1}}{\Gamma(2k+\alpha+2)}$,
5. $I_t^{\alpha} \cos(at) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k (at)^{2k}}{\Gamma(2k+\alpha+1)}$,

Definition (1.2): ***(Weyl Fractional Order Integral, [Oldham, 1974])***

The left fractional order integral of order $\alpha > 0$ of a given function y is defined as:

$${}_{-\infty}I_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t \frac{y(s)}{(t-s)^{1-\alpha}} ds, \qquad (1.29)$$

and the right fractional order integral of order $\alpha > 0$ of a given function y is given by:

$${}_{\infty}I_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\infty} \frac{y(s)}{(s-t)^{1-\alpha}} ds, \tag{1.30}$$

Definition (1.3): ***(Abel-Riemann Fractional Order Integral, [Mittal, 2008])***

The Abel-Riemann (A-R) fractional integral of any order $\alpha > 0$, for a function $y(t)$ with $t \in R^+$ is defined as:

$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \ \alpha > 0, \ t > 0 \tag{1.31}$$

$I^0 = I$ (identity operator) .

The A-R integral possess the semigroup property:

$$I^{\alpha} I^{\beta} = I^{\alpha+\beta}, \text{for all } \alpha, \beta \geq 0 \tag{1.32}$$

1.3.3 Fractional order Derivatives:

In this section some definitions of fractional order derivatives are presented:

Definition (1.4): ***(Riemann-Liouville Fractional Order Derivatives, [Loverro, 2004], [Nishimoto, 1983])***

Let $f : [a, b] \to R$ be a function, α a positive real number, n the integer satisfying $n - 1 \leq \alpha < n$, and Γ is the Euler gamma function. Then, the left and right Riemann–Liouville fractional derivatives of order α are defined by:

$${}_aD_t^{\alpha} y(t) = \frac{d^n}{dt^n} \, {}_aI_t^{n-\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} y(s) ds.$$

and

$${}_bD_t^{\alpha} y(t) = (-1)^n \frac{d^n}{dt^n} \, {}_bI_t^{n-\alpha} y(t)$$

$$= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-s)^{n-\alpha-1} y(s) ds.$$

respectively.

Definition (1.5): (*The A-R Fractional Derivative, [Mittal, 2008]):*

The A-R fractional derivative of order α > 0 is defined as the inverse of the corresponding A-R fractional integral, i.e.,

$$D^{\alpha} I^{\alpha} = I , \tag{1.33}$$

for positive integer m, such that $m - 1 < \alpha \leq m$,

$$(D^{m} I^{m-\alpha}) I^{\alpha} = D^{m}(I^{\alpha} I^{m-\alpha}) = I , \tag{1.34}$$

i.e.,

$$D^{\alpha} y(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^{m}}{dt^{m}} \int_{a}^{t} \frac{y(s)}{(t-s)^{\alpha+1-m}} ds , & m-1 < \alpha < m \\ \\ \frac{d^{m}}{dt^{m}} y(t), & \alpha = m \end{cases} \tag{1.35}$$

Definition (1.6): ***(Caputo Fractional Order Derivative, [Caputo, 1967])***

The Caputo fractional order derivative of a suitable function $y(t)$ is defined as:

$${}^{c}D_{t}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} (\frac{d}{dt})^{n} y(s) ds. \tag{1.36}$$

for $a \leq t \leq b$, where $\alpha \in R^{+}$ and $n = [\alpha]$ ($[\alpha]$ is the integer part of α).

It is remarkable here to mention some of the important properties of the Caputo fractional order derivative as follows:

1. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

2. Caputo's fractional differentiation is linear operator, similar to integer order differentiation:

$${}^{c}D_{t}^{\alpha}(\mu f(t) + \nu g(t)) = \mu \, {}^{c}D_{t}^{\alpha} f(t) + \nu \, {}^{c}D_{t}^{\alpha} g(t),$$

where μ and ν are constant.

3. $(I_{t}^{\alpha} \, {}^{c}D_{t}^{\alpha} y(t)) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0) \frac{t^{k}}{k!}$, for $m - 1 < \alpha \leq m$.

4. ${}^{c}D_t^{\alpha}(I_t^{\beta} y(t)) = {}^{c}D_t^{\alpha-\beta}(y(t)).$

Definition (1.7): ***(Grünwald Fractional Order Derivatives, [Loverro, 2004])***

The Grünwald derivatives of any integer order to any function, can take the form:

$$D_t^{\alpha} = \lim_{N\to\infty} \left\{ \frac{\left(\frac{t}{N}\right)^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} y\left(t - j\,\frac{t}{N}\right) \right\} \tag{1.37}$$

CHAPTER TWO

Existence and Stability of the Solutions of Delay Differential Equations of Fractional Order

2.1. Introduction:

This chapter consists of four sections, in section 2.2, a historical background and review of the existence and stability of the solutions of delay differential equations of fractional order is presented. While in section 2.3, we introduce notations, definitions and preliminary facts. Finally, in section 2.4, the existence and stability of the solution of delay differential equations of fractional order is derived.

2.2 Historical Background

The study of the existence, uniqueness, periodicity, asymptotic behavior, stability, and methods of analytic and numerical solutions of fractional order differential equations have been studied extensively in a large cycle works (see), e.g., [Zhang, 2006], [Ibrahim, 2007], [Xinwei, 2007], [Agarwal, 2008], [Lakshmikantham, 2008], [Agarwal, 2009], [Belmekki, 2009], [Agarwal, 2010], [Andradem 2010], [Ashyralyev, 2011], [Dal, 2011], [Cakir, 2011], [Hicdurmaz, 2011], [De la Sen,2011], [Ibeas,2011], [Yuan,2011], [Berdyshevm 2011] ,[Yuan, 2012], and the references therein.

One of the important aspects in the study of delay differential equations of fractional order, is the investigation of existence and uniqueness of solutions.

Lakshmikantham, [Lakshmikantham,2008] developed the basic theory for delay differential equations of fractional order (FDDEs). In [Zhou, 2009] and [Zhou, 2010] existence and uniqueness for fractional order neutral DDEs have been formulated. Existence of positive solutions for nonlinear FDDE involving

Riemann-Liouville derivative has been addressed in [Ye, 2007]- [Liao, 2009]. Kexue et al. [Kexue, 2011] obtained the existence and uniqueness of the mild solutions for a class of abstract FDDE using solution operator approach.

The existence of a unique solution is proved for a class of nonlinear nonautonomous system of Riemann-Liouville fractional differential systems with different constant delays and nonlocal condition in [Gaafar, 2011]. De la Sen, [De la Sen, 2011] investigated the non-negative solution and the stability and asymptotic properties of the solution of fractional differential dynamic systems involving delayed dynamics with point delays.

El-Sayed et al. [El-Sayed, 1996] proved the existence of the solution of some kind of delay differential equations of fractional order, El-Sayed, Gaafar and Hamadalla [El-Sayed, 2010] discussed the existence and uniqueness of solutions for the non-local non-autonomous system of fractional order differential equations with delays. Zhang [Zhang, 2008] established the existence of a unique solution for delay differential equations of fractional order.

Existence and uniqueness of the solutions of the delay differential equations of fractional order (FDDEs) under various conditions has been established using fixed point theorems, see e.g. [Benchohra, 2008], [Maraaba, 2008], [Abbas, 2011], [Jalilian, 2013], [Yang, 2013] and [Shengli, 2014].

Agarwal et al. [Zhou, 2010] discussed the existence of solutions for the neutral fractional differential equation with bounded delay. By employing the Krasnoselskii's fixed point theorem.

Stability of solution is one of the most basic and interesting problem in control theory. The question of stability is of main interest in the physical and biological systems, such as the fractional Duffing oscillator, fractional predator-prey and rabies models, etc. Recently, the theory of the FDDEs has been studied and some basic results are obtained including stability theory.

Chen and Moore [Chen, 2002] studied stability of 1-dimensional fractional systems with retard time. El-Sayed, Gaafar and Hamadalla [El-Sayed, 2010] discuss the stability of solutions for the non-local non-autonomous system of fractional order differential equations with delays. Najafi et al, [Saberi, 2011], [Refahi, 2012] studied stability analysis of the distributed order differential equations with respect to the non-negative density function. Gao et al. [Zhenghui, 2013] proved the stability of the solutions for nonlinear fractional differential equations with delays and integral boundary conditions.

Time delay plays an important role in mathematical modeling of many real world phenomena. Time delay can have an effect on the stability of a system and occasionally can cause a system to become unstable. To the best of our knowledge, there are relatively few results on the stability of fractional order systems with delay, such as Lazarevic and Spasic [Lazarevic, 2009], Akbari Moornani and Haeri [Moornani, 2010], Wang et al. [Huang, 2011], El-Sayed and Gaafar [Gaafar, 2011] and Kumar and Sukavana [Kumar, 2012].

In [Lazarevic, 2009], a finite-time stability test procedure is proposed for linear nonhomogeneous fractional-order systems with a pure time delay. In [Moornani, 2010], two theorems are given to check the stability of two large classes of fractional order delay systems (retarded and neutral types), respectively.

In [Kumar, 2012], sufficient conditions are established for the approximate controllability of a class of semilinear delay control systems of fractional order.

We restrict our attention in this chapter on the proof of the existence and uniqueness and the stability of the solution of delay differential equations of fractional order of the form:

$$ {}^{c}D_t^{\alpha} y(t) = f\big(t, y(t), y(t-\tau)\big),\ n-1 < \alpha \le n,\ t \in [0, A] \qquad (2.1)$$

$$y(t) = \varphi(t)\ ,\ t < 0, \qquad (2.2)$$

$$y^{(i)}(0) = y_0^i \quad , i = 0,1,2, \dots, n-1 \qquad (2.3)$$

where ${}^{c}D_t^{\alpha}$ is Caputo fractional derivative of order α, $f: [0, A] \times R^2 \rightarrow R$ are continuous functions, t is the independent variable, $y(t)$ is the unknown function, $\varphi(t)$ is the delay function, $\tau \ge 0$ and y_0^i are given constants.

Our aim is to show the existence of a unique solution for (2.1) -(2.3) and its uniform stability by employing the Banach fixed point theorem, to show the asymptotic stability of the zero solution, we transform (2.1) into an integral equation and then using Banach fixed point theorem.

