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The Modified Alternative Direction Iteration Method for Solving Partial Differential Equations with Application to Chronic wounds of Diabetes Patients

A Thesis

Submitted to the College of Science, Al-Nahrian University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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Supervisor Certification

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DEDICATION

To the candle that burned to enlighten my way in life

My Mother

To the man who was the best supporter to me in my life

My Father

To my little diamonds

Zainab & Hussain

To the people who were always encourage me in my life

My Husband, Sisters & Brothers

To my friends who was always helping me

MELAD

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MELAD JAMEEL HMOOD

ABSTRACT

Alternating Direction Implicit method (ADI) was first suggested by Peaceman and Rachford in the mid-50s of the last century for solving systems of algebraic equations in two dimension of spaces[peaceman and rachford,1955], which results from the finite difference discretization method for solving PDEs; [Peaceman and Rachford,1955]. From iterative method's perspective, the ADI method may be considered as a special relaxation method, where a big system is simplified into a number of smaller sub-systems, such that each of them may be solved efficiently and the solution of the whole system is then obtained from the solutions of the sub-systems in an iterative method approach, [Al-Saif and Al-Kanani,2011].

The main theme of this thesis may be directed toward three objective:

The first objective is to explain and clarify, in details, the alternative direction iteration method in a simple way for each type of differential equations, which is produced by rearranging the Crank-Nicholson formulation and discuss the stability, convergent and consistency of the solution using normal time steps. we preposed a new alternative direction iteration method depending on another time step, a new formula derived to give alternative direction iteration method more accurate.

The second objective is to derive and study the system associated with the infected equations of patients with diabetes then the effect of oxygen in the treatment of infected wounds, the obtained system of related equations was of the first dimension, also the alternative direction iteration method could be genarlized to solve the equations of the second dimension. Then we went to re-model the part of this system of equations of the second dimension as a prelude to solve. Finally, the third objective is devoted to solve a system of equations that re-modeled by using the discussed alternative direction iteration method the results that are obtaind to find out how the stability of the system output and the accuracy of the results affect on the stages of treatment.

CONTENTS

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SYMBOLES

PDE	Partial Differential Equation
FDM	Finite Difference Method
FEM	Finite Element Method
TDMA	Tridiagonal Matrix Algorithm
ADI	Alternating Direction Iteration
FDTD	Finite Difference Time Domain
HBOT	Hyperbaric Oxygen Therapy
ECM	Extra Cellular Matrix
D_w	The constant rate of diffusion measured in cm^2/s
k	The oxygen in the wound
β	The constant rate that represent the oxygen enter the wound
$G(t)$	Oxygen increases through the supplemental oxygen given astherapy
γ_{nw} , γ_{bw}	Rates represent oxygen that will be used by bacteria
γ_w	The rate of losing oxygen
ε_b	Constant random motility for bacteria taken from center of wound
γ_b	Natural death of bacteria
D_c	Constant rate of diffusion (cm^2/s)
k_b	The chemoattractant produced by the bacteria
γ_c	The constant rate of chemoattractant decadency
D	The diffusivity rate

INTRODUCTION

Numerical analysis may be considered as the study of algorithms that use numerical treatments for certain problems under consideration. One of the earliest mathematical mainscripts is a Babylonian tablet from the Yale Babylonian Collection, which gives 60 bas numerical approximation of $\sqrt{2}$, the length of the diagonal in a unit square. Being able to compute the sides of a triangle (and hence, being able to compute square roots) is extremely important, for instance, in astronomy, carpentry and construction [Brezis,1998].

Numerical analysis naturally finds applications in all fields of engineering and physical sciences, but in the 21st century, it also has applications in Biological sciences and even in arts had adopted elements of scientific computations where mathematics has been applied to a wide range of applications [Guffey,2015].

In mathematical analysis and applied mathematics, partial derivative is a derivative of any arbitrary order, although the partial term is a misnomer, it accepted widely on this type of derivatives. The complexity of any systems of equations comes from the involving of the nonlinear equations, especially in a matters of applied physics and engineering, medical and others. The importance of the finite difference method (FDM) is being one of the earlest and best ways and the most widely used approaches in solving Partial Differential Equations (PDEs) through simplified approaches and turn it into a grid of dots and rounded differential equations by finite difference equations equivalent to it[Serov,2010].

In this method, the system of equations resulting from the finite difference method will transformed into sub-systems of linear equations produced according to directions and then solved by an iterative procedure with the cooperation of initial and boundary conditions.

As is the case of all medical applications, associated with the treatment of inflammatory diabetics by the pressure and spread of oxygen to the wound for the purpose of eliminating bacteria causing, It is one of the nonlinear complex equations that need to be solved using a numerical method. So, we turned after the submission of the system and remodeled to a second dimension to be resolved using the Alternating Direction Iteration (ADI) method.

We are concerned with the numerical solution of problems in two and three dimensions. For this purpose, a finite difference method, namely (ADI) method which is a type of spiltting method for solving (PDEs) extended, the advantages of this method are unconditionally consistent and stable which gives convergency for the numerical solution.

The thesis is ordered as follows. In chapter one, some definitions and basic concepts related to (PDEs) and some types of methods used for numerical solutions of such equations is introduced.

In chapter two, the ADI method for solving partial differential equations is discussed by taking equation for each type and then implementing the formula for each equation. The consistency, stability and convergent of such equations have been tested. A new formula is implemented here by taking small two time steps, for more accuracy we extend the finite difference formula to drive anew formula of order four, Finally two illustration examples are given.

In chapter three, some biological and mathematical backgrounds concerning with the oxygen therapy treatment of chronic wounds are given, in which, the biological interpretion of the problem is given first, and then the derivation of the mathematical model for one dimensional space is presented with the analytical solution.

Finally, in order to solve the obtained system associated with this treatment by using ADI method, the model will be developed for a two dimensional space and then are solved and debated the results numerically.

1.1 Introduction :

In mathematics, partial differential equation (PDE) is an equation that contains unknown functions of multiple variables known as the independent variables and their derivatives are partial, PDEs are utilized for modeling problems that includes functions which contain a number of independent variables. Where these equations are expressed as a relationship between the function of two or more variables and partial derivatives associated with this job, that having to do with these independent variables [Hoffman,2006].

The research on PDEs goes back to the 18^{th} century, one of the most important phenomenas in the application of PDEs in science and engineering since the second World War [Jeffrey,2009]. Partial differential equations form the basis of many mathematical models of physical, chemical and biological phenomenas and recently they are used in economics, financial forecasting, image processing and other fields [Lorenzo,2009].

Therefore, numerical analysis is a branch of applied mathematics containing a variety of techniques to solve PDEs, such as the finite difference method, spectral method [Smith,1978] finite element method [Gilbert,1974], mesh free method [Duarte and Oden,1996], finite volume method [Leveque,2002], boundary element method [Aliabadi,2011] etc.

The finite element and finite volume methods are widely used in engineering to model problems with complicated geometries; the FDM is often regarded as the simplest method [Saad,2003]; the mesh free method is used to facilitate accurate and stable numerical solutions for PDEs without using a mesh [Hoffman,2006]. During the 18^{th} century, the foundation of the theory of a single first order PDEs and its reduction into a system of ordinary differential equations(ODEs) was carried through a reasonably mature form.

1.2 Partial Differential Equations:

The classical PDEs which serve as model for the later development also appeared first in the 18th and early 19th century [Jeffrey,2009].

In order to classify partial differential equations, it is known that $u_{xy} = u_{yx}$ if u is continous and therefore, the second order PDEs of two independent variables in a general form may be given as :

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0 \qquad \dots (1.1)$$

where the coefficients A, B, C,D,E and F. depend on x and y.

If:

$$A^2 + B^2 + C^2 > 0 \qquad \dots (1.2)$$

over the region of xy-plane, then the PDE is of the second order in this region, is said to be [Hoffman,2006]:

- Elliptic If $B^2 - 4AC < 0$ (e.g. Laplace Eq.)
- Parabolic If $B^2 - 4AC = 0$ (e.g. Heat Eq.)
- Hyperbolic If $B^2 - 4AC > 0$ (e.g. Wave Eq.)

These definitions can be generalized to higher dimensional spaces and higher orders. If no one of the coefficients depends on the dependent variable and all its partial derivatives appearing in the linear form , then it is called linear , as in the following example:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} \qquad \dots (1.3)$$

where x and t are the independent variables, u is the unknown function and ε is the coefficient, otherwise the PDE is called nonlinear if the coefficients depend on the dependent variable, or the derivatives appear in a nonlinear form; as in following example:

$$\frac{\partial u}{\partial x} + f\frac{\partial u}{\partial y} = 0, \quad x \in [a,b], \quad t \geq 0 \qquad \dots (1.4)$$

where x and y are independent variables and f is unknown function, [Serov,2010].

Also, the PDEs may be classified as homogeneous if the unknown function or its derivative appear in each term, otherwise it is called a non-homogeneous,[Hoffman,2006].

In addition, the PDE are the equation that is supplemented by initial and\ or boundary conditions , there are three types of boundary conditions, namely:

1. Dirichlet boundary condition: Numerical values of the function are specific of the boundary of the region .
2. Neumann boundary condition: Specifies the values that the derivative of a solution is taken on the boundary of the domain.
3. Mixed boundary condition: Defines a boundary value problem in which the solution of the given equation is required to satisfy different boundary conditions on disjoint parts of the boundary of the domain, where the condition is stated. Precisely, in a mixed boundary value problem, the solution is required to satisfy a Dirichlet or a Neumann boundary condition in a mutually exclusive way on disjoint parts of the boundary.

1.3 Finite Difference Methods :

The finite difference method (FDM) was first developed by Thomas in 1920 under the title (the method of square) to solve nonlinear hydrodynamic equations; [Smith,1978]. The basic idea of the FDM which is a numerical method for solving differential equations that is to approximate the solution of differential equations, i.e., to find a function (or some discrete approximation to this function) that satisfies certain relationship between various of its derivatives on some given region of space and/or time, along with some boundary conditions along the edges of this domain; [LeVeque,2005].

As one of the earliest and most commonly used numerical methods for computing discrete solutions of differential equations, the finite-difference methods are based on the Taylor series expansion on a set of grids, most commonly, a set of uniformly spaced grids.

The advantages of the FDM lie in two aspects; they are simple in formulation and implementation, and they are highly scalable; [Urroz,2004].

1.4 Explicit Methods:

This type of methods solve directly at a point for all unknown values in the finite difference scheme, it is stable only for certain time step sizes. Stability can be checked using Fourier or Von Neumann analysis [Smith,1978].

A forward difference at a time and a second-order central difference for the space derivative is used to solve the differential equation, for example take the heat equation; [Lorenzo,2009]:

$$u_t = u_{xx} \qquad \text{... (1.5)}$$

we get the equation:

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \qquad \text{... (1.6)}$$

hence we may obtain u_j^{n+1} from the other values as:

$$u_i^{n+1} = (1-2r)u_i^n + ru_{i-1}^n + r\,u_{i+1}^n\,,\ r = \frac{k}{h^2} \qquad \text{... (1.7)}$$

where h and k are the step size discretization in the x and t directions, respectively .

This explicit method is known to be numerically stable and convergent whenever $\frac{k}{h^2} \leq \frac{1}{2}$ and the numerical errors are proportional to the time step and the square of the space step [Paramvir,2008]:

$$\Delta u = O(k) + O(h^2) \qquad ... (1.8)$$

Where $k = \Delta t, h = \Delta x$

1.5 Implicit Methods:

Here there is no explicit formula at each point, only a set of simultaneous equations which must be solved over the whole grid of the discretization mesh most be evaluated [Serov, 2011].

If we use the backward difference at time t_{n+1} and a second-order central difference for the space derivative at position x_i (the backward time, centered space method) one may get the recurrence equation; [Lorenzo,2009]:

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} \qquad ... (1.9)$$

and also one may obtain u_i^{n+1} by solving the system of equations:

$$(1+2r)u_i^{n+1} - ru_{i-1}^{n+1} - r\,u_{i+1}^{n+1} = u_i^n \,, r = \frac{k}{h^2} \qquad ... (1.10)$$

Despite the fact that implicit methods are stable and convergent for all step sizes [Urroz, 2004], but it usually consider more numerically intensive, than the explicit method is required for solving a system of numerical equations on each time step. The errors are linear over the time step and quadratic over the space step [Serov, 2011]:

$$\Delta u = O(k) + O(h^2)$$

Where $k = \Delta t, h = \Delta x$

1.6 Cranks–Nicolson Method:

The Crank Nicolson finite difference scheme was invented by John Crank and Phyllis Nicolson. They originally applied it to the heat equation and they approximated the solution of the heat equation on some set of finite grid points by approximating the derivatives in space x and time t by finite differences; [Duffy,2004].

If we use the central difference at time $t_{n+1/2}$ and a second-order central difference for the space derivative at position x_i we get the equation:

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{1}{2}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{h^2}\right) \quad \dots (1.11)$$

The scheme is always numerically stable and convergent but usually more numerically intensive as it is required for solving a system of numerical equations on each time step. The errors are quadratic over both the time step and the space step [Kettle,1978].

$$\Delta u = O(k^2) + O(h^2)$$

Where $k = \Delta t$, $h = \Delta x$

Usually, the Crank-Nicolson scheme is the most accurate scheme for small time steps. The explicit scheme is less accurate and may be unstable, but is also the easiest to implement and the least numerically intensive. The implicit scheme works as the best for large time steps, [Serov,2011].

***1.7 Tridiagonal Matrix Algorithms*:**

In numerical linear algebra, the tridiagonal matrix algorithm (TDMA), also known as the Thomas algorithm, is a simplified form of Gaussian elimination method that can be used to solve tridiagonal systems of equations. A tridiagonal system for n-unknowns may be written as; [Zargari,2007]

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = r_i \qquad \dots (1.12)$$

where $a_1 = 0$ and $c_n = 0$, $i = 1,2, \dots, n$

The Thomas algorithm is an efficient way for solving tridiagonal matrix systems. It is based on LU decomposition method in which the matrix system Mx = r which rewritten as:

$$LUx = r$$

where L represents a lower triangular matrix and U is an upper triangular matrix; [Thomas,1960].

$$\begin{bmatrix} b_1 & c_1 & \cdots & 0 \\ a_2 & b_2 & \cdots & \vdots \\ \vdots & a_3 & \ddots & c_{n-1} \\ 0 & \vdots & \ddots & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix}$$

A matrix A $= [a_{ij}]$, where *i, j = 1, 2,…,n*; whose nonzero entries lie along the main diagonal, and the immediate sub-diagonal and super-diagonal of the form :

$$A = \begin{bmatrix} b_1 & c_1 & d_1 & \cdots & 0 \\ a_2 & b_2 & c_2 & \cdots & \vdots \\ \vdots & a_3 & b_3 & \ddots & c_{n-1} \\ 0 & \vdots & \vdots & \ddots & b_n \end{bmatrix}$$

is called tridiagonal matrix. Thus A $= [a_{ij}]$, where, $1 \leq i, j \leq n$ is tridiagonal if $a_{ij} = 0$ for $|i - j| > 1$ [*Mikkawy*,2004].

The Thomas algorithm is used because, it is fast and the tridiagonal matrices often occur in practice; [Thomas,1960] .

2. 1 Introduction:

Alternating Direction Implicit method (ADI) was first suggested by Peaceman and Rachford in the mid-50s of the last century for solving systems of algebraic equations in two dimensional of spaced, which results from the finite difference discretization method for solving PDEs; [Peaceman and Rachford,1955]. From the iterative perspective method's, the ADI method may be considered as a special relaxation method, where a big system is simplified into a number of smaller sub-systems, such that, each of them may be solved efficiently and the solution of the whole system is then obtained from the solutions of the sub-systems in an iterative method approach, [Al-Saif and Al-Kanani,2011].

In 1999, Namiki worked on new finite difference time-domain (FDTD) algorithm based on the alternating direction implicit method [Namiki,1999].

Cheon and Hyeongdong presented in 2002 an artical on the analysis of the power plane resonance using the alternating direction implicit method [Cheon and Hyeongdong,2002].

Then in 2006 Lee and Smith presented a paper on alternative approach for implementing periodic boundary conditions in the FDTD method using multiple unit cells [Lee and Smith, 2006].

Multigrid ADI method for two dimensional electromagnetic simulations was also presented by Wang in 2006 [Wang,2006]. Masoud Movahhedi, Abdolali Abdipour, Alexandre Nentchev and Siegfried Selberherr also worked on alternating direction implicit formulation of the finite-element time-domain method in 2007 [Masoud et. al.,2007]. Jihye Shin, Sungsoo S. Kim in 2008 resorting to the use of the ADI method for studies of the dynamical evolution of dense spherical stellar systems, Interpreters used reason for being ADI method reduces the computing time by a factor of two compared to the fully implicit method [Sungsoo S. et.al.,2008], and resolves problems of numerical instability in 2011, Song-Ping Zhu, Wen-Ting Chen introduced a predictor–corrector scheme based on the ADI method for the Heston model, which is considered as a Finance Application[Song-Ping et.al.,2011].

A generalization of the ADI method for solving numerically multidimensional fractional diffusion equations was described by Concezzi and spigler in 2012 [Concezzi et.al.,2012].

Darae Jeong, Junseok Kim provided in 2013 a comparison study of ADI and operator splitting methods on the financial applications [Darae Jeong et.al.,2013].

In 2014, a high-order alternating direction implicit finite difference scheme for the solution of initial-boundary value problems of convection-diffusion type with mixed non-constant coefficients was introduced by Bertram Duringa [Bertram D.,2014].

In numerical analysis, the ADI method is a FDM for solving PDEs; it is an example of an operator splitting method. The idea behind the ADI method is to split the FDEs into two categories, the first one with the x-derivative taken implicitly and the second with the y-derivative taken implicitly [Jeong and Kim, 2013].

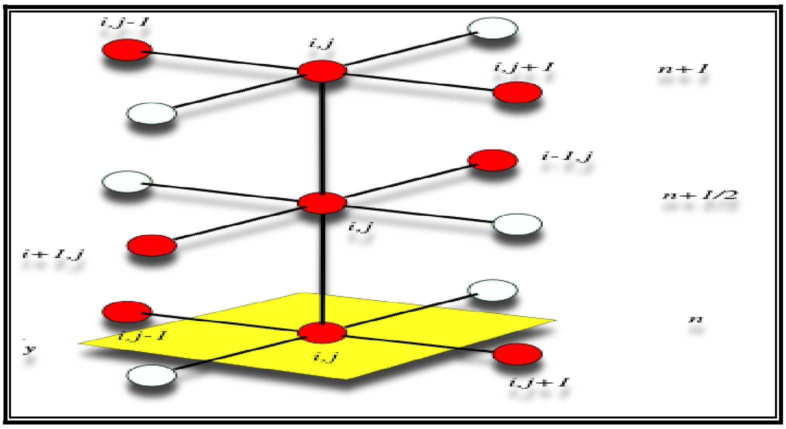


Fig.(2.1) The alternating direction implicit method in finite difference equations [Chang, 1991]

2.2 The ADI Formula for Two Dimensional Non-Homogeneous Parabolic Equations:

As an illustration of the ADI method, consider the two dimensional parabolic equation[Hoffman,2006]:

$$u_t = u_{xx} + u_{yy} - f(x, y) \qquad \text{... (2.1)}$$

where $D = (0, l_1) \times (0, l_2)$, $(x, y) \in D$, $t \in (0, \infty]$, f is a given function with initial condition $u(x, y, 0) = u_0(x, y)$, the boundary conditions are along ∂D, assumed that the spatial discretizations on the x and y directions is k.

The finite difference formulation of equation (2.1) using the central differences for u_{xx} and u_{yy} is:

$$\frac{du_{i,j}}{dt} = \frac{1}{k^2}\left[\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right)\right] - f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.2)}$$

For all $i, j = 1, 2, \dots, N-1$, $N \in \mathbb{N}$

The Crank-Nicholson reformulation of equation (2.2) is obtained by integrating over $[\, t_n , t_{n+1}]$

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{r}{2}[\left(u_{i+1,j}^{n+1} - 2\, u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right) + \left(u_{i,j+1}^{n+1} - 2\, u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right)$$
$$+ \left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) + \left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right)]$$
$$- \Delta t \; f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.3)}$$

where $r = \frac{\Delta t}{k^2}$

Equation (2.3) may be subsequently arranged to give:

$$u_{i,j}^{n+1} - \frac{r}{2}[\left(u_{i+1,j}^{n+1} - 2\, u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right) - \frac{r}{2}\left(u_{i,j+1}^{n+1} - 2\, u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right)$$
$$= u_{i,j}^{n} + \frac{r}{2}\left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) + \frac{r}{2}\left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right)]$$
$$- \Delta t \; f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.4)}$$

The matrix related to the system (2.4) is no longer tridiagonal and so that an iterative procedure is required to find the numerical solution through each time step.

For simplicity, introduce the finite difference operator δ defined by:

$$\delta(x_n) = x_{n+\frac{1}{2}} - x_{n-\frac{1}{2}}$$

then simplifying equation (2.4) to take the following form:

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{r}{2}\left[\delta_x^2 \left(u_{i,j}^{n+1}\right) - \delta_y^2\left(u_{i,j}^{n+1} \right)\right] + \frac{r}{2}[\delta_x^2\left(u_{i,j}^{n}\right) + \delta_y^2(u_{i,j}^{n})]$$
$$- \Delta t \; f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.5)}$$

for all $i , j = 1, 2 \ldots N, \; N \in \mathbb{N}$

Since the direct solution is quite costly, the idea behind using the ADI method is to spilt system (2.5) into two subsystems, yielding:

$$u_{i,j}^{n+1} - r\, \delta_x^2 \left(u_{i,j}^{n+1}\right) = u_{i,j}^{n} + r\, \delta_y^2 \left(u_{i,j}^{n} \right) - \Delta t \; f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.6)}$$

$$u_{i,j}^{n+2} - r\, \delta_x^2 \left(u_{i,j}^{n+2}\right) = u_{i,j}^{n+1} + r\, \delta_y^2 \left(u_{i,j}^{n+1} \right) - \Delta t \; f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.7)}$$

where two time steps $[t_n, t_{n+1}]$ and $[t_{n+1}, t_{n+2}]$ are taken.

Based on this procedure, $\delta_y^2(u_{i,j}^n)$ is explicit and $u_{i,j}^{n+1}$ is determined implicitly and it is remarkable that, the rule is reversed in the second equation (2.7). The important property of this method is that both steps require the solution of the tridiagonal system of linear algebraic equations(2.6) and (2.7). The first system is to find $u_{i,j}^{n+1}$ and the second system is to find $u_{i,j}^{n+2}$.

Equations (2.6) and (2.7) may be written respectively as:

$$(1+2r)u_{I,j}^{n+1} - ru_{i+1,j}^{n+1} - ru_{i-1,j}^{n+1}$$

$$= (1-2r)u_{i,j}^{n} + ru_{i,j+1}^{n} + ru_{i,j-1}^{n} - \Delta t\ f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.8)}$$

$$(1+2r)u_{i,j}^{n+2} - ru_{i+1,j}^{n+2} - ru_{i-1,j}^{n+2}$$

$$= (1-2r)u_{i,j}^{n+1} + ru_{i,j+1}^{n+1} + ru_{i,j-1}^{n+1} - \Delta t\ f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.9)}$$

Hence, the linear system for m unknowns of m equations can written in the following matrix form:

$$AU = F \qquad \text{... (2.10)}$$

where:

$$A = \begin{bmatrix} 1+2r & -r & \cdots & \cdots & 0 & 0 \\ -r & 1+2r & -r & \cdots & \cdots & 0 \\ 0 & -r & 1+2r & -r & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \vdots & -r & 1+2r & -r \\ 0 & 0 & 0 & 0 & -r & 1+2r \end{bmatrix}, \quad U = \begin{bmatrix} u_{1,j}^{n+1} \\ u_{2,j}^{n+1} \\ \vdots \\ \vdots \\ u_{m-1,j}^{n+1} \\ u_{m,j}^{n+1} \end{bmatrix}$$

and $F = \begin{bmatrix} (1-2r)u^{n}_{1,j} + r\left(u^{n}_{1,j+1} + ru^{n}_{1,j-1}\right) - \Delta t \ f^{n+\frac{1}{2}}_{1,j} \\ (1-2r)u^{n}_{2,j} + r\left(u^{n}_{2,j+1} + ru^{n}_{2,j-1}\right) - \Delta t \ f^{n+\frac{1}{2}}_{2,j} \\ \vdots \\ (1-2r)u^{n}_{m-1,j} + r\left(u^{n}_{m-1,j+1} + ru^{n}_{m-1,j-1}\right) - \Delta t \ f^{n+\frac{1}{2}}_{m-1,j} \\ (1-2r)u^{n}_{m,j} + r\left(u^{n}_{m,j+1} + ru^{n}_{m,j-1}\right) - \Delta t \ f^{n+\frac{1}{2}}_{m,j} \end{bmatrix}$

Similarly, applying the same procedure one may solve the second system (2.9).

2.2.1 Consistency of the ADI Method:

We start this subsection with the definition of consistent PDE:

Definition (2.1) [Khouider, 2008]:

Let $F_{i,j}(u) = 0$ represent the difference equation at the $(i,j)th$ mesh point. If u is replaced by U at the mesh points of difference equation, the value of $F_{i,j}(U)$ is called the local truncation error. If this tends to zero as the mesh length tend to zero then the difference equation is said to be consistent with the PDE.

i.e., if the truncation error τ_h is defined by:

$$\tau_h = AU - F \qquad \text{... (2.11)}$$

Then it must satisfies:

$$\lim_{\Delta t, \Delta x, \Delta y \to 0} \tau_h = 0$$

where Δt is a time step, Δx and Δy are the grid spacing in the *x* and *y* directions, respectively.

Also the method is said to be consistent of order (p, q) if:

$$\tau_h = O((\Delta x, \Delta y)^p + (\Delta t)^q)$$

In practice, the exact solution is not known in advanced, but assume that it is smooth and if we rewrite the ADI algorithm (2.6) and (2.7) they are appeared as follows:

$$(1 - r\,\delta_x^2)\,u_{i,j}^{n+1} = \left(1 + r\,\delta_y^2\right)u_{i,j}^{n} - \Delta t\ f_{i,j}^{n+\frac{1}{2}} \quad \text{... (2.12)}$$

$$\left(1 - r\,\delta_y^2\right)u_{i,j}^{n+2} = (1 + r\,\delta_x^2)\,u_{i,j}^{n+1} - \Delta t\ f_{i,j}^{n+\frac{1}{2}} \quad \text{... (2.13)}$$

and after elimination the term at t_{n+1} then(2.12) and (2.13) are given as:

$$(1 - r\,\delta_x^2)\left(1 - r\,\delta_y^2\right)u_{i,j}^{n+2}$$

$$= (1 + r\,\delta_x^2)\left(1 + r\,\delta_y^2\right)u_{i,j}^{n} - 2\Delta t\ f_{i,j}^{n+\frac{1}{2}} \quad \text{... (2.14)}$$

Hence, after carying some simplifications, we may get:

$$\left(1 + r^2\,\delta_x^2\delta_y^2\right)(u^{n+2} - u^n)$$

$$= r\left(\delta_x^2 + \delta_y^2\right)(u^{n+2} + u^n) - 2\Delta t\ f_{i,j}^{n+\frac{1}{2}} \quad \text{... (2.15)}$$

Now, dividing by 2h and replacing r by its definition, we have:

$$\left(1 + r^2\,\delta_x^2\delta_y^2\right)\frac{u^{n+2} - u^n}{2h} = \left(\delta_x^2 + \delta_y^2\right)\frac{u^{n+2} + u^n}{2k^2} - \frac{\Delta t}{h}\ f_{i,j}^{n+\frac{1}{2}} \quad \text{... (2.16)}$$

Therefore, by using the finite difference formula, it yields:

$$\left.\begin{array}{l} \dfrac{u^{n+2} - u^n}{2h} = u^{n+1} + O(h^2) \\ u^{n+2} + u^n = 2u^{n+1} + O(h^2) \end{array}\right\} \quad \text{... (2.17)}$$

Consequently:

$$\left(\delta_x^2 + \delta_y^2\right)\frac{u^{n+2} + u^n}{2k^2} = \frac{\left(\delta_x^2 + \delta_y^2\right)2u_t^{n+1}}{2k^2} - \frac{\Delta t}{h}\ f_{i,j}^{n+\frac{1}{2}} + O(h^2)$$

$$= u_{xx}^{n+1} + u_{yy}^{n+1} - f + O(h^2, k^2) \quad \text{... (2.18)}$$

Finally:

$$\frac{r^2\, \delta_x^2\delta_y^2(\, u^{n+2} - \, u^{n})}{2h} = \frac{h^2}{k^4}\, \delta_x^2\delta_y^2(u_t^{n+1}) + O(h^2)$$

$$= \, h^2 u_{txxyy}^{n+1} + O(h^2, k^2) \qquad \text{... (2.19)}$$

Assembling component parts for getting:

$$u_t^{n+1} + O(h^2) + h^2 u_{txxyy}^{n+1} + O(r^2h^2)$$

$$= u_{xx}^{n+1} + u_{yy}^{n+1} - f + O(h^2, k^2) \qquad \text{... (2.20)}$$

which may rearrange to give :

$$u_t^{n+1} - u_{xx}^{n+1} - u_{yy}^{n+1} + f = \, O(h^2, k^2) \qquad \text{... (2.21)}$$

Then the ADI method for two dimensional non-homogeneous parabolic equations is unconditionally consistent.