2.3 Preliminaries [Lax, 2002.]

In order to prove the existence and uniqueness theorem in addition to the stability theorem for the solution of delay differential equations of fractional order some basic concepts are needed and it will be given in this section.

Definition 2.1: ***(Normed space)***

Let V be a vector space. Then A norm on V over the field F is a function $\|.\|: V \rightarrow [0, \infty) \subseteq R$

which satisfies the following properties:

(i) $\|x + y\| \le \|x\| + \|y\|, (all\ x, y \in V)$, (Triangle inequality).

(ii) $\|\lambda x\| = |\lambda| \|x\|$, for all $\lambda \in F, x \in V$, (scaling property).

(iii) $\|x\| = 0 \iff x = 0$ (for $x \in V$).

A vector space V over F together with a chosen norm $\|.\|$ is called a normed space (over F) and we write $(V, \|.\|)$.

Definition 2.2: ***(complete)***

The space V is said to be complete if whenever $\{v_n\}$ is a Cauchy Sequence in V, that is $\|v_n - v_m\| \to 0$ as $n, m \to \infty$, then there exists a $v \in V$ such that $\|v_n - v\| \to 0$ as $n \to \infty$, that is $\{v_n\}$ is convergent sequence in V.

Definition 2.3: ***(Banach space)***

A Banach space is complete normed vector space.

Definition 2.4: ***(contraction mapping)***

Let $f: X \to X$ be a function on the metric space (X, d). Then f is a contraction if there exists a constant $0 \leq k < 1$ such that

$d\big(f(x), f(y)\big) \leq kd(x, y)$, for all $x, y \in X$.

Theorem 2.1: ***(Banach fixed point theorem)***

Let f be a contraction on the complete metric space X. Then f has a unique fixed point $x \in X$.

2.4 Main Results

The existence and stability of the solution of delay differential equations of fractional order (2.1) -(2.3) are given in the subsections (2.4.1) and (2.4.2) respectively.

2.4.1 Existence and uniqueness Theorem

In this section we shall prove the existence and uniqueness of the solution of the equation given by (2.1)-(2.3) as follows:

Let Y is the class of all continuous functions defined on R^+.

For $y \in Y$, the norm is defined by

$$\|y\| = \sup_{t \in R^+} \{e^{-Nt} |y(t)|\}, \quad y \in Y.$$

where $N \in R^+$ will be chosen later.

Definition 2.5: ***(Lipschitz condition)***

Let $f : R^+ \times R^2 \to R$ be a continuous function and satisfy the Lipschitz condition

$$\left|f\big(t, x(t), y(t-\tau)\big) - f\big(t, u(t), v(t-\tau)\big)\right| \leq h|x-u| + k|y-v| \qquad (2.4)$$

for all $x, y, u, v \in R$, where $h, k > 0$.

Lemma (2.1):

The function $y(t)$ is a solution of the problem (2.1) -(2.3) if and only if

$$y(t) = \begin{cases} \sum_{k=0}^{n-1} y_0^k \frac{t^k}{k!} + I_t^\alpha f(t, y(t), y(t-\tau)), & t > 0 \\ \phi(t), & t < 0 \end{cases} \qquad (2.5)$$

Proof:

For $t > 0$, applying the operator I_t^α to the both sides of the Eq. (2.1), leads to:

$$y(t) - \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{t^k}{k!} = I_t^\alpha f(t, y(t), y(t-\tau)).$$

Hence

$$y(t) = \sum_{k=0}^{n-1} y^{(k)}(0^+) \frac{t^k}{k!} + I_t^\alpha f(t, y(t), y(t-\tau)). \qquad (2.6)$$

Substituting eq.(2.3) in to eq. (2.6), we get

$$y(t) = \sum_{k=0}^{n-1} y_0^k \frac{t^k}{k!} + I_t^\alpha f(t, y(t), y(t-\tau)).$$

Theorem 2.2:

Let $f : R^+ \times R^2 \to R$ be a continuous function and satisfy the Lipschitz condition (2.4) and if

$\tau^{\alpha}[h + ke^{1}] < 1$, where $h, k > 0$.

Then the delay differential equations of fractional order (2.1) -(2.3) have a unique solution.

Proof:

For $t > 0$, integrating both sides of equation (2.1), we obtain

$$y(t) - \sum_{k=0}^{n-1} y^{(k)}(0^{+}) \frac{t^{k}}{k!} = I_{t}^{\alpha} f(t, y(t), y(t-\tau)).$$

Then

$$y(t) = \sum_{k=0}^{n-1} y^{(k)}(0^{+}) \frac{t^{k}}{k!} + I_{t}^{\alpha} f(t, y(t), y(t-\tau)). \quad (2.7)$$

Substituting eq. (2.3) into eq. (2.7), we get

$$y(t) = \sum_{k=0}^{n-1} y_{0}^{k} \frac{t^{k}}{k!} + I_{t}^{\alpha} f(t, y(t), y(t-\tau)). \quad (2.8)$$

Now, construct a mapping $F: Y \to Y$ be defined by

$$Fy = \sum_{k=0}^{n-1} y_{0}^{k} \frac{t^{k}}{k!} + I_{t}^{\alpha} f(t, y(t), y(t-\tau)).$$

Then

$$|Fx - Fy| = |I_{t}^{\alpha} f(t, x(t), x(t-\tau)) - I_{t}^{\alpha} f(t, y(t), y(t-\tau))|$$

$$= \left| \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(t, x(s), x(s-\tau)\big) ds \right.$$

$$\left. - \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(t, y(s), y(s-\tau)\big) ds \right|$$

$$= \left| \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{ f\big(t, x(s), x(s-\tau)\big) \right.$$

$$\left. - f(t, y(s), y(s-\tau)) \} ds \right|$$

$$\leq \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f\big(t, x(s), x(s-\tau)\big) - f\big(t, y(s), y(s-\tau)\big) \right| ds$$

$$\leq \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{ h|x(s) - y(s)| + k|x(s-\tau) - y(s-\tau)| \} ds$$

$$\leq h\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds$$

$$+k\int_0^\tau \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s-\tau)-\varphi(s-\tau)|\}ds$$

$$+k\int_\tau^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s-\tau)-y(s-\tau)|\}ds$$

$$\leq h\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds$$

$$+k\int_\tau^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s-\tau)-y(s-\tau)|\}ds$$

and

$$e^{-Nt}|Fx-Fy| \leq h\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)}e^{-Ns}|x(s)-y(s)|ds$$

$$+k\int_\tau^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau)}e^{-N(s-\tau)}|x(s-\tau)-y(s-\tau)|\}ds$$

$$\leq h\sup_{t\in R^+}\{e^{-Nt}|x(t)-y(t)|\}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)}ds$$

$$+k\int_\tau^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau)}e^{-N(s-\tau)}|x(s-\tau)-y(s-\tau)|\}ds$$

Set $\vartheta = s-\tau \rightarrow d\vartheta = ds$

at $s = t \qquad \rightarrow \vartheta = t-\tau$

at $s = 0 \qquad \rightarrow \vartheta = -\tau$

Hence

$$e^{-Nt}|Fx-Fy| \leq h\sup_{t\in R^+}\{e^{-Nt}|x(t)-y(t)|\}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)}ds$$

$$+k\int_{-\tau}^{t-\tau} \frac{(t-\vartheta-\tau)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\vartheta)}e^{-N\vartheta}|x(\vartheta)-y(\vartheta)|d\vartheta$$

Set $v = N(t-s) \rightarrow dv = -Nds$

at $s = t \qquad \rightarrow v = 0$

at $s = 0 \qquad \rightarrow v = Nt$

at $\vartheta = t-\tau \qquad \rightarrow v = 0$

at $\vartheta = -\tau \qquad \rightarrow v = Nt$

Hence

$$e^{-Nt}\,|Fx - Fy| \leq h \sup_{t\in R^+}\{e^{-Nt}|x(t) - y(t)|\}\,\frac{1}{N^\alpha}\int_0^{Nt}\frac{v^{\alpha-1}}{\Gamma(\alpha)}\; e^{-v}\,dv$$

$$+k\sup_{t\in R^+}\{e^{-Nt}|x(t) - y(t)|\}\,\frac{1}{N^\alpha}\int_0^{Nt}\frac{v}{\Gamma(\alpha)}^{\alpha-1}\; e^{-N(v+\tau)}\,dv$$

$$\leq \frac{h}{N^\alpha}\sup_{t\in R^+}\{e^{-Nt}|x(t) - y(t)|\}$$

$$+k\frac{e^{-N\tau}}{N^\alpha}\sup_{t\in R^+}\{e^{-Nt}|x(t) - y(t)|\}\int_0^{Nt}\frac{v}{\Gamma(\alpha)}^{\alpha-1}\; e^{-v}\,dv$$

$$\leq \frac{h}{N^\alpha}\sup_{t\in R^+}\{e^{-Nt}|x(t) - y(t)|\}$$

$$+k\frac{e^{-N\tau}}{N^\alpha}\sup_{t\in R^+}\{e^{-Nt}|x(t) - y(t)|\}$$

$$\leq \frac{h}{N^\alpha}\|x - y\| + \frac{ke^{-N\tau}}{N^\alpha}\|x - y\|$$

$$\leq \frac{h + ke^{-N\tau}}{N^\alpha}\|x - y\|.$$

Therefore,

$$\sup_{t\in R^+}\{e^{-Nt}|Fx - Fy|\} \leq \frac{h + ke^{-N\tau}}{N^\alpha}\|x - y\|$$

Now, choose N large enough such that $\frac{h+ke}{N^\alpha} < 1$

Let us choose $N = \frac{1}{\tau}$. We get

$$\sup_{t\in R^+}\{e^{-Nt}|Fx - Fy|\} \leq \tau^\alpha[h + ke^1]\|x - y\|$$

We have $\tau^\alpha[h + ke^1] < 1$. So, the map. $F: Y \rightarrow Y$ is a contraction and it has a fixed point $y = Fy$. By using Banach fixed point theorem there exists a unique $y \in Y$ which is a solution of the problem (2.1) –(2.3).

2.4.2 Stability of a unique solution for fractional order delay differential equations

In this section, we shall study the stability of the solution of delay differential equations of fractional order given by (2.1) -(2.3).

Given that $\tilde{y}(t)$ is the solution of delay differential equations of fractional order

$$(\tilde{G})\quad \begin{cases} {}^{c}D_t^{\alpha} y(t) = f\big(t, y(t), y(t-\tau)\big),\ n-1<\alpha\le n,\ t>0 \\ y(t)=\tilde{\varphi}(t)\ ,\ t<0, \\ y^{(i)}(0)=\tilde{y}_0^{i}\quad ,i=0,1,2,\dots,n-1 \end{cases}$$

Definition 2.6 [Zhenghui, 2013]:

The solution of delay differential equations of fractional order (2.1)-(2.3) is stable if for any $\epsilon>0$, there exists $\delta>0$ such that for any two solutions $y(t)$ and $\tilde{y}(t)$ of delay differential equations of fractional order (2.1)-(2.3) and $\tilde{G}$ respectively, one has $\|\varphi(t)-\tilde{\varphi}(t)\|\le\delta$, then $\|y(t)-\tilde{y}(t)\|\le\epsilon$ for all $t\ge 0$.

Definition 2.7 [Zhenghui, 2013]:

System (2.1) -(2.3) is said to be asymptotically stable if the solution $y(t)$ tends to zero as $t\to\infty$.

Theorem 2.3:

The solution of delay differential equations of fractional order (2.1)-(2.3) is uniformly stable.

Proof:

Let $y(t)$ and $\tilde{y}(t)$ be the solutions of delay differential equations of fractional order (2.1)-(2.3) and $\tilde{G}$ respectively, then for all $t>0$, from equation (2.4), we have

$$|y(t) - \tilde{y}(t)| = |I_t^{\alpha} f(t, y(t), y(t-\tau)) - I_t^{\alpha} f(t, \tilde{y}(t), \tilde{y}(t-\tau))|$$

$$= \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(t, y(s), y(s-\tau)\big)ds\right.$$

$$\left.- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(t, \tilde{y}(s), \tilde{y}(s-\tau)\big)ds\right|$$

$$= \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{f\big(t, y(s), y(s-\tau)\big) - f\big(t, \tilde{y}(s), \tilde{y}(s-\tau)\big)\}ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(t, y(s), y(s-\tau)\big) - f\big(t, \tilde{y}(s), \tilde{y}(s-\tau)\big)\right|ds$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{h|y(s) - \tilde{y}(s)| + k|y(s-\tau) - \tilde{y}(s-\tau)|\}ds$$

$$\leq h \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s) - \tilde{y}(s)|ds$$

$$+k \int_0^{\tau} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s-\tau) - \tilde{\varphi}(s-\tau)|\}ds$$

$$+k \int_{\tau}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s-\tau) - \tilde{y}(s-\tau)|\}ds$$

and

$$e^{-Nt} |y(t) - \tilde{y}(t)|$$

$$\leq h \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns}|y(t) - \tilde{y}(t)|ds$$

$$+k \int_0^{\tau} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau)} e^{-N(s-\tau)}|\varphi(s-\tau) - \tilde{\varphi}(s-\tau)|ds$$

$$+k \int_{\tau}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau)} e^{-N(s-\tau)}|y(s-\tau) - \tilde{y}(s-\tau)|ds$$

Set $\vartheta = s - \tau \quad \rightarrow \; d\vartheta = ds$

at $s = t \qquad \rightarrow \vartheta = t - \tau$

at $s = 0 \qquad \rightarrow \vartheta = -\tau$

at $s = \tau \qquad \rightarrow \vartheta = 0$

Hence

$e^{-Nt}\,|y(t) - \tilde{y}(t)|$

$$\leq h\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\; e^{-N(t-s)}\,e^{-Ns}|y(t)\;-\tilde{y}(t)|ds$$

$$+k\int_{-\tau}^{0}\frac{(t-\vartheta-\tau)^{\alpha-1}}{\Gamma(\alpha)}\; e^{-N(t-\vartheta)}\,e^{-N\vartheta}|\varphi(\vartheta)-\tilde{\varphi}(\vartheta)|d\vartheta$$

$$+k\int_{0}^{t-\tau}\frac{(t-\vartheta-\tau)^{\alpha-1}}{\Gamma(\alpha)}\; e^{-N(t-\vartheta)}\,e^{-N\vartheta}|y(\vartheta)-\tilde{y}(\vartheta)|d\vartheta$$

$$\leq h\sup_{t\in R^{+}}\{e^{-Nt}|y(t)\;-\tilde{y}(t)|\}\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\; e^{-N(t-s)}\,du$$

$$+k\sup_{t\in R^{+}}\{e^{-Nt}|\varphi(t)-\tilde{\varphi}(t)|\}\int_{-\tau}^{0}\frac{(t-\vartheta-\tau)^{\alpha-1}}{\Gamma(\alpha)}\; e^{-N(t-\vartheta)}\,d\vartheta$$

$$+k\sup_{t\in R^{+}}\{e^{-Nt}|y(t)\;-\tilde{y}(t)|\}\int_{0}^{t-\tau}\frac{(t-\vartheta-\tau)^{\alpha-1}}{\Gamma(\alpha)}\; e^{-N(t-\vartheta)}\,d\vartheta$$

Set $v = N(t-s) \;\rightarrow\; dv = -Nds$

at $s = t \qquad \rightarrow v = 0$

at $s = 0 \qquad \rightarrow v = Nt$

at $\vartheta = t-\tau \qquad \rightarrow v = 0$

at $\vartheta = -\tau \qquad \rightarrow v = Nt$

at $\vartheta = 0 \qquad \rightarrow v = N(t-\tau)$

Hence

$e^{-Nt}\,|y(t) - \tilde{y}(t)|$

$$\leq h\sup_{t\in R^{+}}\{e^{-Nt}|y(t)\;-\tilde{y}(t)|\}\,\frac{1}{N^{\alpha}}\int_{0}^{Nt}\frac{v^{\alpha-1}}{\Gamma(\alpha)}e^{-v}\,dv$$

$$+k\sup_{t\in R^+}\{e^{-Nt}|\varphi(t)-\tilde{\varphi}(t)|\}\, e^{-N\tau}\frac{1}{N^\alpha}\int_{N(t-\tau)}^{Nt}\frac{v^{\alpha-1}}{\Gamma(\alpha)}e^{-v}\,dv$$

$$+k\sup_{t\in R^+}\{e^{-Nt}|y(t)\ -\tilde{y}(t)|\}\, e^{-N\tau}\frac{1}{N^\alpha}\int_{0}^{N(t-\tau)}\frac{v^{\alpha-1}}{\Gamma(\alpha)}e^{-v}\,dv$$

$$\leq \frac{h}{N^\alpha}\sup_{t\in R^+}\{e^{-Nt}|y(t)\ -\tilde{y}(t)|\}\ +\frac{ke^{-N\tau}}{N^\alpha}\sup_{t\in R^+}\{e^{-Nt}|y(t)-\tilde{y}(t)|\}$$

$$+\frac{ke^{-N\tau}}{N^\alpha}\sup_{t\in R^+}\{e^{-Nt}|\varphi(t)-\tilde{\varphi}(t)|\}$$

$$\leq \frac{h}{N^\alpha}\|y\ -\tilde{y}\|+\frac{ke^{-N\tau}}{N^\alpha}\|y\ -\tilde{y}\|+\frac{ke^{-N\tau}}{N^\alpha}\|\varphi-\tilde{\varphi}\|$$

$$\leq \frac{h+ke^{-N\tau}}{N^\alpha}\|y\ -\tilde{y}\|+\frac{ke^{-N\tau}}{N^\alpha}\|\varphi-\tilde{\varphi}\|$$

Then

$$\sup_{t\in R^+}\{e^{-Nt}|y(t)\ -\tilde{y}(t)|\}\ \leq \frac{h+ke^{-N\tau}}{N^\alpha}\|y\ -\tilde{y}\|+\frac{ke^{-N\tau}}{N^\alpha}\|\varphi-\tilde{\varphi}\|$$

It follows that

$$\left[1-\frac{h+ke^{-N\tau}}{N^\alpha}\right]\|y\ -\tilde{y}\|\leq\frac{ke^{-N\tau}}{N^\alpha}\|\varphi-\tilde{\varphi}\|$$

we choose $N\ =\frac{1}{\tau}$, and we have

$$\left[1-\tau^\alpha\left[h+ke\ \ \right]\right]\|y\ -\tilde{y}\|\leq\tau^\alpha ke\ \ \|\varphi-\tilde{\varphi}\|$$

therefore, for $\epsilon>0$, we can find

$$\delta=\left[\tau^\alpha ke\ \ \right]^{-1}\left[1-\tau^\alpha\left[h+ke\ \ \right]\right]\|y\ -\tilde{y}\|$$

$$\delta=\left[\tau^\alpha ke\ \ \right]^{-1}\left[1-\tau^\alpha\left[h+ke\ \ \right]\right]\epsilon>0$$

Such that if $\|\varphi(t)-\tilde{\varphi}(t)\|\leq\delta$, then $\|y(t)-\tilde{y}(t)\|\leq\epsilon$ for all $t\geq 0$, which proves that the solution $y(t)$ of problem (2.1) -(2.3) is uniformly stable.

CHAPTER THREE

A modified Approach for Solving Delay Differential Equations of Fractional Order Using Generalized Hat Functions

3.1 Introduction

In this chapter generalized Hat functions operational matrices will be presented together with the method of steps in order to find the numerical solution of delay differential equations of fractional order.

This chapter consists of five sections, in section 3.2, a review of the generalized Hat functions is described, while in section 3.3, operational matrices of the integration for generalized Hat functions is presented. In section 3.4, we will have focused on the numerical solution of the delay differential equation of fractional order using the method of steps with the operational matrices of integration for generalized Hat functions. Finally, in section 3.5, some illustrative examples have been introduced in order to show the accuracy and efficiency of the proposed method.