The local truncation error is given by:

$$\tau = U_t^{n+1} - U_{xx}^{n+1} - U_{yy}^{n+1} + f \qquad \text{... (2.22)}$$

In addition, obtainning a relation between the local error τ and the global error E:

$$E \, = U_{i,j} - \, u_{i,j} \qquad \text{... (2.23)}$$

$$\|E\|_\infty = \max_{1\le i,j\le m}\left|E_{i,j}\right|$$

$$= \max_{1\le i,j\le m}\left|U_{i,j} - \, u_{i,j}\right|$$

which is just the largest possible error.

We subtract the equation (2.23) from the equation (2.10) that defines u, to obtain:

$$\text{AE} = \, -\tau \qquad \text{... (2.24)}$$

2.2.2 Stability of the ADI Method:

Stability means that the error caused by a small perturbation in the numerical solution remains bounded [Gilberto and Urroz, 2004].

If we look at (2.24) we have:

$$A^k E^k = -\tau^k \qquad \text{... (2.25)}$$

Where k describe a mesh spacing of a grid point.

The matrix A^k is an m × m matrix with:

$$k = \frac{1}{m+1}$$

So that, its dimension will grown as $k \to 0$. Let $(A^k)^{-1}$ be the inverse of the matrix A^k, we have:

$$E^k = -(A^k)^{-1}\,\tau^k$$

Taking the norm:

$$\left\|E^k\right\|_\infty = \left\|(A^k)^{-1}\,\tau^k\right\|_\infty$$

$$\leq \left\|(A^k)^{-1}\right\|_\infty \left\|\tau^k\right\|_\infty$$

If we let $(A^k)^{-1}$ to be bounded by some constant independent of k, then:

$$\left\|E^k\right\|_\infty \leq C\left\|\tau^k\right\|_\infty$$

and so that, $\left\|E^k\right\|_\infty$ goes to zero as fast as $\left\|\tau^k\right\|_\infty$ goes to zero; this means that if $(A^k)^{-1}$ exists for all k, and if there was a constant C, the finite difference equation which gives a system of equations that satisfy :

$$A^k u^k = f^k$$

which is said to be stable [Leveque, 2005].

The basic idea for the stability analysis for PDEs consists of expanding the solutions of the equation as a complex Fourier series and analyzing a generic component of the solution; [Khouider, 2008].

The stability of the two dimensional ADI algorithm given by equations (2.12) and (2.13) can be investigated by the substitution:

$$u_{s,q}^n = \lambda^n e^{i(\alpha s+\beta q)n}$$

where $i = \sqrt{-1}$, λ^n is the amplitude of the n-th component, $\beta = \frac{2\pi}{T}$ = angular frequency of the *n-th* component, T = period of the *n-th* component, $\alpha = \frac{2\pi}{l}$ = wave number of the n-th componint and l = wave length; [Gilberto and Urroz, 2004]; the calculation based on the above observation implies that:

$$\delta_x^2 u = \left(e^{i\alpha k} - 2 + e^{-i\alpha k}\right)u$$

$$= 2(cos\alpha k - 1)u$$

$$= -4(sin^2 \alpha k/2)u \qquad \dots (2.26)$$

$$\delta_y^2 u = \left(e^{i\beta k} - 2 + e^{-i\beta k}\right)u$$

$$= 2(cos\beta k - 1)u$$

$$= -4(sin^2 \beta k/2)u \qquad \dots (2.27)$$

hence, by eliminating the u^{n+1} from our algorithm we get:

$$(1 - r\,\delta_x^2)(1 - r\,\delta_y^2)u^{n+2} = (1 + r\,\delta_x^2)(1 + r\,\delta_y^2)u^n - 2\Delta t\ f_{i,j}^{n+\frac{1}{2}} \qquad \dots (2.28)$$

Now, applying (2.26) and (2.27) in equation (2.28), we have:

$$\lambda\left(1 + 4r(\sin^2 \alpha k/2)(1 + 4r\sin^2 \beta k/2)\right)$$

$$= 1 + 16r^2\sin^2(\alpha k/2)\sin^2(\beta k/2) - 2\Delta t\ f_{i,j}^{n+\frac{1}{2}} \qquad \dots (2.29)$$

Then, if

$$\lambda = \frac{1 + 16r^2 sin^2(\alpha k/2) sin^2(\beta k/2) - 2\Delta t\ f_{i,j}^{n+\frac{1}{2}}}{\left(1 + 4r(\sin^2 \alpha k/2)(1 + 4r\sin^2 \beta k/2)\right)}$$

and since a simple comparesion between the nominator and demonator will implies that $0 < \lambda < 1$, thus it can be seen that the two dimensional ADI algorithm is unconditionally stable [Mampaey,1989].

2.2.3 Convergent of the ADI Method:

Convergence means that the finite-difference solution approaches the exact solution of the PDE as the increments Δx, Δy, Δt tends to zero [Khouider, 2008]. Thus, a method is said to be convergent if:

$$\|E^k\|_\infty \le \|(A^k)^{-1}\|_\infty \|\tau^k\|_\infty$$
$$\le C\|\tau^k\|_\infty \to 0 \quad , \qquad k \to 0$$

which represents a combination of the conditions of consistency and stability, [Leveque, 2005].

We have introduced above in equations (2.21) and (2.29) the ADI method for solving of the two dimensional non homogeneous parabolic equation (2.1) which is consistent and unconditionally stable, and may be considered as the sufficient conditions for convergent according to the Lax equivalence theorem.

Theorem (2.1) (Lax equivalence) [Khouider,2008]:

The approximate numerical solution to a well posed linear problem converges to the exact solution of the continuous equation if and only if the numerical scheme is consistent and stable

The prove of this theorem given in appendix A.

2.3 Implementation of the ADI Method for Two Dimensional Elliptic Equation:

Consider the two dimensional Poisson's equation[Hoffman,2006]:

$$u_{xx} + u_{yy} = f(x,y) \ , x \in [0,L], \ \ y \in [0,M] \qquad \text{... (2.23)}$$

with the boundary conditions:

$$u(0,y) = g_1(y) \, , u(L,y) = g_2(y) \ , \ u(x,0) = \varphi_1(x) \, , \ u(x,M) = \varphi_2(x)$$

where φ_1 , φ_2 , g_1 and g_2 are given continuous functions, L, M > 0.

Assuming that the spatial discretization on the x and y directions equals to k, then the finite difference formulation using central difference of Poisson equation is:

$$\frac{u_{i-1,j} - 2\,u_{i,j} + u_{i+1,j}}{k^2} + \frac{u_{i,j-1} - 2\,u_{i,j} + u_{i,j+1}}{k^2} = f_{i,j} \qquad \text{... (2.24)}$$

The Crank-Nicolson reformulation of equation (2.24) is:

$$\left(u_{i+1,j}^{n+1} - 2\,u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right) + \left(u_{i,j+1}^{n+1} - 2\,u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right) +$$

$$\left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) + \left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right) = k^2 f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.25)}$$

which is subsequently may be rearranged to give:

$$\left(u_{i+1,j}^{n+1} - 2\,u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right) + \left(u_{i,j+1}^{n+1} - 2\,u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right) =$$

$$-\left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) - \left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right) + k^2 f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.26)}$$

The related matrix of the system (2.26) is no longer tridiagonal and so, an iterative procedure is required to advance the solution through each time step.

For simplify, introduce the finite difference operator δ is introduced by the following definition:

$$\delta_x^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}$$

$$\delta_y^2 u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2}$$

then equation (2.26) becomes:

$$\delta_x^2\left(u_{i,j}^{n+1}\right) + \delta_y^2\left(u_{i,j}^{n+1}\right) = -[\delta_x^2\left(u_{i,j}^{n}\right) + \delta_y^2\left(u_{i,j}^{n}\right)] + r f_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.27)}$$

Therefore, by using the ADI method, the FDE may be splitted into two equations, namely:

$$\delta_x^2 \left(u_{i,j}^{n+1}\right) = -\delta_y^2\left(u_{i,j}^{n} \right) + rf_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.28)}$$

$$\delta_y^2\left(u_{i,j}^{n+2} \right) = -\delta_x^2\left(u_{i,j}^{n+1}\right) + rf_{i,j}^{n+\frac{1}{2}} \qquad \text{... (2.29)}$$

Noting that, for j =1, 2, 3... n −1; equation (2.28) yields a tridiagonal system of equations and can be easily solved by using Gaussian elimination method. Similarly; for i =1, 2... n −1; equation (2.29) also yields a tridiagonal system of equations. Each of the two obtained system of equations (2.28) and (2.29) are used alternately.

***2.3.1 Consistency of the ADI Method*:**

As it is worked in the parabolic equation, the consistency of the ADI method for the elliptic equation may be carried out similarly, for this purpose consider the related FDE:

$$2(\delta_x^2+\delta_y^2)\left(u_{i,j}^{n+2} + u_{i,j}^{n}\right) - 2rf_{i,j}^{n+\frac{1}{2}} = 0 \qquad \text{... (2.30)}$$

Then dividing by 2k we obtain:

$$(\delta_{x+}^2\delta_y^2) \; \frac{u_{i,j}^{n+2}+u_{i,j}^{n}}{2k} - \frac{r}{2k} f_{i,j} = 0 \qquad \text{... (2.31)}$$

and by using the finite difference formula:

$$u_{i,j}^{n+2} + u_{i,j}^{n} = 2u_{i,j}^{n+1} + O(k^2)$$

we get:

$$u_{xx}^{n+1} + u_{yy}^{n+1} - f = O(k^2) \qquad \text{... (2.32)}$$

which means that the ADI method for the two-dimensional elliptic equation is unconditionally consistent.

2.3.2 Stability of the ADI Method:

Similarly using Fourier series method, the stability of the two-dimensional elliptic equation can be investigated by the substitution:

$$u_{s,q}^{n} = \lambda^{n} e^{i(\alpha s+\beta q)k}$$

where

$$\delta_x^2 u = -4(\sin^2 \alpha k/2)u$$

$$\delta_y^2 u = -4(\sin^2 \beta k/2)u$$

Therefore:

$$-16\lambda(\sin^2 \alpha k/2) - 8\lambda(\sin^2 \beta k/2) = 0 \qquad \text{... (2.33)}$$

Then $\lambda = 0$, which means that the two dimensional ADI method is unconditionally stable and by using Lax equivalence theorem it can completely prove that the ADI method for elliptic equation is convergent [Mampaey,1989].

2.4 Implementation of ADI Method for Two Dimensional Hyperbolic Equation:

Consider the two dimensional wave equation defined in the rectangular domain $\Omega \subset R^2$ [Hoffman,2006]:

$$\frac{\partial^2 u}{\partial t^2} = \nu^{2}(u_{xx} + u_{yy}) \qquad \text{... (2.34)}$$

with the initial conditions given by:

$$u(x,y,0) = u_0(x,y) \quad ,(x,y) \in \Omega$$

$$\frac{\partial u}{\partial t}(x,y,0) = u_t(x,y) \quad ,(x,y) \in \Omega$$

and with boundary conditions:

$$u(x,y,t) = f(x,y,t), \quad (x,y) \in \partial\Omega, \; t \in (0,T]$$

Then, the finite difference formulation of equation(2.34) is:

$$u_{i,j}^{n+1} - 2\,u_{i,j}^{n} + u_{i,j}^{n-1} =$$

$$\frac{\Delta t^2 \nu^2}{k^2}\left[\left(u_{i+1,j}^{n} - 2\,u_{i,j}^{n} + u_{i-1,j}^{n}\right) + \left(u_{i,j+1}^{n} - 2\,u_{i,j}^{n} + u_{i,j-1}^{n}\right)\right] \text{... (2.35)}$$

where $\Delta x = \Delta y = k$

The Crank-Nicholson reformulation of equation (2.35) is:

$$u_{i,j}^{n+1} - 2\, u_{i,j}^{n} + u_{i,j}^{n-1} =$$
$$\frac{\Delta t^2 \nu^2}{k^2}\Big[\big(u_{i+1,j}^{n+1} - 2\, u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\big) + \big(u_{i,j+1}^{n+1} - 2\, u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\big) +$$
$$\big(u_{i+1,j}^{n} - 2\, u_{i,j}^{n} + u_{i-1,j}^{n}\big) + \big(u_{i,j+1}^{n} - 2\, u_{i,j}^{n} + u_{i,j-1}^{n}\big)\Big] \qquad \text{... (2.36)}$$

Define the operator δ, such that

$$\delta_x^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}$$

$$\delta_y^2 u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1,}}{(\Delta y)^2}$$

Then, equation (2.36) is reduced to:

$$u_{i,j}^{n+1} = 2\, u_{i,j}^{n} - u_{i,j}^{n-1} + r\big[\delta_x^2\,(u_{i,j}^{n+1}) + \delta_y^2(u_{i,j}^{n+1}) + \delta_x^2\,(u_{i,j}^{n}) +$$
$$\delta_y^2\,(u_{i,j}^{n})\big] \qquad \text{... (2.37)}$$

where $r = \frac{\Delta t^2 \nu^2}{k^2}$

The implementation of the ADI scheme is beginning with the finite difference scheme:

$$u_{i,j}^{n+1} - 2\, u_{i,j}^{n} + u_{i,j}^{n-1} = r\big[\delta_x^2\,(u_{i,j}^{n}) + \delta_y^2(u_{i,j}^{n})\big] \qquad \text{... (2.38)}$$

A further time step gives

$$u_{i,j}^{n+2} - 2\, u_{i,j}^{n+1} + u_{i,j}^{n} = r\big[\delta_x^2\,(u_{i,j}^{n+1}) + \delta_y^2(u_{i,j}^{n+1})\big] \qquad \text{... (2.39)}$$

This procedure gives the two – step algorithm:

$$u_{i,j}^{n+1} - r\delta_x^2\,(u_{i,j}^{n+1}) = 2\, u_{i,j}^{n} + r\delta_y^2(u_{i,j}^{n}) - u_{i,j}^{n-1} \qquad \text{... (2.40)}$$

$$u_{i,j}^{n+2} - r\delta_y^2(u_{i,j}^{n+2}) = 2\, u_{i,j}^{n+1} + r\delta_x^2\,(u_{i,j}^{n+1}) - u_{i,j}^{n} \qquad \text{... (2.41)}$$

for all $i, j = 1, 2 \ldots N$, $N \in \mathbb{N}$.

2.4.1 Consistency of the ADI Method:

The ADI algorithm for the hyperbolic equation may be rewritten as:

$$(1 - r\delta_x^2)u_{i,j}^{n+1} = (2 + r\delta_y^2)u_{i,j}^{n} - u_{i,j}^{n-1}$$

$$(1 - r\delta_y^2)u_{i,j}^{n+2} = (2 + r\delta_x^2)u_{i,j}^{n+1} - u_{i,j}^{n}$$

and after elimination the term at t_{n-1} and t_{n+1}, we obtain:

$$(1 - r\delta_x^2)(1 - r\delta_y^2)u_{i,j}^{n+2} = (2 + r\delta_y^2)(2 + r\delta_x^2)u_{i,j}^{n} \qquad \dots (2.42)$$

Similarly, as in the parabolic and elliptic equations, the finite difference formula will be used, to get:

$$\frac{\partial^2 u}{\partial t^2} - \nu^2(u_{xx} + u_{yy}) = O(h^2, k^2) \qquad \dots (2.43)$$

This means that the method is consistent.