3.2 Generalized Hat functions, [Tripathi, 2013]

Usually the Hat functions are defined on the domain $[0,1]$. These are continuous functions with shape of Hats, when plotted on two dimensional planes. We consider the more general case for which the domain of definition is $[0, A]$. The interval $[0, A]$ is divided into n equidistant subintervals,$[ih, (i+1)h]$ of equal lengths h where $h = \frac{A}{n}$ .The generalized Hat function's family of first $(n + 1)$ Hat functions is defined as follows:

$$\psi_0(t) = \begin{cases} \frac{h-t}{h}, & 0 \le t < h \\ 0, & otherwise \end{cases} \tag{3.1}$$

$$\psi_i(t) = \begin{cases} \frac{t-(i-1)h}{h}, & (i-1)h \le t < h \\ \frac{(i+1)h-t}{h}, & ih \le t < (i+1)h,\ i = 1,2,\dots,n-1 \\ 0, & otherwise \end{cases} \tag{3.2}$$

$$\psi_n(t) = \begin{cases} \frac{t-(A-h)}{h}, & A-h \le t \le A \\ 0, & otherwise \end{cases} \tag{3.3}$$

According to the definition of Hat functions:

$$\psi_i(jh) = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases} \tag{3.4}$$

and

$$\psi_i(t)\psi_j(t) = 0, \quad |i-j| \ge 2 \tag{3.5}$$

and

$$\sum_{i=0}^{n} \psi_i(t) = 1$$

3.2.1 Function Approximation

An arbitrary function $f \in L_2\,[0, A]$ is approximated in vector form as:

$$f(t) = \sum_{i=0}^{n} f_i\psi_i(t) = F_{n+1}^T\psi_{n+1}(t) = \psi_{n+1}^T(t)F_{n+1}, \tag{3.6}$$

where

$$F_{n+1} \triangleq [f_0, f_1, f_2, \ldots, f_n]^T, \tag{3.7}$$

and

$$\Psi_{n+1}(t) \triangleq [\psi_0(t), \psi_1(t), \psi_2(t), \ldots, \psi_n(t)]^T. \tag{3.8}$$

The important aspect of using generalized Hat functions in the approximation of function $f(t)$, lies in the fact that the coefficients f_i in the eq. (3.6), are given by

$$f_i = f(ih), \; i = 0,1,2, \ldots, n. \tag{3.9}$$

From relation (3.5), we have:

$$\Psi(t)\Psi(t)^T = \begin{bmatrix} \psi_0^2(t) & \psi_0(t)\psi_1(t) & & & & \\ \psi_0(t)\psi_1(t) & \psi_0^2(t) & \psi_0(t)\psi_1(t) & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \psi_0(t)\psi_1(t) \\ & & & & \psi_0(t)\psi_1(t) & \psi_0^2(t) \end{bmatrix} \tag{3.10}$$

3.3 Operational Matrices of the Integration for generalized Hat Functions

The integer order and fractional order operational matrices of integration for generalized Hat functions is given in the subsections (3.3.1) and (3.3.2) respectively.

3.3.1 Integer Order generalized Hat Functions Operational Matrix of Integration

Since $\int_0^t \psi_i(\tau)d\tau \in L_2[0, A]$, eq. (3.6) is used to approximate it in the terms of generalized Hat functions as

$$\int_0^t \psi_i(\tau)d\tau \simeq \sum_{j=0}^{n} a_{ij}\, \psi_j(t)\ , i = 0,1,2, \dots, n\ . \qquad (3.11)$$

Using eq. (3.9), we calculate the coefficients a_{ij} as

$$a_{ij} = \int_0^{ih} \psi_i(\tau)d\tau, \quad j = 0,1,2, \dots, n. \qquad (3.12)$$

The coefficients a_{ij} will form a $(n+1)\times(n+1)$ matrix P_{n+1} with $(i+1, j+1)^{th}$ entry as a_{ij} , for $i = 0,1,2, \dots, n$, $j = 0,1,2, \dots, n$. Using the values of a_{ij}'s from eq. (3.12), we obtain the matrix P_{n+1} as:

$$P_{n+1} = \left(\frac{h}{2}\right) \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(n+1)\times(n+1)} \qquad (3.13)$$

The matrix P_{n+1} is called the integer order Hat functions operational matrix of integration.

It plays a pivotal role in the determination of $\int_0^t f(\tau)d\tau$ for an arbitrary $f \in L_2[0, A]$.

With the help of eqs. (3.8) and (3.11), we have

$$\int_0^t \Psi_{n+1}(\tau)d\tau = P_{n+1}\Psi_{n+1}(\mathrm{t}). \qquad (3.14)$$

3.3.2 Block Pulse Functions [Yi, 2013]

The block pulse functions are form a complete set of orthogonal functions [Wang, 1983] which defined on the interval $[0, A)$ by:

$$\varphi_i(t) = \begin{cases} 1, & ih \le t < (i+1)h \\ 0, & otherwise, \end{cases}$$

where , $i = 1,2,\dots,n-1$ and $h = \frac{A}{n}$

The functions φ_i's are disjoint and have compact support $[ih,(i+1)h]$.The following properties of block pulse functions will be used in the sequel.

$$\varphi_i(t)\varphi_j(t) = \begin{cases} \varphi_i(\mathrm{t}), & i = j \\ 0, & i \ne j \end{cases}$$

$$\int_0^A \varphi_i(\tau)\,\varphi_j(\tau)d\tau = \begin{cases} \frac{A}{n}, & i = j \\ 0, & i \ne j \end{cases}$$

Remark 3.1:

Each of the Hat functions ψ_i's may be expanded into n-block pulse functions according as:

$$\psi_i(t) \simeq \sum_{j=0}^{n-1} \Upsilon_{ij}\,\varphi_j(t),\ i = 0,1,2,\dots,n\ , \tag{3.15}$$

where

$$\Upsilon_{ij} = \frac{1}{h}\int_0^A \psi_i(\tau)\varphi_j(\tau)d\tau,$$

Thus

$$\Psi_{n+1}(t) \triangleq [\psi_0(t),\psi_1(t),\psi_2(t),\dots,\psi_n(t)]^{\,T} = \Omega_n\Phi_n, \tag{3.16}$$

where

$$\Phi_n(t) \triangleq [\varphi_0(t),\varphi_1(t),\varphi_2(t),\dots,\varphi_{n-1}(t)]^{\,T}$$

and Ω_n is a $(n+1)\times(n)$ matrix whose $(i+1, j+1)^{th}$ entry as Υ_{ij}, for $i = 0,1,2,\dots,n$, $j = 0,1,2,\dots,n-1$. Calculating the values of Υ_{ij}, the general shape of the matrix Ω_n is given by

$$\Omega_n = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}_{(n+1)\times(n)}$$

Kilicman and Al Zhour, [Kilicman, 2007] have given the block pulse operational matrix of the fractional integration F^{α} as follows:

$$(I_t^{\alpha}\Phi_n)(t) \approx F^{\alpha}\Phi_n(t) \tag{3.17}$$

where

$$F^{\alpha} = \left(\frac{A}{n+1}\right)^{\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 0 & \zeta_1 & \zeta_2 & & \zeta_n \\ 0 & 1 & \zeta_1 & \cdots & \zeta_{n-1} \\ 0 & 0 & 1 & & \zeta_{n-2} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

where $\zeta_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, $k = 1,2,\dots,n-1$.

Next, we derive the generalized Hat functions operational matrix of the fractional integration. Let

$$\int_0^t \psi_{n+1}(\tau)d\tau = P_{n+1}^{\alpha}\Psi_{n+1}(\mathrm{t}). \tag{3.18}$$

where matrix P_{n+1}^{α} is called the generalized Hat functions operational matrix of the fractional integration.

Using eq.'s (3.16) and (3.17), we have:

$$(I_t^{\alpha}\psi_i)(t) \approx (I_t^{\alpha}\Omega_n\Phi_n)(t)$$
$$\approx \Omega_n(I_t^{\alpha}\Phi_n)(t)$$
$$\approx \Omega_n F^{\alpha}\Phi_n(t) \tag{3.19}$$

From eq.'s (3.18) and (3.19), we get:

$$P_{n+1}^{\alpha}\Psi_{n+1}(\mathrm{t}) = P_{n+1}^{\alpha}\Omega_n\Phi_n$$
$$= \Omega_n F^{\alpha}\Phi_n(t)$$

Then, the generalized Hat functions operational matrix of the fractional integration P_{n+1}^{α} is given by:

$$P_{n+1}^{\alpha} = \Omega_n F^{\alpha}{\Omega_n}^{-1} \tag{3.20}$$

3.3.3 Fractional Order Generalized Hat Functions Operational Matrix of Integration

When we deal with the differential equations of fractional order, the operational matrices of fractional order integral are plays an important role. In this subsection, we shall present the fractional order of generalized Hat functions operational matrices of integration. Several definitions of fractional order integration have been proposed [Loverro, 2004]. We formulate problem in terms of the Riemann- Liouville fractional order integration, which is defined as

$$(I_t^{\alpha}f)(\mathrm{t}) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau$$
$$= \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t),\ 0 \le t < A, \tag{3.21}$$

where $t^{\alpha-1} * f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$. If eq. (3.6) used to approximate $f(t)$ as $f(t)$=$C_{n+1}^{T}\Psi_{n+1}(t)$ then the Riemann – Liouville fractional integral of $f(t)$ becomes

$$(I_t^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t) \simeq C_{n+1}^T \frac{1}{\Gamma(\alpha)} \{t^{\alpha-1} * \Psi_{n+1}(t)\}$$

Hence

$$(I_t^{\alpha} f)(t) = C_{n+1}^T (I_t^{\alpha} \Psi_{n+1})(t) \tag{3.22}$$

Where

$(I_t^{\alpha} \Psi_{n+1})(t) \triangleq [(I_t^{\alpha}\psi_0)(t), (I_t^{\alpha}\psi_1)(t),(I_t^{\alpha}\psi_2)(t),\ldots,(I_t^{\alpha}\psi_n)(t)]^T$. (3.23)

The above expression computed through computation of $\frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * \psi_i(t)\}$, as is enables us to calculate the Riemann- Liouville fractional integral of $f(t)$, by using linearity property of the integral transform.

Applying the definition of the convolution for $\psi_i(t)$ from eq. (3.21) we have

$$(I_t^{\alpha}\psi_{n+1})(\mathrm{t}) = \frac{1}{\Gamma(\alpha)}\{t^{\alpha-1} * \psi_i(t)\}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\psi_i(t)d\tau \tag{3.24}$$

If we expand $(I_t^{\alpha}\psi_i)(\mathrm{t})$ in terms of generalized Hat functions as

$$(I_t^{\alpha}\psi_i)(\mathrm{t}) \simeq \sum_{j=0}^{n} b_{ij}\,\psi_j(t),\ i = 0,1,2,\ldots,n. \tag{3.25}$$

Where the coefficient b_{ij} is value of $(I_t^{\alpha}\psi_i)(\mathrm{t})$ at j^{th} node point (jh), then from eq. (3.24) we have

$$b_{ij} = (I_t^{\alpha}\psi_i)(jh) = \frac{1}{\Gamma(\alpha)}\int_0^{jh}(jh-\tau)^{\alpha-1}\psi_i(\tau)d\tau, j = 0,1,2,\ldots,n. \tag{3.26}$$

From eq.'s (3.1) – (3.3) and (3.26) we get

$$b_{0j} = \begin{cases} 0\,, & j = 0\,, \\ \frac{h^{\alpha}}{\Gamma(\alpha+2)}\left((j-1)^{\alpha+1} + j^{\alpha}(1-j+\alpha)\right), & j = 1,2,3,\ldots,n \end{cases} \tag{3.27}$$

and for $i = 1,2,3,\ldots,n,\quad j = 1,2,3,\ldots,n,$

$$b_{ij}=\begin{cases} 0\,, & i>j,\\ \frac{h^{\alpha}}{\Gamma(\alpha+2)}, & i=j,\\ \frac{h^{\alpha}}{\Gamma(\alpha+2)}\left((j-i+1)^{\alpha+1}-2(j-i)^{\alpha+1}+(j-i-1)^{\alpha+1}\right) & ,i<j. \end{cases} \tag{3.28}$$

Thus the fractional order generalized Hat functions operational matrix of integration P_{n+1}^{α} is $(n+1)\times(n+1)$ dimensional matrix whose $(i+1,j+1)^{th}$ entry as b_{ij} . It is computed using the values of b_{ij} from eq.'s (3.27) and (3.28), and is given by

$$P_{n+1}^{\alpha}=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\begin{bmatrix} 0 & \zeta_1 & \zeta_2 & & \zeta_n \\ 0 & 1 & \xi_1 & \cdots & \xi_{n-1} \\ 0 & 0 & 1 & & \xi_{n-2} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{(n+1)\times(n+1)} \tag{3.29}$$

where $\zeta_k=k^{\alpha}(\alpha-k+1)+(k-1)^{\alpha+1},\ k=1,2,\dots,n.$
and $\xi_k=(k+1)^{\alpha+1}-2k^{\alpha+1}+(k-1)^{\alpha+1}.\ \ k=1,2,\dots,n-1.$
As a special case, when $\alpha=1$, P_{n+1}^{α} becomes the same as P_{n+1}.

This establishes that fractional order of generalized Hat functions operational matrix of integration P_{n+1} is a generalization of integer order generalized Hat functions operational matrix of integration P_{n+1}.

3.4 The Approach

In this section, we shall approximate solution of the following delay differential equations of fractional order:

$$^{c}D_t^{\alpha}y(t)=F(t,y(t),y(\phi(t))),\ n-1<\alpha\leq n,\ t>0 \tag{3.30}$$

$$y(t) = \psi(t)\ ,\ t \in [-\tau, 0], \tag{3.31}$$

$$y^{(i)}(0) = y_0^{(i)} \quad , i = 0,1,2, \dots , n-1 \tag{3.32}$$

Where ${}^{c}D_t^{\alpha}$ is Caputo fractional derivative of order α, F is a nonlinear operator, t is the independent variable, $y(t)$ is the unknown function, $\phi(t)$ is the delay function $\psi(t)$ is given functions and $y_0^{(i)}$ are given constants.

First we convert the delay differential equation of fractional order to fractional non-delay differential equation by applying the method of steps, as

$${}^{c}D_t^{\alpha} y(t) = F(t, y(t), \psi(\phi(t))),\ n-1 < \alpha \leq n,\ t > 0 \tag{3.33}$$

$$y^{(i)}(0) = y_0^{(i)} \quad , i = 0,1,2, \dots , n-1\ . \tag{3.34}$$

Now in order to solve eq.'s (3.30) and (3.32) by using the operational matrices of generalized Hat functions, we approximate ${}^{c}D_t^{\alpha} y(t)$ and $y(t)$ in terms of generalized Hat functions as follows

$$(\ {}^{c}D_t^{\alpha} y)(t) = C_{n+1}^{T} \Psi_{n+1}(t). \tag{3.35}$$

And upon operating I_t^{α} to the both sides of equation (3.35) leads to

$$y(t) = C_{n+1}^{T} P_{n+1}^{\alpha} \Psi_{n+1}(t) + \sum_{k=0}^{n-1} y^{(k)}(0^{+}) \frac{t^k}{k!}\ , \tag{3.36}$$

where

$$\Psi_{n+1}(t) \triangleq [\psi_0(t), \psi_1(t), \psi_2(t), \dots, \psi_n(t)]^{T},$$

and

$$C_{n+1}(\mathrm{t}) \triangleq [c_0, c_1, c_2, \dots, c_n]^{T}.$$

Hence

$$F\Big(t, y(t), y\big(\phi(t)\big)\Big)$$

$$= F(t, C_{n+1}^{T} P_{n+1}^{\alpha} \Psi_{n+1}(t)\ +\ \sum_{k=0}^{n-1} y^{k}(0) \frac{t^k}{k!}, \psi(\phi(t))). \tag{3.37}$$

Substituting eq.'s (3.35) and (3.37) into eq. (3.33) gives

$$C_{n+1}^{T}\Psi_{n+1}(t) = F(t, C_{n+1}^{T}P_{n+1}^{\alpha}\Psi_{n+1}(t) + \sum_{k=0}^{n-1} y^{k}(0)\frac{t^{k}}{k!}, \psi(\phi(t))) \qquad (3.38)$$

Also, by substituting eq. (3.11) into eq. (3.34), we get

$$y^{(i)}(0) = C_{n+1}^{T}\Psi_{n+1}(0) = y_0^{(i)} \quad , i = 0,1,2, \dots, n-1 \qquad (3.39)$$

Solving eq.'s (3.38)- (3.39), the coefficients C_{n+1}^{T} will be obtained. Then using eq. (3.36), one can get the output response $y(t)$.

3.5 Illustrative Examples

In this section, we shall solve linear and nonlinear delay differential equations of fractional order by the proposed method given in section (3.4), and compare the results that we have been obtained with the existing methods such as Chebyshev wavelets method [Saeed, 2015] and with the exact solution.

we refer $\boldsymbol{y_{hat}}$ to represent the solution by generalized Hat functions, $\boldsymbol{y_{ch}}$ to represent the solution by Chebyshev wavelets method and $\boldsymbol{y_{exact}}$ to represent the exact solution.

Example (3.1):

Consider the delay differential equations of fractional order with nonlinear delay function

$$^{c}D_t^{\alpha}y(t) = 1 - 2y^2\left(\frac{t}{2}\right), \quad 0 < \alpha \le 1, \; 0 < t \le 1 \qquad (3.40)$$

$$y(t) = \sin(t), \quad -1 \le t \le 0 \qquad (3.41)$$

$$y(0) = 0 \qquad (3.42)$$

The exact solution, when $\alpha = 1$, is $y(t) = \sin(t)$.

First we convert the delay differential equation of fractional order to fractional order non-delay differential equation by applying the method of steps, as

$$ {}^{c}D_t^{\alpha} y(t) = 1\text{-}2sin^2\left(\frac{t}{2}\right), 0 < \alpha \leq 1,\ 0 < t \leq 1 \tag{3.43}$$

$$ y(0) = 0\ . \tag{3.44}$$

Now we approximate ${}^{c}D_t^{\alpha} y(t)$ in eq. (3.43), in terms of generalized Hat functions as follows

$$ (\ {}^{c}D_t^{\alpha} y)(t) = C_{n+1}^T \Psi_{n+1}(t) \tag{3.45}$$

Hence

$$ y(t) = C_{n+1}^T P_{n+1}^{\alpha} \Psi_{n+1}(t) \tag{3.46}$$

Also writing the term $1 - 2sin^2\left(\frac{t}{2}\right)$ in eq. (3.43) in terms of generalized Hat functions leads to

$$ 1\text{-}2sin^2\left(\frac{t}{2}\right) = F_{n+1}^T \Psi_{n+1}(t)\,, \tag{3.47}$$

where

$$ F_{n+1} \triangleq [f_0, f_1, f_2, \ldots, f_n]^{\ T}, $$

and

$$ f_i = 1 - 2sin^2\left(\frac{ih}{2}\right),\ \ i = 0,1,2,\ldots,n. $$

Substituting eq.'s (3.45) and (3.47) into eq. (3.43), we have

$$ C_{n+1}^T \Psi_{n+1}(t) = F_{n+1}^T \Psi_{n+1}(t), \tag{3.48}$$

which implies that

$$ C_{n+1}^T = F_{n+1}^T \tag{3.49}$$

Then using eq. (3.46), one can get the output response $y(t)$.

For $n = 8$, it seems from table (3.1) that the results obtained from the proposed method when $\alpha = 1$ provides better results as compared with the Chebyshev wavelet method and with the exact solution.

(Table 3.1)

Comparison of the approximate solution of example (3.1) using the proposed method and Chebyshev wavelet method when $\alpha = 1$ with the exact solution.

t	y_{ch} $\alpha = 1$	y_{hat} $\alpha = 1$	y_{exact} $\alpha = 1$
0	0	0	0
0.125	0.124	0.124	0.124
0.250	0.246	0.247	0.247
0.375	0.355	0.365	0.366
0.500	0.464	0.478	0.479
0.625	0.581	0.584	0.585
0.750	0.682	0.680	0.681
0.875	0.755	0.766	0.767
1	0.846	0.840	0.841

Following figure (3.1) represent the approximate solution of example (3.1) using the proposed method for different values of α and with the exact solution when $\alpha = 1$.