2.4.2 Stability of the ADI Method:

Similarly as in the two dimensional parabolic and elliptic equations, the stability of the two dimensional hyperbolic equation may be investigated by the substitution [Mampaey,1989].

$$u_{s,q}^{n} = \lambda^n e^{i(\alpha s + \beta q)k}$$

where,

$$\delta_x^2 u = -4(\sin^2 \alpha k/2)u$$

$$\delta_y^2 u = -4(\sin^2 \beta k/2)u$$

To get:

$$\lambda\big(1 - 4r(\sin^2 \alpha k/2)\big)(1 - 4r\sin^2 (\beta k/2)) =$$

$$\big(2 + 4r(\sin^2 \alpha k/2)\big)(2 + 4r\sin^2 (\beta k/2) \qquad \dots (2.44)$$

Then by definetion,

$$\lambda = \frac{\big(2 + 4r(\sin^2 \alpha k/2)\big)(2 + 4r\sin^2 (\beta k/2)}{\big(1 - 4r(\sin^2 \alpha k/2)\big)(1 - 4r\sin^2 (\beta k/2))}$$

Then $0 < \lambda < 1$. Thus, the two dimensional hyperbolic ADI method is unconditionally stable and with the unconditional consistency, which implies that the convergent of our algorithm have been proved.

2.5 A New Formulation for the ADI Method by using Another Time Steps:

Consider the two dimensional wave equetion[Hoffman,2006]:

$$\frac{\partial^2 u}{\partial t^2} = v^2\left(u_{xx} + u_{yy}\right), (x, y) \in \Omega = [0,1]^2, t \in (0, T]$$

with initial and boundary conditions:

$$u(x, y, t) = f\,(x, y, t)$$
$$u(x, y, 0) = u_0(x, y, t)$$
$$\frac{\partial u(x,y,0)}{\partial t} = \varphi(x, y)$$

where f, u_0 and φ are given functions and $\partial\Omega$ is a boundary in the domain Ω.

Now, after replacing this equation by the finite difference approximations, one may get:

$$\left(u_{i,j}^{n+1} - 2\, u_{i,j}^{n} + u_{i,j}^{n-1}\right) = \frac{\Delta t^2 v^2}{k^2}\left[\left(u_{i+1,j}^{n} - 2\, u_{i,j}^{n} + u_{i-1,j}^{n}\right) + \left(u_{i,j+1}^{n} - 2\, u_{i,j}^{n} + u_{i,j-1}^{n}\right)\right] \quad \ldots (2.45)$$

To apply the new formula of the (ADI) method we spilt equation (2.45) as follows:

$$\left(u_{i,j}^{n+\frac{1}{2}} - 2\,u_{i,j}^{n} + u_{i,j}^{n-\frac{1}{2}}\right) = \frac{r}{2}\left[\delta_x^2\left(u_{i,j}^{n+\frac{1}{2}}\right) + \delta_y^2\left(u_{i,j}^{n}\right)\right] \qquad \ldots (2.46)$$

$$\left(u_{i,j}^{n+1} - 2\,u_{i,j}^{n+\frac{1}{2}} + u_{i,j}^{n}\right) = \frac{r}{2}\left[\delta_x^2\left(u_{i,j}^{n+\frac{1}{2}}\right) + \delta_y^2\left(u_{i,j}^{n+1}\right)\right] \qquad \ldots (2.47)$$

Carying some simplifications:

$$u_{i,j}^{n+\frac{1}{2}} - 0.5r\left[\delta_x^2\left(u_{i,j}^{n+\frac{1}{2}}\right)\right] = 2\left(u_{i,j}^{n}\right) + 0.5\,r\,\delta_y^2\left(u_{i,j}^{n}\right) - u_{i,j}^{n-\frac{1}{2}} \qquad \ldots (2.48)$$

$$u_{i,j}^{n+1} - 0.5r\left[\delta_x^2\left(u_{i,j}^{n+1}\right)\right] = 2\left(u_{i,j}^{n+\frac{1}{2}}\right) + 0.5\,r\,\delta_y^2\left(u_{i,j}^{n+\frac{1}{2}}\right) - u_{i,j}^{n} \qquad \ldots (2.49)$$

Rearrange equations (2.48) and (2.49) leads to:

$$[1 - 0.5r\,\delta_x^2]\left(u_{i,j}^{n+\frac{1}{2}}\right) = [\,2 + 0.5\,r\,\delta_y^2\,]\,u_{i,j}^{n} - u_{i,j}^{n-\frac{1}{2}} \qquad \ldots (2.50)$$

$$[1 - 0.5r\,\delta_x^2]\left(u_{i,j}^{n+1}\right) = \left[2 + 0.5\,r\,\delta_y^2\right]u_{i,j}^{n} - u_{i,j}^{n-\frac{1}{2}} \qquad \ldots (2.51)$$

This formula is used to spilt the finite difference equation by using two small time steps $[t_{n-\frac{1}{2}}, t_n]$ and $[t_n, t_{n+\frac{1}{2}}]$, with i , j = 1, 2 …N, $N \in \mathbb{N}$; which give two equations used alternately and by substituting the initial and boundary conditions, finally, the result will be a tridiagonal system which solved easily.

2.6 Increase the Accuracy of the ADI Method of Two Dimensional Equations:

Consider the two dimensional diffusion equations[Hoffman,2006]:

$$u_t = u_{xx} + u_{yy} \quad , \quad t \in (0, \infty] \ , (x, y) \in D$$

where $D = (0, l_1) \times (0, l_2)$, with initial condition:

$$u(x, y, 0) = u_0(x, y) \quad , (x, y) \in D$$

and the boundary conditions are at along ∂D .

In classical (ADI) method, a second order (centered with time and backward with space) finite difference formulation usually used to obtain a scheme, the formulation obtained in this way will be unconditionally stable with a consistent condition which will be a sufficient for a method to be convergent, the resulting formula will be a second order with time and space and as it shown in sections (2.2), (2.3) and (2.4) for each type of PDE.

For more accuracy, a new finite difference formulation will be used here which will be centered with time and space and as follows:

the finite difference formula:

$$u_t = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t}$$

$$u_{xx} = \frac{-u_{i+2,j}^{n} + 16u_{i+1,j}^{n} - 30u_{i,j}^{n} + 16u_{i-1,j}^{n} - u_{i-2,j}^{n}}{(\Delta x)^2}$$

$$u_{yy} = \frac{-u_{i,j+2}^{n} + 16u_{i,j+1}^{n} - 30u_{i,j}^{n} + 16u_{i,j-1}^{n} - u_{i,j-2}^{n}}{(\Delta y)^2}$$

These two formulations for u_{xx} and u_{yy} are the centeral differences with $O(x^4)$ and $O(y^4)$ respectively.

Replacing both sides in diffusion equation by their central difference approximations, to get :

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{-u_{i+2,j}^{n} + 16u_{i+1,j}^{n} - 30u_{i,j}^{n} + 16u_{i-1,j}^{n} - u_{i-2,j}^{n}}{(\Delta x)^2} + \frac{-u_{i,j+2}^{n} + 16u_{i,j+1}^{n} - 30u_{i,j}^{n} + 16u_{i,j-1}^{n} - u_{i,j-2}^{n}}{(\Delta y)^2} \quad \ldots (2.52)$$

Then, after considering $\Delta x = \Delta y = h$, and simplifying equation (2.52), it will become:

$$u_{i,j}^{n+1} - u_{i,j}^{n} = \frac{\Delta t}{h^2}\left[\left(-u_{i+2,j}^{n} + 16u_{i+1,j}^{n} - 30u_{i,j}^{n} + 16u_{i-1,j}^{n} - u_{i-2,j}^{n}\right) + \left(-u_{i,j+2}^{n} + 16u_{i,j+1}^{n} - 30u_{i,j}^{n} + 16u_{i,j-1}^{n} - u_{i,j-2}^{n}\right)\right] \ldots (2.53)$$

In the formulation of the ADI method, equation (2.53) is rearranged by taking two time steps as follows:

$$u_{i,j}^{n+1} - u_{i,j}^{n} = r\left[\left(-u_{i+2,j}^{n} + 16u_{i+1,j}^{n} - 30u_{i,j}^{n} + 16u_{i-1,j}^{n} - u_{i-2,j}^{n}\right) + \left(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} - 30u_{i,j}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}\right)\right] \quad \ldots (2.54)$$

$$u_{i,j}^{n+2} - u_{i,j}^{n+1} = r\left[-u_{i+2,j}^{n+2} + 16u_{i+1,j}^{n+2} - 30u_{i,j}^{n+2} + 16u_{i-1,j}^{n+2} - u_{i-2,j}^{n+2}\right) + \left(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} - 30u_{i,j}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}\right)] \quad \ldots (2.55)$$

Simplify equations (2.54) and (2.55) with some rearrangement, yields to:

$$u_{i,j}^{n+1} - 30r\, u_{i,j}^{n+1} = u_{i,j}^{n} - 30r\, u_{i,j}^{n} + r\left(-u_{i+2,j}^{n} + 16u_{i+1,j}^{n} + 16u_{i-1,j}^{n} - u_{i-2,j}^{n}\right) + r\left(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}\right) \quad \ldots (2.56)$$

$$u_{i,j}^{n+2} - 30r\, u_{i,j}^{n+2} = u_{i,j}^{n+1} - 30r\, u_{i,j}^{n+1} + r(-u_{i+2,j}^{n+2} + 16u_{i+1,j}^{n+2} + 16u_{i-1,j}^{n+2} - u_{i-2,j}^{n+2}) + r(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}) \quad \text{... (2.57)}$$

Which after some simplification become:

$$(1-30r)u_{i,j}^{n+1} = (1-30r)u_{i,j}^{n} + r(-u_{i+2,j}^{n} + 16u_{i+1,j}^{n} + 16u_{i-1,j}^{n} - u_{i-2,j}^{n}) + r(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}) \quad \text{... (2.58)}$$

$$(1-30r)u_{i,j}^{n+2} = (1-30r)u_{i,j}^{n+1} + r(-u_{i+2,j}^{n+2} + 16u_{i+1,j}^{n+2} + 16u_{i-1,j}^{n+2} - u_{i-2,j}^{n+2}) + r(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}) \quad \text{... (2.59)}$$

for i , j = 1, 2 …N, $N \in \mathbb{N}$; equations (2.58) and (2.59) will used alternatively and the resulting systems will be tridiagonal that may be solved easily.

Next, numerical examples will be considered for an illustration purpose:

Example (2.1):

The two dimensional, steady-state conduction problem, with the initial and boundary condition, with $0 \le x, y \le 1$, is given as follows:

$u(x,0) = 0$, $u(x,1) = 100$

$u(0,y) = 0$, $u(1,y) = 50$

To determine the temperature u along the surface using the ADI method, the steady state condition problem is[Hoffman,2006]:

$$u_{xx} + u_{yy} = 0 \quad \text{... (2.59)}$$

In order to solve this problem, the ADI method given in equations (2.28) and (2.29) will be applied, to get:

$$u_{i+1,j}^{n+1} - 4u_{i,j}^{n+1} + u_{i-1,j}^{n+1} = -u_{i,j-1}^{n} - u_{i,j+1}^{n} \quad \text{... (2.60)}$$

To start the iteration, set $n = 0$, for the first row, $j = 1$, then our equation gives :

$$u_{i+1,1}^{(1)} - 4u_{i,1}^{(1)} + u_{i-1,1}^{(1)} = -u_{i,0}^{(0)} - u_{i,2}^{(0)} \quad \text{... (2.61)}$$

First step calculation with $i = 1$ and $i = 2$ give two more equations:

$$u_{2,1}^{(1)} - 4u_{1,1}^{(1)} + u_{0,1}^{(1)} = -u_{1,0}^{(0)} - u_{1,2}^{(0)} \quad ...(2.62)$$

$$u_{3,1}^{(1)} - 4u_{2,1}^{(1)} + u_{1,1}^{(1)} = -u_{2,0}^{(0)} - u_{2,2}^{(0)} \quad ... (2.63)$$

Now, by substituting the values of the boundary conditions, and by assuming that $u_{12} = 100$, $u_{22} = 100$, the above equations yields:

$$u_{2,1}^{(1)} - 4u_{1,1}^{(1)} + 0 = -100$$

$$4u_{1,1}^{(1)} - 100 = u_{2,1}^{(1)} \quad ... (2.64)$$

$$50 - 4u_{2,1}^{(1)} + u_{1,1}^{(1)} = -100 \quad ... (2.65)$$

By substituting the value of $u_{2,1}^{(1)}$ from equation (2.64) in equation (2.65):

$$50 - 4(4u_{1,1}^{(1)} - 100) + u_{1,1}^{(1)} = -100$$

$$u_{1,1}^{(1)} = 78.5714\,, \quad u_{2,1}^{(1)} = 214.2857$$

To compute the values over the second row, set $j = 2$ in equation (2.60). Thus:

$$u_{i+1,2}^{(1)} - 4u_{i,2}^{(1)} + u_{i-1,2}^{(1)} = -u_{i,1}^{(0)} - u_{i,3}^{(0)} \quad ... (2.66)$$

Now, with $i = 1$ and $i = 2$ equation (2.66) gives:

$$u_{2,2}^{(1)} - 4u_{1,2}^{(1)} + u_{0,2}^{(1)} = -u_{1,1}^{(0)} - u_{1,3}^{(0)} \quad ... (2.67)$$

$$u_{3,2}^{(1)} - 4u_{2,2}^{(1)} + u_{1,2}^{(1)} = -u_{2,1}^{(0)} - u_{2,3}^{(0)} \quad ... (2.68)$$

By substituting the boundary values and solving the above equations, we get

$$u_{2,2}^{(1)} - 4u_{1,2}^{(1)} + 0 = -100$$

$$u_{2,2}^{(1)} = 4u_{1,2}^{(1)} - 100 \quad ... (2.69)$$

$$50 - 4u_{2,2}^{(1)} + u_{1,2}^{(1)} = -0 - 100 \quad ... (2.70)$$

$$-4\left(4u_{1,2}^{(1)} - 100\right) + u_{1,2}^{(1)} = -1 \quad ... (2.71)$$

$$u_{1,2}^{(1)} = 78.5714\,, \quad u_{2,2}^{(1)} = 214.2857$$

Completing the computations on the two rows, and then alternate the direction to compute the value of the solution on the columns, starting with the first one, for this purpose by using the formula:

$$u_{i,j+1}^{n+2} - 4u_{i,j}^{n+2} + u_{i,j-1}^{n+2} = -u_{i-1,j}^{n+1} - u_{i+1,j}^{n+1} \qquad \text{... (2.72)}$$

with $n = 0$, $i = 1$ and $j = 1,2$ respectively:

$$u_{1,2}^{(2)} - 4u_{1,1}^{(2)} + u_{1,0}^{(2)} = -u_{0,1}^{(1)} - u_{2,1}^{(1)} \qquad \text{... (2.73)}$$

$$u_{1,3}^{(2)} - 4u_{1,2}^{(2)} + u_{1,1}^{(2)} = -u_{0,2}^{(1)} - u_{2,2}^{(1)} \qquad \text{... (2.74)}$$

Now, substituting the boundary values in the equations (2.73) and (2.74):

$$u_{1,2}^{(2)} - 4u_{1,1}^{(2)} = -214.2857 \qquad \text{... (2.75)}$$

$$-4u_{1,2}^{(2)} + u_{1,1}^{(2)} = -314.2857 \qquad \text{... (2.76)}$$

The above system of equations may be solved by computing the tridiagonal matrix, which will give the system :

$$\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} u_{1,1}^{(2)} \\ u_{1,2}^{(2)} \end{bmatrix} = \begin{bmatrix} -214.2857 \\ -314.2857 \end{bmatrix}$$

and hence:

$$u_{1,1}^{(2)} = 78.09522 \quad , u_{1,2}^{(2)} = 98.09523$$

Now, for $i = 2$, $j = 1,2$, we get :

$$u_{2,2}^{(2)} - 4u_{2,1}^{(2)} + u_{2,0}^{(2)} = -u_{1,1}^{(1)} - u_{3,1}^{(1)} \qquad \text{... (2.77)}$$

$$u_{2,3}^{(2)} - 4u_{2,2}^{(2)} + u_{2,1}^{(2)} = -u_{1,2}^{(1)} - u_{3,2}^{(1)} \qquad \text{... (2.78)}$$

Substituting the boundary values in the equations (2.77) and (2.78):

$$u_{2,2}^{(2)} - 4u_{2,1}^{(2)} + 0 = -78.5714 - 50$$

$$100 - 4u_{2,2}^{(2)} + u_{2,1}^{(2)} = -78.5714 - 50$$

$$u_{2,2}^{(2)} - 4u_{2,1}^{(2)} = -128.5714 \qquad \text{... (2.79)}$$

$$-4u_{2,2}^{(2)} + u_{2,1}^{(2)} = -228.5714 \qquad \text{... (2.80)}$$

Solve the above system of equations by computing tridiagonal matrix:

$$\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} u_{2,1}^{(2)} \\ u_{2,2}^{(2)} \end{bmatrix} = \begin{bmatrix} -128.5714 \\ -228.5714 \end{bmatrix}$$

$$u_{2,1}^{(2)} = 27.8095 , \qquad u_{2,2}^{(2)} = 85.5238$$

The iterations are continued to improve the solution function values obtained first along the discretization rows and then along the discretization columns, and so on ,e.g. we continue our process for n = 1,2,3 ...

***Example (2.2)* :**

Consider the two dimensional parabolic equation:

$$u_t = u_{xx} + u_{yy} - f(x,y)$$

with the exact solutionis given by:

$$u(x,y,t) = \exp(-t)\sin(\pi x)\sin(\pi y), (x,y) \in D, \ 0 \le t \le 1$$

where $D = (0,1)^2$. With initial and boundary conditions:

$$u(x,y,0) = 0 \quad ,(x,y) \in D$$

$$u(x,y,t) = 0 \ , \ 0 \le t \le 1$$

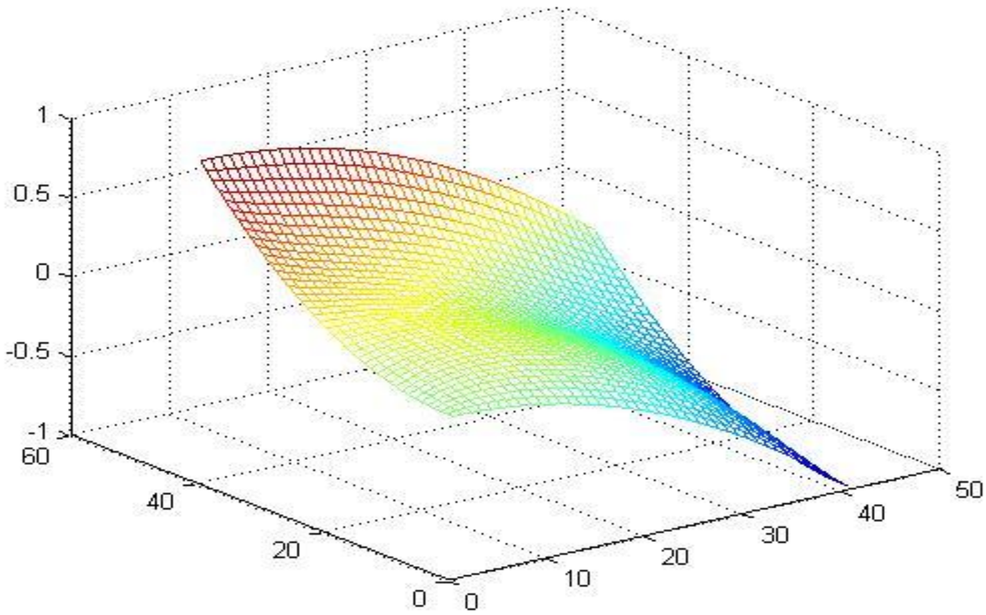
By using the two dimensional ADI formula for parabolic equation

$$u_{i,j}^{n+1} - r\,\delta_x^2\left(u_{i,j}^{n+1}\right) = u_{i,j}^{n} + r\,\delta_y^2\left(u_{i,j}^{n}\right) - \Delta t\ f_{i,j}^{n+\frac{1}{2}}$$

$$u_{i,j}^{n+2} - r\,\delta_x^2\left(u_{i,j}^{n+2}\right) = u_{i,j}^{n+1} + r\,\delta_y^2\left(u_{i,j}^{n+1}\right) - \Delta t\ f_{i,j}^{n+\frac{1}{2}}$$

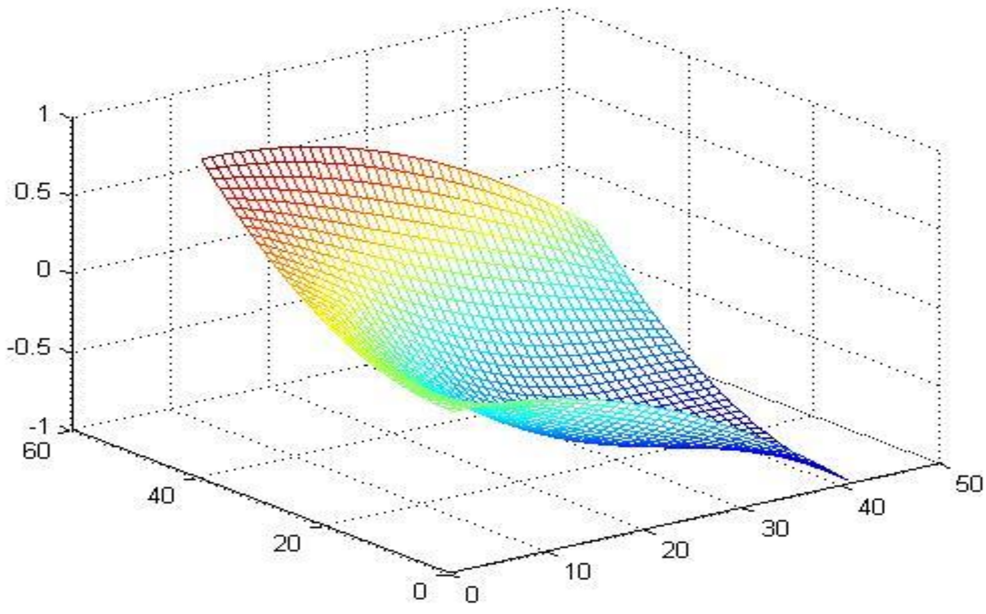
Which is used alternately with i, j=1,2,..., $n-1$

Here, Dirichlet boundary conditions are taken. One can see that when having large grid sizes, with few nodes, it will be better to have few and larger time steps. The interesting result is that there seem to be an optimal ratio between the grid size, the direction and the time steps.



Figure(2.2)*Numerical solution of example (2.2) by using ADI method with ; n = 40 ;*

final time = 0.8, h = (b-a)/n , i,j= 1,2,....



Figure(2.3)exact solution for example(2.2)by using ADI method with; n = 40 ;

final time = 0.8, h = (b-a)/n , i,j= 1,2,....

3.1 Introduction:

Chronic wounds are wounds which fail to proceed through normally therapy processes[Russell,2013] that represent a significant soconomic problem, because those peoples who suffering from chronic wounds test large pain affects quality of life and mental condition in which the treatment of the symptoms of the disease costs a money and take a lot of time, [John,et.al.,2013]

Despite recent advances in the knowledge of wound heal, the treatment of chronic wounds remains limited basically because there are many different causes of wound chronic like diabetes, venous arterial occlusion, venous in sufficiency, etc.,[Russell, 2013].

Furthermore, there is much discussion about the best way for treatment of those wounds, the required treatment differes depending on the causes of the wound chronic, [Guffey,2015].

In order to full understanding of the situation of chronic wound, and to determine what prevents them from healing, time consumption and the difficult technology, the mathematical modelling may give a clear vision into the wound healing process [Jennifer,2009]. Mathematical modelling has the ability to generate a theoretical prediction that cannot be expected before that.

Hyperbaric Oxygen Therapy (HBOT) is the process of intermittent breathing 100% oxygen at the same time increased while the pressure of the treatment room of more than one atmosphere, [Stephen, 2015]. There have been several reports that HBOT aids the healing of ulcerated wounds. One way in which this benefit to be achieved is through an increase in the tissue oxygen tension in the injured region, [Jennifer,2009].

The aim of this chapter is to present the one dimensional mathematical modelling and then developing of this model for two dimension and explain how the effect of oxygen therapy technique in treatment of a chronic wound, prelude for solving the system in two dimension by using our approach the ADI method, we present in some detail the motivation behind its use.

This chapter consist of eight sections. Section one present an introduction for the problem, section two present a biological concepts with respect to problem, section three present mathematical model of the problem, section four introduce one dimensional model of the problem, section five containes the dimensionless of the problem, section sex present the analytical results, section seven present the development of the model in two dimension, finally, section eight contains numerical method and results.

3.2 Biological Concepts of the Problem:

The successful wound healing is throught four stages; namely hemostasis, inflammation, proliferation and remodeling [John, 2013]. Each stage may be defined as follows:

Hemostasis; means that when exposure to infected blood tissue capillaries cut currents inside the wound, and is aimed at the immediate response of the body to impede blood loss, blood flow in the wound carries platelets and fibrinogen, both of them important in the healing process [Jennifer, 2009].

Inflammation; is featuring by the arriving of phagocytic to the wound site, [Stephen,2015]. After one day of injury, the attraction of neutrophils in the wound by the chemoattractant already been issued in foreign phagocytose bacteria and particle coagulation process, [Jennifer,2009].

Proliferation; here during the phase proliferative of wound healing the dominant cell is the fibroblast, which is responsible for making the extra cellular matrix (ECM) and collagen, [Russell,2013]. As the inflammatory phase progresses and the cells needed for repair and regeneration reach the injury site, the proliferative phase begins four crucial events during proliferative phase; namely angiogenesis, granulation tissue formation, wound contraction and epithelialization [John,2013]. The proliferation of fibroblasts depends on the amount of oxygen available and these cells can live only duties required in the wound where enough oxygen available [Jennifer,2009]. It is stimulated by chemoattractant, such as platelet derived growth factor to produce the collagen, [Russell, 2013].

Remodeling, it could be the final stage delayed for several months or even years during the remodeling, and the absence of hypoxia results in a significant decrease in the density of blood vessels and increase cellular cells, [Jennifer, 2009]. Also during the remodeling phase and the formation of the wound is shrinking wound cells completely become a force wound after an increase of about 20% of normal tensile strength, and after three weeks of the injury will be measuring up to 80% and get this within two years, [John,2013].

Figure 3.1 illustrate the stages of the successfull healing wound.

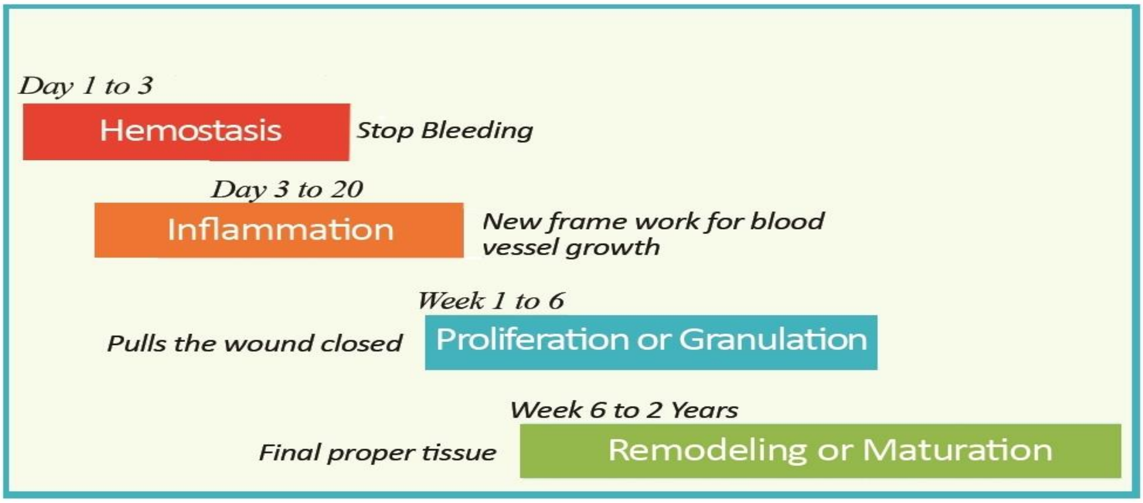


Figure (3.1): Stages of wound healing [Jinnefer,2009]

3.3 Mathematical Modelling of the Problem:

Over the last century, applied mathematics has been use for a wide range of real life and biological fields There is no surprise that mathematics played an important and key role in understanding the growth, activity and the proliferation of bacteria in nature, chronic wound and the effect of oxygen in the treatment of these wounds, [Shuaa,2011].

Mathematical modelling can provide independent insight into biological process and has the potential to generate theoretical predictions, which could not been anticipated in advance, therapy stimulating further biomedical research and reducing the need for time-consuming, technically difficult and often costly experiments, [Jennifer,2009].

Those mathematical modeling researchs give an insight into wound healing mechanisms look at the relative importance of the processes and information that are also useful in the treatment of methodologies that will be carried out by medical staff, [Russell,2013]. Mathematical models can be used for a number of different reasons, such as the development of scientific understanding and testing the effect of changes in the system, [Shuaa, 2011].