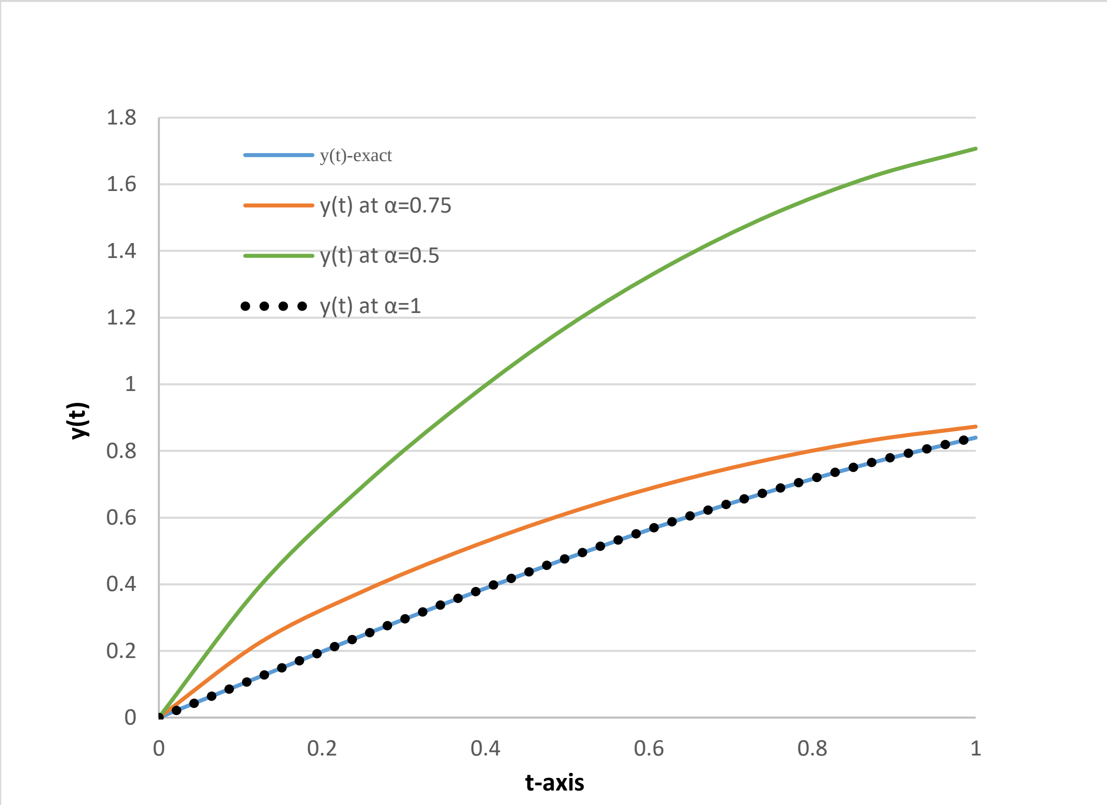

Fig. 3.1: Solution by the proposed method at different values of α compared with the exact solutions at $\alpha = 1$.

Example (3.2):

Consider the delay differential equation of fractional order

$$ {}^{c}D_{t}^{\alpha} y(t) - y\left(\frac{t}{2}\right) = 0, \quad 0 < \alpha \leq 1, \; 0 < t \leq 1 \tag{3.50}$$

$$ y(t) = 1 + t, \; -1 \leq t \leq 0 \tag{3.51}$$

$$ y(0) = 1 \tag{3.52}$$

The exact solution is $y(t) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{\frac{1}{2}k(k-1)}}{k!} t^k$.

First we convert the delay differential equation of fractional order to fractional order non-delay differential equation by applying the method of steps, as:

$$ {}^{c}D_{t}^{\alpha} y(t) = 1 + \frac{t}{2}, \; 0 < \alpha \leq 1, \; 0 < t \leq 1 \tag{3.53}$$

$$y(0) = 1 \tag{3.54}$$

Now we approximate ${}^{c}D_t^{\alpha} y(t)$ in eq. (3.53), in terms of generalized Hat functions as follows

$$({}^{c}D_t^{\alpha} y)(t) = C_{n+1}^T \Psi_{n+1}(t) \tag{3.55}$$

Hence

$$y(t) = C_{n+1}^T P_{n+1}^{\alpha} \Psi_{n+1}(t) + 1 \tag{3.56}$$

Also writing the term $1 + \frac{t}{2}$ in eq. (3.53) in terms of generalized Hat functions leads to

$$1 + \frac{t}{2} = F_{n+1}^T \Psi_{n+1}(t), \tag{3.57}$$

Where

$$F_{n+1} \triangleq [f_0, f_1, f_2, \ldots, f_n]^T,$$

and

$$f_i = 1 + \frac{ih}{2}, \quad i = 0,1,2, \ldots, n.$$

Substituting eq.'s (3.55) and (3.57) into eq. (3.53), we have

$$C_{n+1}^T \Psi_{n+1}(t) = F_{n+1}^T \Psi_{n+1}(t), \tag{3.58}$$

which implies that

$$C_{n+1}^T = F_{n+1}^T \tag{3.59}$$

Then using eq. (3.56), one can get the output response $y(t)$.

For $n = 8$, it seems from table (3.2) that the results obtained from the proposed method when $\alpha = 1$ provides better results as compared with the Chebyshev wavelet method and with the exact solution.

(Table 3.2)

Comparison of the approximate solution of example (3.2) using the proposed method and Chebyshev wavelet method when $\alpha = 1$ with the exact solution.

t	y_{ch} $\alpha = 1$	y_{hat} $\alpha = 1$	y_{exact} $\alpha = 1$
0	1	1	1
0.125	1.13	1.13	1.12
0.250	1.28	1.26	1.26
0.375	1.44	1.41	1.41
0.500	1.62	1.56	1.56
0.625	1.82	1.72	1.72
0.750	2.03	1.89	1.90
0.875	2.26	2.06	2.08
1	2.50	2.25	2.27

Following figure (3.2) represent the approximate solution of example (3.2) using the proposed method for different values of α and with the exact solution when $\alpha = 1$.

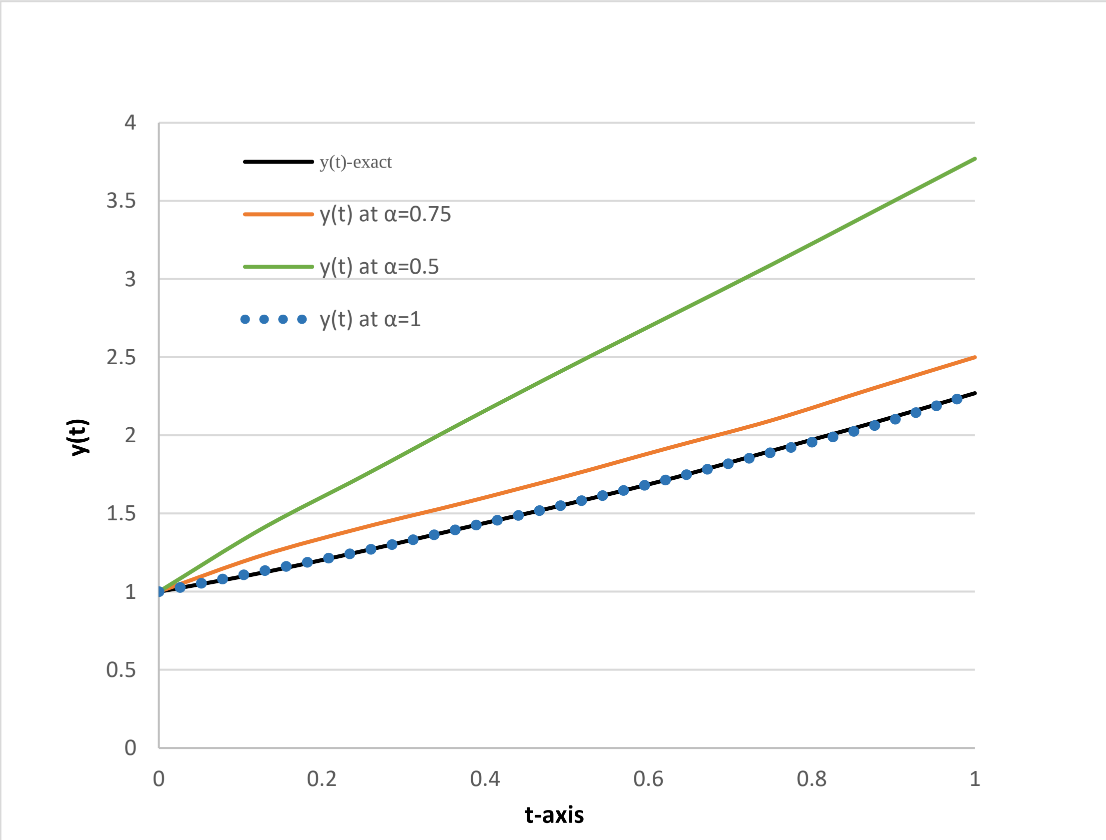

Fig. 3.2: Solution by the proposed method at different values of α compared with the exact solutions at $\alpha = 1$.

Example 3.3:

Consider the following delay differential equation of fractional order of second order

$$ {}^{c}D_t^{\alpha} y(t) = \frac{3}{4} y(t) + y\left(\frac{t}{2}\right) + 2 - t^2, \ 1 < \ \alpha \ \le 2, 0 < t \le 1 \qquad (3.60) $$

$$ y(t) = t^2, \ -1 \le \ t \ \le 0 \qquad (3.61) $$

$$ y^{(i)}(0) = 0 \quad , i = 0,1 \qquad (3.62) $$

The exact solution of the above equation when $\alpha = 2$, is $y(t) = t^2$.

First we convert the delay differential equation of fractional order to the fractional order non-delay differential equation by applying the method of steps, as

$${}^{c}D_t^{\alpha}y(t) - \frac{3}{4}y(t) = 2 - \frac{3\,t^2}{4},\ 1 < \ \alpha\ \leq 2,\ 0 < t \leq 1 \qquad (3.63)$$

$$y^{(i)}(0) = 0 \quad , i = 0{,}1 \qquad (3.64)$$

Now we approximate ${}^{c}D_t^{\alpha}y(t)$ in eq. (3.63), in terms of generalized Hat functions as follows

$$(\ {}^{c}D_t^{\alpha}y)(t) = C_{n+1}^{T}\Psi_{n+1}(t) \qquad (3.65)$$

Hence

$$y(t) = C_{n+1}^{T}P_{n+1}^{\alpha}\Psi_{n+1}(t) \qquad (3.66)$$

Also writing the term $2 - \frac{3t^2}{4}$ in eq. (3.63) in terms of generalized Hat functions leads to

$$2 - \frac{3t^2}{4} = F_{n+1}^{T}\Psi_{n+1}(t)\,, \qquad (3.67)$$

where

$$F_{n+1} \triangleq [f_0, f_1, f_2, \dots, f_n]^{\,T},$$

and

$$f_i = 2 - \frac{(ih)^2}{4},\ \ i = 0{,}1{,}2, \dots , n.$$

Substituting eq.'s (3.65) - (3.67) into eq. (3.63), we have

$$C_{n+1}^{T}\Psi_{n+1}(t) - \frac{3}{4}\,C_{n+1}^{T}P_{n+1}^{\alpha}\Psi_{n+1}(t) = F_{n+1}^{T}\Psi_{n+1}(t), \qquad (3.68)$$

Then using eq. (3.66), one can get the output response $y(t)$.