Figure 3.2 illustrate the steps followed in studying such type of problem formulation and analysis.

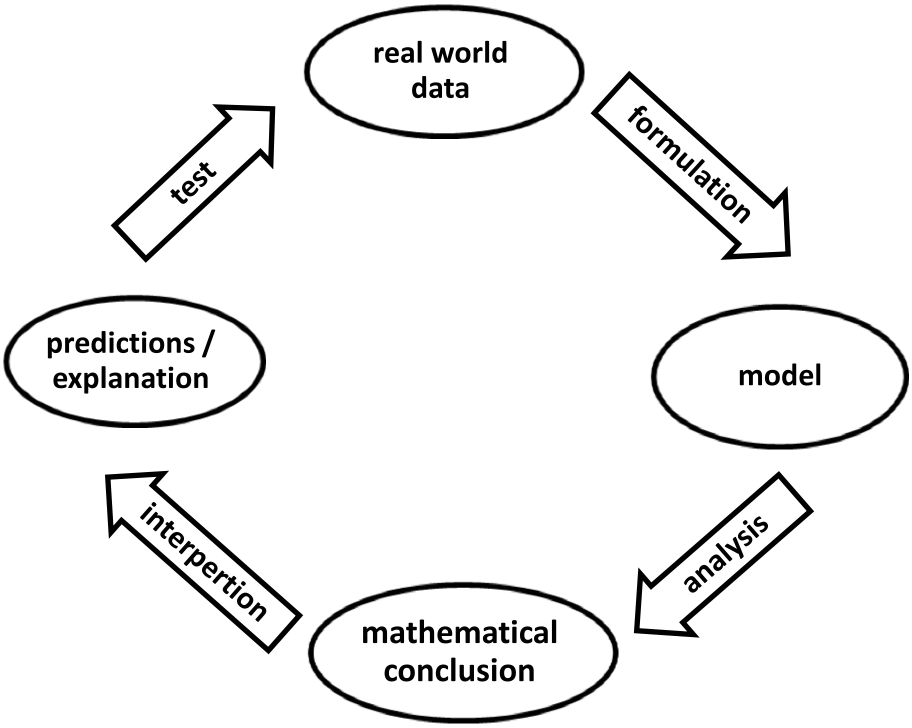


Figure (3.2): Stages of modelling (Vetterling, 1987)

3.4 The one Dimensional Model [Guffey,2015]:

The model describes the interactions between oxygen, bacteria and a chemoattractant in a radially symmetric wound under hyperbaric oxygen therapy. The wound is considered to be one-dimensional with $x=0$ at the center to $x=L$, $a \leq x \leq L$, measured in centimeters. Let t denote the time from the start of treatment measured in seconds. The variable w represents the concentration of oxygen in grams per centimeter, and let b denotes the concentration of the chemoattractant released by bacteria.

3.4.1 Oxygen equation:

Oxygen is the key element in this treatment, where this process is under pressure oxygen to the wound festering certain quantities with some chemical effects are scaled work of the bacteria causing the inflammation. Thus, the model for oxygen parameter in the wound can be formulated as follows[Guffey,2015]:

$$\frac{\partial w}{\partial t} = D_w \frac{\partial^2 w}{\partial x^2} + \beta + kG(t) - \gamma_{nw} nw - \gamma_{bw} bw - \gamma_w w \quad \ldots (3.1)$$

where:

D_w is the constant rate of diffusion measured in cm^2/s.

k represents the oxygen in the wound.

β the constant rate that represent the oxygen enter the wound.

$G(t)$ Oxygen increases through the supplemental oxygen given astherapy.

γ_{nw} , γ_{bw} rates represent oxygen that have been used by bacteria.

γ_w the rate of losing oxygen.

Figure(3.3): A hyperbaric chamber machine used to provide oxygen therapy to the patients.

[John G., et. al., *2013*]

3.4.2 Bacterial equation:

Injury, bad situation and deaths associated with lack of healing of skin wounds and chronic with high numbers of people suffering from obesity and chronic diseases, has been emphasizing the role of bacterial communities in chronic wounds in recent years, particularly in the context of prolonging the duration of inflammatory treatment[Guffey,2015]:

$$\frac{\partial b}{\partial t} = \varepsilon_b \frac{\partial^2 b}{\partial x^2} + k_b b\left(1 - \frac{b}{b_0}\right) - b\,\frac{w\delta + k_{nr} n}{kw + w\gamma_{rb} b + \gamma_r} - \gamma_b b \qquad \text{... (3.2)}$$

ε_b Constant random motility for bacteria taken from center of wound

γ_b Natural death of bacteria

Figure (3.4): Progression of healing in a chronic wound [John G., et. al., *2013*]

Figure (3.5): Progression of healing in a chronic wound to patient after two weeks without using HBOT

Figure (3.6): Progression of healing in a chronic wound to patient after four weeks without using HBOT

Figure(3.7): A machine used to provide oxygen to patients in figures (3.5) and (3.6)

3.4.3 Chemoattractant equation:

Chemotaxis is the phenomenon whereby somatic cells, bacteria, and other single-cell or multicellular organisms direct their movements according to certain chemicals in their environment. This is important for bacteria to find food, Chemoattractant is represents inorganic or organic substances possessing chemotaxis-inducer effect in motile cells. Effects of chemoattractant are elicited via described or hypothetic chemotaxis receptors, the chemoattractant moiety of a ligand is target cell specific and concentration dependent [Julius A.,1974].

The chemoattractant equation for our biological problem is as follows[Guffey,2015]:

$$\frac{\partial c}{\partial t} = D_c \frac{\partial^2 c}{\partial x^2} + k_b b - \gamma_c c \qquad \text{... (3.3)}$$

where

D_c Constant rate of diffusion (cm^2/s)

k_b Represent the chemoattractant produced by the bacteria

γ_c Represent the constant rate of chemoattractant decadency

3.4.4 Initial and boundary conditions:

The intial and boundary conditions of one dimensional problem may be defined as follows:

The selection of w(x) and its conditions by assumption that the oxygen level is stabilized in the wound boundary at the first six hours. Thus, at the end of this hours the conditions will be:

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \quad , \quad \left.\frac{\partial b}{\partial x}\right|_{x=0} = 0 \quad , \quad \left.\frac{\partial c}{\partial x}\right|_{x=0} = 0$$

The boundary conditions after this six hours will taken by assuming that the blood vessels are located directly after ending of this hours outside the chronic wound region and with closed region to keep oxygen level at the skin capacity.

$$\mathrm{w(L,t) = w_0} \quad , \qquad \left.\frac{\partial \mathrm{b}}{\partial \mathrm{x}}\right|_{\mathrm{x=L}} = 0$$

$$\mathrm{w(x,0) = L} \quad , \qquad \mathrm{b(x,0) = b_0}\left(\frac{\mathrm{x-L}}{\mathrm{L}}\right)^2 e^{-\left(\frac{\mathrm{x}}{\mathrm{L}}\right)^2}$$

$$\left.\frac{\partial \mathrm{c}}{\partial \mathrm{x}}\right|_{\mathrm{x=L}} = 0 \quad , \qquad \mathrm{c(x,0) = c_0}\left(\frac{\mathrm{x-L}}{\mathrm{L}}\right) e^{-\left(\frac{\mathrm{x}}{\mathrm{L}}\right)^2}$$

3.5 Non- Dimensionless of Equations:

After non-dimensionlizing system (3.1) - (3.3) as it is introduced in [Guffey, 2015], we get finally:

$$\frac{\partial w}{\partial t} = D_w \frac{\partial^2 w}{\partial x^2} + \beta + kG(t) - \gamma_{nw} n w - \gamma_{bw} b w - \gamma_w w \qquad \text{... (3.4)}$$

$$\frac{\partial b}{\partial t} = \varepsilon_b \frac{\partial^2 b}{\partial x^2} + k_b b(1 - b_o) - b\frac{w\delta + k_{nr} n}{kw + w\gamma_{rb} b + 1} - \gamma_b b \qquad \text{... (3.5)}$$

$$\frac{\partial c}{\partial t} = D_c \frac{\partial^2 c}{\partial x^2} + k_b b - \gamma_c c \qquad \text{... (3.6)}$$

where

$$G(t) = \begin{cases} 1 & when\ oxygen\ is\ administerad \\ 0 & otherwise \end{cases}$$

With the initial and boundary condition

$$\left.\frac{\partial \mathrm{b}}{\partial \mathrm{x}}\right|_{\mathrm{x=0}} = 0 \ , \quad \left.\frac{\partial \mathrm{b}}{\partial \mathrm{x}}\right|_{\mathrm{x=L}} = 0 \quad , \quad \mathrm{b(x,0)} = (1 - x)^2 e^{-\left(\frac{x}{2}\right)^2}$$

$$\left.\frac{\partial \mathrm{w}}{\partial \mathrm{x}}\right|_{\mathrm{x=0}} = 0 \ , \quad \mathrm{w(L,t) = w_0} \quad , \quad \mathrm{w(x,0) = L}$$

$$\left.\frac{\partial \mathrm{c}}{\partial \mathrm{x}}\right|_{\mathrm{x=0}} = 0 \ , \quad \left.\frac{\partial \mathrm{c}}{\partial \mathrm{x}}\right|_{\mathrm{x=L}} = 0 \quad , \quad \mathrm{c(x,0)} = (1 - x)^2 e^{-\left(\frac{x}{2}\right)^2}$$

3.6 Analytical Results:

Analytical results may provid valuable insight to the biological implications of the model. First, the wound will be examined without the pressure of bacteria in order to provide a biological description of the oxygen and chemoattractant, where we must examine the steady states of the system. Consider the ordinary differential equations satisfied by travelling wave solution of equation (3.4) and (3.6) where b is assumed to be zero travelling wave arise frequently in the context of wound healing. Biologically this implies that a wave front of cells move with a constant speed and constant shape. It is assumed that there is a solution for each equation of the form f(ϕ), where $\phi = x + at$ and a is the speed of propagation of the travelling wave. without presence of bacteria, let:

$$w(x,t) = f(\phi) \ , \quad c(x,t) = g(\phi)$$

Then equations (3.4) and (3.6) will becomes :

$$a \frac{df}{d\phi} = D_w \frac{d^2 f}{d\phi^2} + \beta - \gamma_{nw} f g - \gamma_w f \qquad \text{... (3.7)}$$

$$a \frac{dg}{d\phi} = D_c \frac{d^2 g}{d\phi^2} - \gamma_c g \qquad \text{... (3.8)}$$

To examine the steady states of this system, this system will transformed to a first order system by setting:

$$x_1 = f$$

$$x_2 = \frac{df}{d\phi}$$

$$x_3 = g$$

$$x_4 = \frac{dg}{d\phi}$$

Substituting this change of variables into the system and separate the derivatives to one side and remove the terms with spatial derivatives, implies that:

$$a\,x_2^{'} = \beta - \gamma_{nw}x_1x_3 - \gamma_w x_1 \qquad \text{... (3.9)}$$

$$a\,x_4^{'} = -\gamma_c x_3 \qquad \text{... (3.10)}$$

and upon dividing on a, implies

$$x_2^{'} = \frac{\beta}{a} - \frac{\gamma_{nw}}{a}x_1x_3 - \frac{\gamma_w}{a}x_1 \qquad \text{... (3.11)}$$

$$x_4^{'} = -\frac{\gamma_c}{a}x_3 \qquad \text{... (3.12)}$$

Afer setting the derivatives to zero, equations (3.11) and (3.12) will gives the steady state $(x_1, x_3) = \left(\frac{\beta}{\gamma_w}, 0\right)$. The Jacobian matrix for this system is:

$$J = \begin{bmatrix} -\frac{\gamma_{nw}}{a}x_3 - \frac{\gamma_w}{a} & -\frac{\gamma_{nw}}{a}x_1 \\ 0 & -\frac{\gamma_c}{a} \end{bmatrix}$$

Then the eigenvalues at $J(x_1, x_3) = J(\frac{\beta}{\gamma_w}, 0)$ will be $\{-\frac{\gamma_w}{a}, -\frac{\gamma_c}{a}\}$.

The real part of the eigenvalues are negative, therefore, the steady state is stable. Which means that, the chemoattractant of the system will tend to zero and the oxygen level will be stable naturally in the wound to an average equals to $\frac{\beta}{\gamma_w}$, without bacteria in the system.

Now, to study the steady states of the system, substitute x_1, x_2, x_3 and x_4 into the system, implies that:

$$x_1' = x_2$$

$$x_2' = \frac{\gamma_{nw}}{D_w} x_1 x_3 - \frac{a}{D_w} x_2 + \frac{\gamma_w}{D_w} - \frac{\beta}{D_w}$$

$$x_3' = x_4$$

$$x_4' = \frac{\gamma_c}{D_n} x_3 - \frac{\gamma_c}{D_n} x_2 x_4 - \frac{a}{D_n} x_1 + \frac{\gamma_c}{D_n} x_3 \left(\frac{a}{D_w} x_2 - \frac{\gamma_w}{D_w} x_1 x_3 + \frac{\beta}{D_w}\right)$$

Solving for x_1, x_2, x_3 and x_4, where the Jacobian matrix at the critical point $\left(\frac{\beta}{\gamma_w}, 0,0,0\right)$

$$J\left(\frac{\beta}{\gamma_w}, 0,0,0\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma_{nw}}{D_w} x_3 & \frac{-a}{D_w} & \frac{\gamma_{nw}}{D_w} x_1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\gamma_c}{D_n}\frac{\gamma_w}{D_w} x_3 - \frac{a}{D_n} & -\frac{\gamma_c}{D_n} x_4 + \frac{\gamma_c a}{D_n D_w} x_3 & \frac{\gamma_c}{D_n} - \frac{\gamma_w}{D_w} x_1 & -\frac{\gamma_c}{D_n} x_2 \end{bmatrix}$$

And therefore the characteristic polynomial is :

$$p(\alpha) = a_0 \alpha^4 + a_1 \alpha^3 + a_2 \alpha^2 + a_3 \alpha + a_4$$

$$= \alpha^4 + \frac{aD_w + aD_n}{D_n D_w} \alpha^3 + \frac{a^2 - \gamma_w D_n - \gamma_c D_w}{D_n D_w} \alpha^2 + \frac{a(\gamma_w + \gamma_c)}{D_n D_w} \alpha + \frac{(\gamma_w \gamma_c)}{D_n D_w}$$

All parameter values are positive, therefore $a_1, a_2, a_3, a_4 > 0$. Then, according to Routh- Hurwitz theorem the system is stable, which means that during the inflammatory stage, oxygen level will be vary spatially in the wound.

3.7 Model Development:

To develop the model from one dimensional space to two dimensional cases, we must describe the interactions in space and time of the biological factors in the healing process, and therefore, three component are considerd namely *w* which represents oxygen, b which represent the bacteria and *c* which represent chemoattractant.