For $n = 8$, it seems from table (3.3) that the results obtained from the proposed method when $\alpha = 2$ provides better results as compared with the Chebyshev wavelet method and with the exact solution.

(Table 3.3)

Comparison of the approximate solution of example (3.3) using the proposed method and Chebyshev wavelet method when $\alpha = 2$ with the exact solution.

t	y_{ch} $\alpha = 2$	y_{hat} $\alpha = 2$	y_{exact} $\alpha = 2$
0	0	0	0
0.125	0.004	0.016	0.016
0.250	0.054	0.083	0.063
0.375	0.180	0.177	0.141
0.500	0.517	0.325	0.250
0.625	0.619	0.473	0.391
0.750	0.828	0.636	0.563
0.875	1.073	0.809	0.766
1	1.356	0.987	1

Following figure (3.3) represent the approximate solution of example (3.3) using the proposed method for different values of α and with the exact solution when $\alpha = 2$.

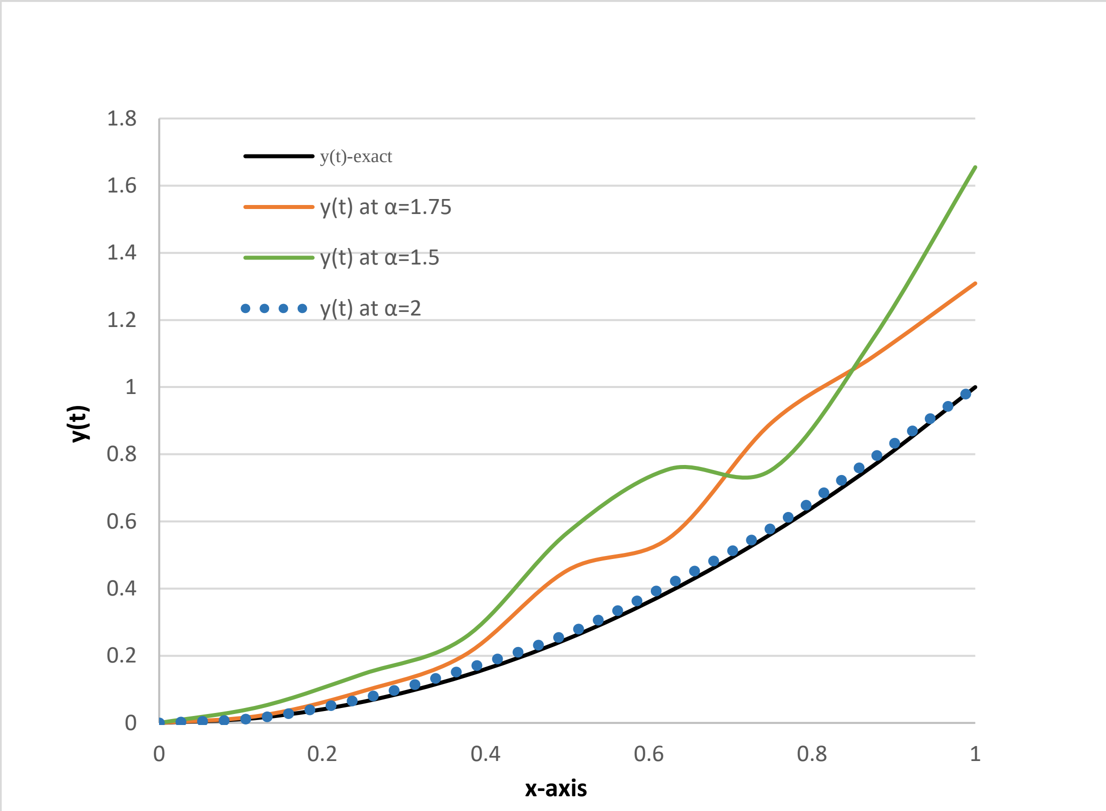

Fig. 3.3: Solution by the proposed method at different values of α compared with the exact solutions at $\alpha = 2$.

Example 3.4:

Consider the delay differential equations of fractional order with nonlinear delay function

$${}^{c}D_t^{\alpha} y(t) = -2y^2(t-0.01) + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + 2(t-0.01)^2,$$

$$0 < \alpha \leq 1 \,, 0 < t \leq 1 \qquad (3.69)$$

$$y(t) = t, -1 \leq t \leq 0 \qquad (3.70)$$

$$y(0) = 0 \qquad (3.71)$$

The exact solution, when $\alpha = 1$, is $y(t) = t$.

First we convert the delay differential equation of fractional order to the fractional order non-delay differential equation by applying the method of steps, as

$$ {}^{c}D_t^{\alpha} y(t) = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}. \; 0 < \alpha \le 1. \quad 0 < t \le 1 \tag{3.72}$$

$$ y(0) = 0 \tag{3.73}$$

Now we approximate ${}^{c}D_t^{\alpha} y(t)$ in eq. (3.72), in terms of generalized Hat functions as follows

$$ ({}^{c}D_t^{\alpha} y)(t) = C_{n+1}^{T} \Psi_{n+1}(t) \tag{3.74}$$

Hence

$$ y(t) = C_{n+1}^{T} P_{n+1}^{\alpha} \Psi_{n+1}(t) \tag{3.75}$$

Also writing the term $\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}$ in eq. (3.72) in terms of generalized Hat functions leads to

$$ \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} = F_{n+1}^{T} \Psi_{n+1}(t) \, , \tag{3.76}$$

where

$$ F_{n+1} \triangleq [f_0, f_1, f_2, \ldots, f_n]^{T}, $$

and

$$ f_i = \frac{1}{\Gamma(2-\alpha)} (ih)^{1-\alpha} \, , \; i = 0,1,2, \ldots, n. $$

Substituting eq.'s (3.74) and (3.76) into eq. (3.72), we have

$$ C_{n+1}^{T} \Psi_{n+1}(t) = F_{n+1}^{T} \Psi_{n+1}(t), \tag{3.77}$$

which implies that

$$ C_{n+1}^{T} = F_{n+1}^{T} \tag{3.78}$$

Then using eq. (3.75), one can get the output response $y(t)$.

For $n = 8$, it seems from table (3.4) that the results obtained from the proposed method when $\alpha = 1$ provides better results as compared with the Chebyshev wavelet method and the exact solution.

(Table 3.4)

Comparison of the approximate solution of example (3.4) using the proposed method and Chebyshev wavelet method when $\alpha = 1$ with the exact solution.

t	y_{ch} $\alpha = 1$	y_{hat} $\alpha = 1$	y_{exact} $\alpha = 1$
0	0	0	0
0.125	0.1250	0.1250	0.1250
0.250	0.2501	0.2500	0.2500
0.375	0.3752	0.3750	0.3750
0.500	0.5002	0.5000	0.5000
0.625	0.6253	0.6250	0.6250
0.750	0.7504	0.7500	0.7500
0.875	0.8754	0.8750	0.8750
1	1.0040	1	1

Following figure (3.4) represent the approximate solution of example (3.4) using the proposed method for different values of α and with the exact solution when $\alpha = 1$.

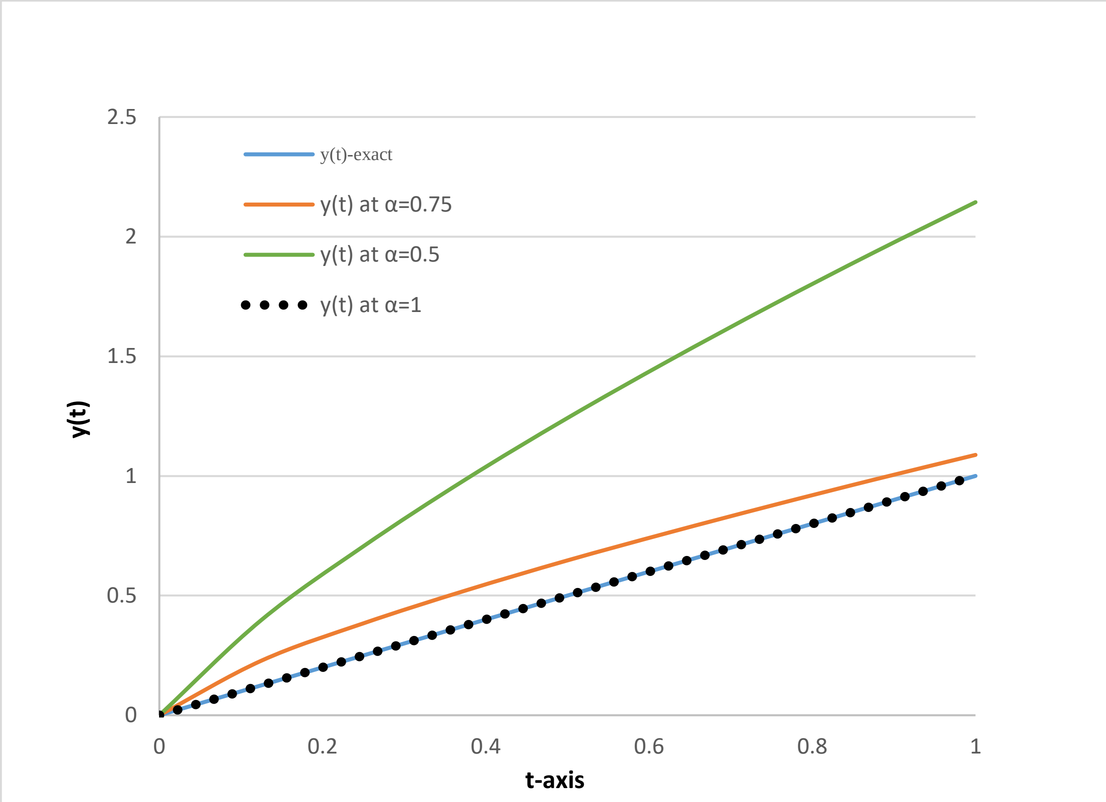

Fig. 3.4: Solution by the proposed method at different values of α compared with the exact solutions at $\alpha = 1$.