Now, for $0 < x < l_1$, $0 < y < l_2$The boundary conditions are given by:

$f(0,y) = y \quad , \quad f(x,0) = x$

$f(l_1, y) = l_1 + y \quad , \quad f(x, l_2) = x + l_2$

Consider the diffusion equation represented as in the following:

$$\frac{\partial f}{\partial t} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

where D represents the diffusivity rate.

Before the development of this biological system, which consists of three nonlinear partial differential equations, we point out that, the oxygen would be the same amount that was indicated to it in the form of one dimensional model, where the oxygen concentration is increased by the rate β , and the chemoattractant acts to stimulate the endothelial cells of vessels in the nearby healthy tissue to leave their parent vessel; however, it is observed that the low levels of oxygen can result in sever impairment of wound tissue granulation.

Modelling will exist by assuming that the death rate of bacteria γ_b are increased in the region of hypoxia so that:

$$\gamma_b \propto \frac{1}{\gamma_{rb} b + 1}$$

Our system of equations will be:

1. Oxygen equation:

The development and modification of equation (3.1) for two dimensional case is:

$$\frac{\partial w}{\partial t} = D_w \underbrace{\nabla^2 w}_{diffusion} + \underbrace{\beta}_{\substack{rate\ of\\ oxygen\\ enter}} + k \underbrace{G(t)}_{\substack{rate\ of\\ oxygen\\ increasing\\ supply}} \underbrace{-\gamma_{nw} nw - \gamma_{bw} bw}_{\substack{rate\ of\\ oxygen\\ used\ by\\ bacteria}} - \underbrace{\gamma_w w}_{\substack{lossing\\ oxygen}} \quad \ldots (3.10)$$

2. Bacteria equation:

In two dimensional case, the bacterial equation (3.2) take the form:

$$\frac{\partial b}{\partial t} = \underbrace{\epsilon_b}_{\substack{constant\\ random\\ motilty}} \underbrace{\nabla^2 b}_{diffusion} + k_b b(1-b_o) - b\frac{w\delta + k_{nr}n}{kw + w\gamma_{rb}b + 1} - \underbrace{\gamma_b b}_{\substack{dnatural\\ death\\ of\\ bacteria}} \quad \ldots (3.12)$$

3. Chemoattractant equation:

The two dimensional formulation of equation (3.3) is:

$$\frac{\partial c}{\partial t} = D_c \underbrace{\nabla^2 c}_{diffusion} + \underbrace{k_b b}_{\substack{rate\ of\\ chemoattractent\\ from\\ bacteria}} - \underbrace{\gamma_c}_{\substack{rate\ of\\ chemoattractent\\ decadency}} c \quad \ldots (3.11)$$

where: $\nabla^2 = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2}$

4. Intial and boundary conditions:

The initial and boundary conditions may be developed as:

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \quad , \quad \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0$$

$$w(1, y, t) = 1 \ , \ w(x, 1, t) = 1 \ , \ w(x, y, 0) = 1$$

$$\left.\frac{\partial b}{\partial x}\right|_{x=0} = 0 \quad , \quad \left.\frac{\partial b}{\partial y}\right|_{y=0} = 0 \quad , \quad b(x, y, 0) = (1 - x - y)^2\, e^{-\left(\frac{xy}{2}\right)^2}$$

$$\left.\frac{\partial c}{\partial x}\right|_{x=0} = 0 \quad , \quad \left.\frac{\partial c}{\partial y}\right|_{y=0} = 0$$

$$\left.\frac{\partial}{\partial x}\right|_{x=1} = 0 \ , \quad \left.\frac{\partial c}{\partial y}\right|_{y=1} = 0 \quad , \quad c(x, y, 0) = (1 - x - y)^2\, e^{-\left(\frac{xy}{2}\right)^2}$$

For computer solution purpose, parameter values used in the numerical solution are listed in table 3.1 which are:

Table (3.1): Parameter values used in numerical solution [Russell, 2013]

parameter	***Non-dimensional***	***Dimensional***
w_o	*1*	*5.4**$10^{-6} g * cm^{-1}$
b_o	*1*	*1**$10^{-3} g * cm^{-1}$
c_o	*1*	*1*
D_w	*1*	*5**$10^{-6} cm^2 * s^{-1}$
D_c	*1.5*	*1.5*
$\in_b$	*0.0001*	*0.0001*
β	*0.2284*	*6.1667**$10^{-12} cm^{-1} * g * s^{-1}$
G	*0*	*0*
γ_{nw}	*37*	*0.185*$g^{-1} * s^{-1}$
γ_{bw}	*22.7872*	*22.7872*
γ_w	*2.4667*	*0.01233*$* 10^{-12}\ s^{-1}$
k_b	*1.26*	*7.13*$* 10^{-5}\ s^{-1}$
γ_c	*0.9*	*0.9*
k_c	*10*	*10*
δ	*0.7992*	*0.7992*
γ_{rb}	*3.73*	*3.73*
γ_b	*5*	*2.5*$* 10^{-6}\ s^{-1}$
k_{nr}	*2*	*2*

3.8 Numerical Method and Results:

The system of equations (3.10), (3.11) and (3.12) can solve easily by using two dimensional non homogeneous parabolic equations (2.8) and (2.9) as follows, where $r = \frac{\Delta t}{K^2}$:

Oxygen equation:

$$(1+2r)w_{i,j}^{n+1} - rw_{i+1,j}^{n+1} - rw_{i-1,j}^{n+1}$$

$$= (1-2r)w_{i,j}^{n} + rw_{i,j+1}^{n} + rw_{i,j-1}^{n} + \beta + kG(t) - \Delta t\,\gamma_{nw}\, n_{i,j}^{n+\frac{1}{2}}$$

$$w_{i,j}^{n+\frac{1}{2}} - \Delta t \gamma_{bw} b_{i,j}^{n+\frac{1}{2}}\, w_{i,j}^{n+\frac{1}{2}} - \gamma_w w_{i,j}^{n+\frac{1}{2}}$$

$$(1+2r)w_{i,j}^{n+2} - rw_{i+1,j}^{n+2} - rw_{i-1,j}^{n+2}$$

$$= (1-2r)w_{i,j}^{n+1} + \ rw_{i,j+1}^{n+1} \ + rw_{i,j-1}^{n+1} - \beta + kG(t) - \Delta t\,\gamma_{nw}\, n_{i,j}^{n+\frac{1}{2}}$$

$$w_{i,j}^{n+\frac{1}{2}} - \Delta t \gamma_{bw} b_{i,j}^{n+\frac{1}{2}}\, w_{i,j}^{n+\frac{1}{2}}$$

Bacteria equation:

$$(1+2r)b_{i,j}^{n+1} - rb - rb_{i-1,j}^{n+1}$$

$$= (1-2r)b_{i,j}^{n} + rb_{i,j+1}^{n} + rb_{i,j-1}^{n} + \Delta t\, k_b\, b_{i,j}^{n+\frac{1}{2}} \left(1 - b_{i,j}^{n+\frac{1}{2}}\right)$$

$$- b_{i,j}^{n+\frac{1}{2}} \frac{w_{i,j}^{n+\frac{1}{2}}\delta + k_{nr}n}{kw_{i,j}^{n+\frac{1}{2}} + w_{i,j}^{n+\frac{1}{2}}\gamma_{rb} b_{i,j}^{n+\frac{1}{2}} + 1} - \Delta t \gamma_b b_{i,j}^{n+\frac{1}{2}}$$

$$(1+2r)b_{i,j}^{n+2} - rb_{i+1,j}^{n+2} - rb_{i-1,j}^{n+2}$$

$$= (1-2r)b_{i,j}^{n+1} + \ rb_{i,j+1}^{n+1} \ + rb_{i,j-1}^{n+1} + \Delta t\, k_b\, b_{i,j}^{n+\frac{1}{2}} \left(1 - b_{i,j}^{n+\frac{1}{2}}\right)$$

$$- b_{i,j}^{n+\frac{1}{2}} \frac{w_{i,j}^{n+\frac{1}{2}}\delta + k_{nr}n}{kw_{i,j}^{n+\frac{1}{2}} + w_{i,j}^{n+\frac{1}{2}}\gamma_{rb} b_{i,j}^{n+\frac{1}{2}} + 1} - \Delta t \gamma_b b_{i,j}^{n+\frac{1}{2}}$$

Chemoattractant equation:

$$(1+2\mathrm{r})\mathrm{C}_{\mathrm{i,j}}^{\mathrm{n+1}} - \mathrm{rC}_{\mathrm{i+1,j}}^{\mathrm{n+1}} - r\mathrm{C}_{\mathrm{i-1,j}}^{\mathrm{n+1}}$$

$$= (1-2\mathrm{r})\mathrm{C}_{\mathrm{i,j}}^{\mathrm{n}} + \mathrm{rC}_{\mathrm{i,j+1}}^{\mathrm{n}} + \mathrm{rC}_{\mathrm{i,j-1}}^{\mathrm{n}} + \Delta\mathrm{t}\ \ k_b\ b_{i,j}^{n+\frac{1}{2}} - \Delta\mathrm{t}\gamma_c C_{i,j}^{n+\frac{1}{2}}$$

$$(1+2\mathrm{r})\mathrm{C}_{\mathrm{i,j}}^{\mathrm{n+2}} - \mathrm{rC}_{\mathrm{i+1,j}}^{\mathrm{n+2}} - \mathrm{rC}_{\mathrm{i-1,j}}^{\mathrm{n+2}}$$

$$= (1-2\mathrm{r})\mathrm{C}_{\mathrm{i,j}}^{\mathrm{n+1}} + \ \mathrm{rC}_{\mathrm{i,j+1}}^{\mathrm{n+1}} \ + \mathrm{rC}_{\mathrm{i,j-1}}^{\mathrm{n+1}} + \Delta\mathrm{t}\ \ k_b\ b_{i,j}^{n+\frac{1}{2}} - \Delta\mathrm{t}\gamma_c C_{i,j}^{n+\frac{1}{2}}$$

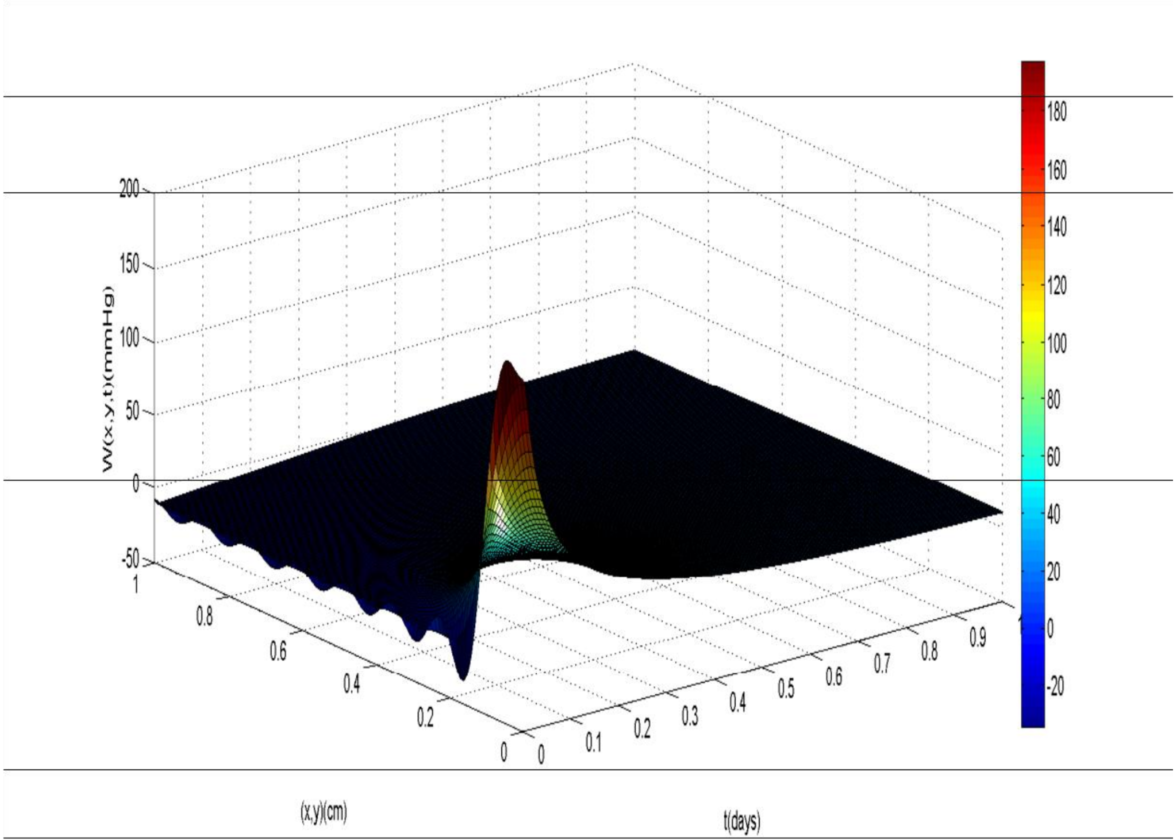


Figure (3.5)

Numerical Solutionof system(3.10)-(3.12) by using ADI method; after(1)day with ,n= 10, h = (b-a)/n ,
i,j= 1,2,... Solutions are produced using MATLAB

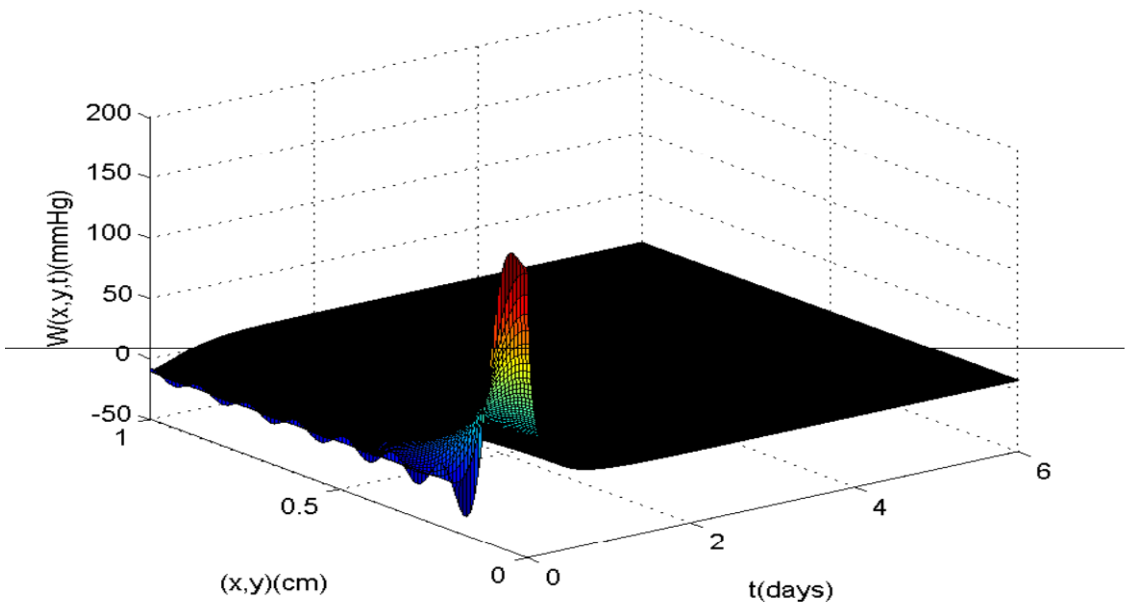


Figure (3.6)