Example (3.5):

Consider the following fractional order delay differential equation

$${}^{c}D_t^{\alpha} y(t) = -y(t) - y(t - 0.3) + e^{-t+0.3},$$

$$2 < \alpha \leq 3 \,, 0 < t \leq 1 \quad (3.79)$$

$$y(t) = e^{-t}, \; -1 \leq t \leq 0 \quad (3.80)$$

$$y(0) = 1, y'(0) = -1, y''(0) = 1 \quad (3.81)$$

The exact solution of the above equation when $\alpha = 3$, is $y(t) = e^{-t}$.

First we convert the fractional delay differential equation to the fractional non-delay differential equation by applying the method of steps, as:

$${}^{c}D_t^{\alpha} y(t) + y(t) = 0, \;\; 2 < \alpha \leq 3, \; 0 < t \leq 1 \quad (3.82)$$

$$y(0) = 1, y'(0) = -1, y'(0) = 1 \tag{3.83}$$

Now we approximate ${}^{c}D_t^{\alpha}y(t)$ in eq. (3.82), in terms of generalized Hat functions as follows

$$({}^{c}D_t^{\alpha}y)(t)=C_{n+1}^{T}\Psi_{n+1}(t) \tag{3.84}$$

Hence

$$y(t)=C_{n+1}^{T}P_{n+1}^{\alpha}\Psi_{n+1}(t)+1-t+\frac{t^2}{2} \tag{3.85}$$

Also writing the term $1-t+\frac{t^2}{2}$ into eq. (3.82) in terms of generalized Hat functions leads to

$$1-t+\frac{t^2}{2} = F_{n+1}^{T}\Psi_{n+1}(t)\,, \tag{3.86}$$

where

$$F_{n+1} \triangleq [f_0, f_1, f_2, \dots, f_n]^{T},$$

and

$$f_i = 1 - ih + \frac{(ih)^2}{2},\ \ i = 0,1,2,\dots,n.$$

Substituting eq.'s (3.86) into eq. (3.85), we get

$$y(t)=C_{n+1}^{T}P_{n+1}^{\alpha}\Psi_{n+1}(t)+F_{n+1}^{T}\Psi_{n+1}(t) \tag{3.87}$$

Substituting eq.'s (3.87) and (3.84) into eq. (3.82), we have

$$C_{n+1}^{T}\Psi_{n+1}(t) + [C_{n+1}^{T}P_{n+1}^{\alpha}\Psi_{n+1}(t) + F_{n+1}^{T}\Psi_{n+1}(t)\,] = 0 \tag{3.88}$$

From eq. (3.88), one can obtain the coefficients C_{n+1}^{T}. Then using eq. (3.85), one can get the output response $y(t)$.

For $n = 8$, it seems from table (3.5) that the results obtained from the proposed method when $\alpha = 3$ provides better results as compared with the Chebyshev wavelet method and the exact solution.

(Table 3.5)

Comparison of the approximate solution of example (3.5) using the proposed method and Chebyshev wavelet method when $\alpha = 3$ with the exact solution.

t	y_{ch} $\alpha = 3$	y_{hat} $\alpha = 3$	y_{exact} $\alpha = 3$
0	1	1	1
0.125	0.882	0.882	0.882
0.250	0.779	0.778	0.778
0.375	0.691	0.687	0.687
0.500	0.617	0.606	0.606
0.625	0.558	0.535	0.535
0.750	0.513	0.472	0.472
0.875	0.482	0.416	0.416
1	0.458	0.367	0.367

Following figure (3.5) represent the approximate solution of example (3.5) using the proposed method for different values of α and the exact solution when $\alpha = 3$.

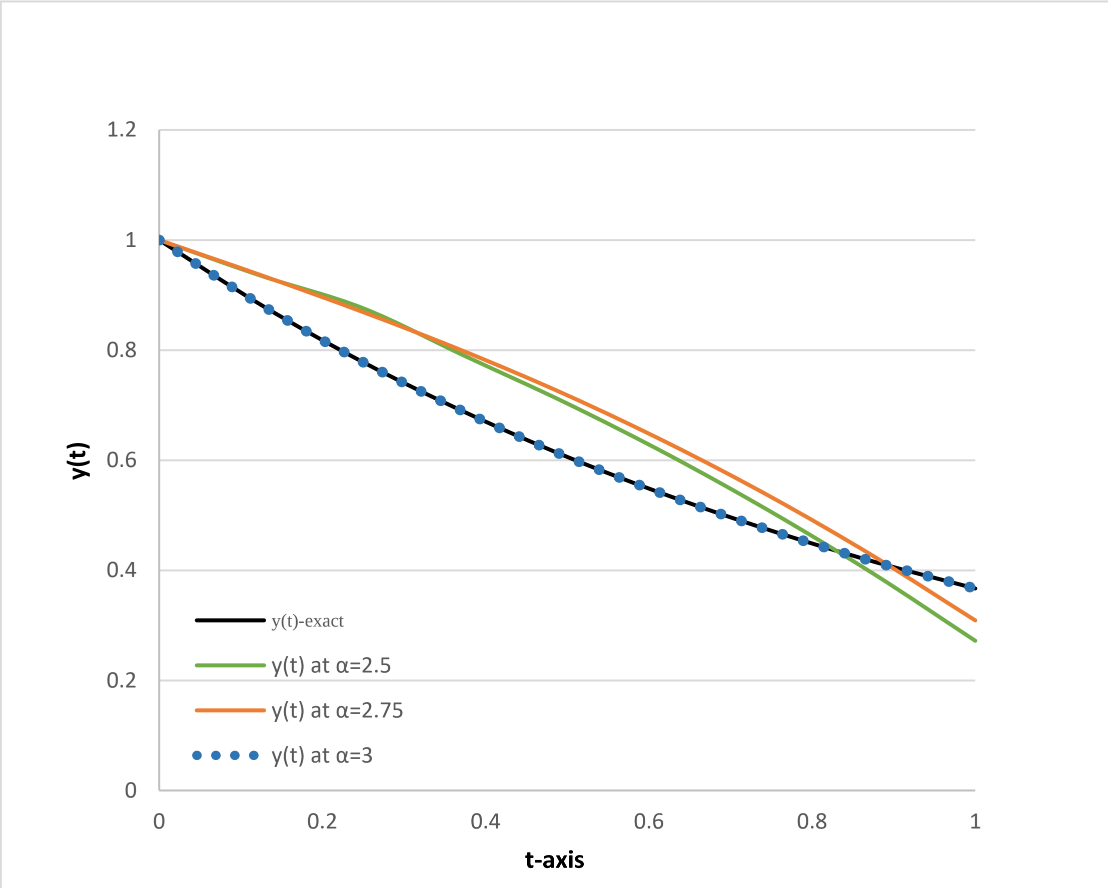

Fig. (3.5): Solution by the proposed method at different values of α compared with the exact solutions at $\alpha = 3$.

Conclusions and Future Works

From the present study, we conclude the following:

1- The existence and uniqueness theorem of the solution of delay differential equations of fractional order and the stability theory of the solution of such equations are proved in chapter two.

2- A generalized Hat functions operational matrices together with the method of steps have been presented to be an efficient method for solving delay differential equations of fractional order.

3- One can conclude from the results of the numerical examples that the proposed method gave us a good agreement with the exact solutions.

4- The obtained results of the numerical examples that have been presented in chapter three are compared with the solutions obtained by some other numerical methods such as including Chebyshev wavelet method and from the results we conclude that the present method gives more accurate values as compared to the Chebyshev wavelet method.

For future works we recommended the following:

1- Numerical solution of variable order fractional delay differential equations using the proposed method.

2- Numerical solution of integro-delay differential equations of fractional order using generalized Hat basis functions.

3- Solving system of delay differential equations of fractional order using generalized Hat functions.

4- Numerical solution of delay partial differential equations of fractional order using Hat basis function together with another basis function such as Haar basis function.

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الملخص

الغرض الرئيسي لهذه الأطروحة هو دراسة وإيجاد الحلول العددية للمعادلات التفاضلية التباطؤية ذات الرتب الكسرية ويمكن تقسيم هذا الغرض إلى هدفين وكالتالي:

الهدف الأول هو اثبات نظرية الوجود والوحدانية واستقرارية الحلول للمعادلات التفاضلية التباطؤية ذات الرتب الكسرية.

الهدف الثاني هو إيجاد الحلول العددية للمعادلات التفاضلية التباطؤية ذات الرتب الكسرية باستخدام مصفوفة العمليات لدوال (Hat).

تم اقتراح أسلوب يتضمن دمج طريقة الخطوات ودوال (Hat) لحل المعادلات التفاضلية التباطؤية ذات الرتب الكسرية.

هذه الأسلوب يقوم بتحويل المعادلات التفاضلية التباطؤية ذات الرتب الكسرية إلى معادلات تفاضلية غير تباطؤية ذات الرتب الكسرية بواسطة أستخدام طريقة الخطوات ومن ثم أستخدام مصفوفة العمليات لدوال (Hat) لتحويل المعادلت التفاضلية الغير تباطؤية ذات الرتب الكسرية الخطية وغير الخطية الى منظومة معادلات الجبرية ومن ثم إيجاد الحل لها.

تم أعطاء بعض الأمثلة التوضـيحية وتم مقارنة نتائج هذه الأمثلة مع بعض الطرق الموجودة مثل طريقة مويجات تشيبيشيف والحل المضبوط من أجل توضيح دقة وكفاءة الطريقة المقترحة.

جمهــورية العـــراق
وزارة التعليم العالي والبحث العلمي
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قسم الرياضيات وتطبيقات الحاسوب

الحلول العددية للمعادلات التفاضلية التباطؤية ذات الرتب الكسرية

رسالة

مقدمة الى كلية العلوم-جامعة النهرين
هي جزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل
قتيبـــة وادي إبراهـيم
(بكالوريوس علوم، جامعة النهرين، ٢٠١٤)

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