*Numerical Solutionof system(3.10)-(3.12) by using ADI method; after(25)days with e=*0.658842875481342*,n= 10, h = (b-a)/n , i,j= 1,2,... Solutions are produced using MATLAB*

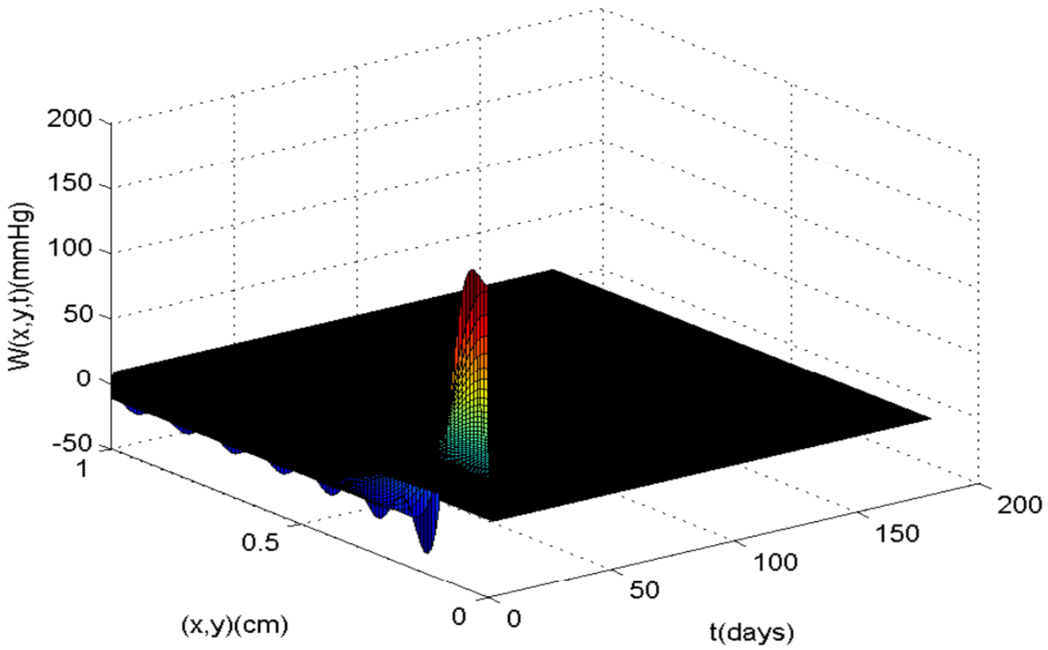


figure (3.7)

Numerical Solutionof system(3.10)-(3.12) by using ADI method; after(6)months with ,n= 10, h = (b-a)/n , i,j= 1,2,... Solutions are produced using MATLAB

4.1 Conclusions:

In chapter two, we consider three types of equations then derived a second order ADI formula for this equations with the given initial and boundary conditions, unconditional consistent and stability was discussed for each types, and by this two conditions the convergent become easy to prove. Then by using a small time steps we derive another formula of ADI method for wave equation, then, for a more accuracy the forth order formula was developed here.

Finally, we solve examples by this derived formula with the using of MATLAB program.

In chapter three, we introduce a biological model of the treatment of bacterial infection in a chronic wound by using the oxygen therapy technique, the biological and mathematical model was discussed .Also, the stability of this system was proved to one dimensional system in an analytical result. Then the system was developed in two dimensions in order to solve it by the ADI method.

4.2 Recommendations

There are a number of extensions to the methods presented in this thesis that could be pursued in the future. In particular, the following areas could lead to fruitful research:

1. Investigate our numerical method (ADI method) with other boundary conditions, numerical schemes presented in this thesis can be adapted to other boundary conditions (non-homogeneous Dirichlet or Neumann conditions) with nonlinear source terms

2. Investigate the consistency and stability of the developed high-accuracy (ADI) formula.

3. Implementation of our numerical methods in higher-dimensional problems with nonlinear fractional differential equation, come with the associated increase in computational complexity associated with larger matrices, but these may be addressed by the aforementioned high-efficiency technique.

4. Solving the full system realted with the treatment of chronic wounds with oxygen therapy technique, which contains six nonlinear equations by using the (ADI) method

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APPENDIX A
Equivalence Theorem (Lax-Richtmyer)

The Fundamental Theorem of Numerical Analysis. For consistent numerical approximations, stability and convergence are equivalent.

Lax proved for IVPs. The theorem applies as well to BVPs, approximations to functions and integrals, and PDEs. Here we will prove that for consistent numerical schemes, stability implies convergence. The converse is more difficult to prove, and will be proved in the context of PDEs, using functional analysis. Approximate Lu = f by $L_\Delta\ u_\Delta = f_\Delta$. We will assume the problems are close, and prove that the solutions are close. Consistency implies f_Δ → f and L_Δu → Lu as n → 1, Δt → 0, with nΔt = t fixed. Stability implies L_Δ^{-1} remains uniformly bounded. Then as n→1, Δt → 0, with n Δ = t fixed, u_Δ converges to u:

$$u - u_\Delta = L_\Delta^{-1}\ \ (L\Delta u - Lu) + L_\Delta^{-1}\ (f - f_\Delta) \to 0$$

Let's look at the details for the IVP du/dt = au, $u(0) = u_0$ (proof due to Strang). The exact solution is u(t) = $e^{at}u_0 \equiv H^n\ u_0$, where H = exp{aΔt} is the exact growth factor. The approximate solution is u_n = $G^n u_0$. Stability implies

$$|G^n| \le e^{kn\Delta t} = e^{kt}$$

where K is a positive constant independent of n. (For A-stability, K = 0 and e^{kt} = 1.) Consistency implies

$$|G - H| \le C\Delta t^{p+1},\ p > 0$$

Then we'll prove convergence:

$$|G^n u_0 - H^n u_0| \to 0$$

as n→ ∞, Δt → 0, with n Δt = t fixed. We use a telescoping identity

$$G^n - H^n = G^n - G^{n-1}H + G^{n-1}H - G^{n-2}H^n + \ldots + GH^{n-1} - H^n$$

$$= G^{n-1}\ (G - H) + G^{n-2}\ (G - H)H + \ldots + (G - H)\ H^n \qquad (\star)$$

Every term has a factor G–H, and |G–H| ≤ $C\Delta t^{p+1}$ by consistency. Every term has a power of G (possibly G^0) which is bounded by stability. Every term has a power of H (possibly H^0) which is bounded since the continuous problem is well-posed. There are n = t/Δt terms in Eq. (⋆). Therefore as as n→ ∞, Δt → 0, with nΔt = t fixed,

$$|G^n - H^n| \le \frac{t}{\Delta t} e^{kt} C\Delta t^{p+1} = O(\Delta t^p) \to 0 \qquad (\star\star)$$

The telescoping series (⋆) is exactly how error accumulates in a difference equation.

$G^n - H^n = \sum_{j=1}^{n} G^{n-j}\,(\mathrm{G} - \mathrm{H})\,H^{j-1}$

H^{j-1}propagates the exact solution to time level j − 1; (G − H) is the local error going from timelevel j −1 to j; and G^{n-j}propagates this error forward with the difference method to time level n.

Note that Eq. (⋆⋆) implies that although the local error e_l = LTE ≡ Δt τ to leading order in Δt, the (global) error

$$e \sim t\, e^{kt}\, \max_{j=1\ldots n}\left|\tau_j\right|$$

where here ∼means of the same order Δt^p in Δt. Thus e ∼τ ; in other words, the (global) error and the (global) truncation error are of the same order Δt^p in Δt, but the constants in front of Δt^p are different.

APPENDIX B

Fisher's equation

Consider the following ODE model for population growth:

$u'(t) = a\big(u(t)\big)u(t)\ \ ,\ \ u(t) = u_o\ \text{ at }\ t = 0$

where $u(t)$ denotes the population size at time t, and $a(u)$ plays the role of the population dependent growth rate. The model asserts that the population changes at a rate that is proportional to the present population size. Moreover, if we suppose

$$a(u) = \alpha\left(1 - \frac{u(t)}{u_\infty}\right)$$

for constant positive parameters, α, u_∞, then the population grows when the population is smaller than the `limiting value" u_∞ and the population size decreases if u exceeds this value. Thus the model predicts behaviour that is qualitatively consistent with the way we observe populations to behave. This autonomous differential equation has critical points at u = 0 and u = u_∞ and it is easy to see that the critical point at zero is unstable while the other is a stable critical point. Then the population u(t) will tend to the limiting value u_∞ as t tends to infnity, regardless of the initial population size. This equation is a special case of the more general autonomous equation, $u'(t) = F(u(t))$. Now the partial differential equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = F\big(u(t)\big), x \in R^n \ , \ \ t > 0$$

can be viewed as an attempt to incorporate the mechanism of diffusion into the population model. We are going to discuss equations of this form in the case n = 1 where the equation can be written more generally as

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 b(u(x,t))}{\partial x^2} = F\big(u(t)\big)\,, \qquad b' > 0$$

Equations of this form arise in a variety of biological applications and in modelling certain chemical reactions and are referred to as reaction. diffusion equations. To clear the reaction diffusion equation with positive constant parameters, D, α and u_∞:

$$\frac{\partial u(x,t)}{\partial t} - D\frac{\partial^2 u(x,t)}{\partial x^2} = \alpha u(x,t)\left(1 - \frac{u(x,t)}{u_\infty}\right)$$

This is know as **Fisher's equation** and it is usually viewed as a population growth model. The various parameters in the equation have the following dimensions

D denotes diffusivity (L^2T^{-1})

α denotes growth rate (T^{-1});

u_∞ denotes carrying capacity

(number of individuals)

To reduction to dimensionless form, it is often useful to rewrite the partial differential equation in terms of dimensionless variables. We define

$\tau = \alpha t$ time scaled to the growth rate

$z = x\sqrt{\frac{\alpha}{D}}$ distance scaled to diffusion length

$v = \frac{u(x,t)}{u_\infty}$ population scaled to carrying capacity

and then the equation becomes:

$$\frac{\partial v(z,\tau)}{\partial \tau} - D\frac{\partial^2 v(x,\tau)}{\partial z^2} = v(1-v)$$

Now, in order to investigate the existence of travelling wave solutions, we suppose $v(z,\tau) = V(z - c\tau)$ with V(s) tending to constant values as *s* tends to plus orminus infnity. Then

$$-cV^{'}(s) - V^{''}(s) = V(s)\big(1 - V(s)\big), \ \ -\infty < s = z - c\tau < \infty$$

This second order equation reduces to the following autonomous dynamical system:

$$V'(s) = W(s)$$
$$W'(s) = -cW(s) - V(s)(1 - V(s))$$

This system has critical points at (0, 0) and (1, 0). Since

$$J_{(V,W)} = \begin{bmatrix} 0 & 1 \\ 2V-1 & -c \end{bmatrix}$$

We can classify the critical points according to the eigenvalues of this matrix.

at (0,0) $\quad \lambda_{\pm} = \frac{1}{2}(-c \pm \sqrt{c^2 - 4})$

a stable node if c > 2 and stable focus if 0 < c < 2,

at (1; 0) $\quad \lambda_{\pm} = \frac{1}{2}(-c \pm \sqrt{c^2 - 4})$

A saddle point for all values of c. In the case 0 < c < 2, the origin is a stable focus and the orbits of the system are curves in the (V;W) - plane.

$$\left\{ \begin{matrix} V = V(s), & V_{(-\infty)} = V_0 \\ W = W(s), & V_{(-\infty)} = V_0 \end{matrix} \right\} -\infty < s < \infty$$

With (V (s), W(s)) → (0; 0) as s → ∞. Since the origin is a focus, the orbits are such that V and W assume both positive and negative values as the curve spirals toward the origin. Negative values for V are not physically meaningful in the population interpretation of V = u. Therefore, we conclude that there are no relevant travelling wave solutions for wave speeds between zero and 2, In the case c > 2 the origin is a stable node and the orbits in the fourth quadrant that are attracted to the origin approach the node with V positive and W negative. Then these are physically relevant orbits. If there exists an orbit with Vo = 1;Wo = 0 that is attracted to the origin, then this orbit, which is in fact a heteroclinic orbit joining the two critical points, corresponds to a travelling wave solution to the Fisher's equation. The component V = V (s) of the heteroclinic orbit is a smooth function such that V 0(s) = W(s) < 0 for all s. In addition, V (s) tends to 1 as s tends to minus infnity and V (s) tends to 0 as s tends to plus infnity so that 0 and 1 are the state values ahead of and behind the wave, respectively.

الملخص

تضمنت هذه الرسالة ثلاث اهداف رئيسة منها:

الهدف الأول هو لدراسة و توضيح طريقة الاتجاه التكراري المتناوب

(Alternative Direction Iteration method) لحل المعادلات التفاضلية الجزئية في البعد الفضاءات الثنائية و لكل نوع من أنواع المعادلات التفاضلية وذلك عن طريق تجزئة معادلة كرانك-نيكولسون (Crank-Nicholson) الناتجة من معادلة الفروقات المنتهية أضافة الى مناقشة

الاستقرار(Stability) والتقارب (convergence) والاتساق (consistency) ومن ثم إعطاء صيغة جديدة اعتمادا على خطوة زمن مختلفة. تعتبر طريقة الاتجاه التكراري المتناوب من الدرجة الثانية و لزيادة دقة هذه الطريقة تم ايجاد تقنية عددية جديدة تكون بنفس خصائص المعادلة من الدرجة الثانية لكن بدقة اكثر.

الهدف الثاني لهذه الرسالة هو أشتقاق و دراسة نظام المعادلات المرتبطة بألتهابات الجروح للمرضى الذين يعانون من مرض السكري (patients with diabetes) وتأثير الأكسجين في علاج الجروح المصابة عن طريق تقليل فعالية ونشاط البكتريا المسببة لالتهاب، ودراسة استقرارية الحل التحليلي لهذا النظام الذي يكون عادة في فضاء البعد الأول. وبما ان طريقة الاتجاه التكراري المتناوب يمكن استخدامها لحل المعادلات في فضاءات البعد الثاني فأكثر، فقد اعيدت نمذجة النظام المرتبط بهذه المعالجة الى فضاء البعد الثاني تمهيدا لحله بأستخدام هذه الطريقة.

وأخيرا، فإن الهدف الثالث من هذا البحث هو لحل نظام من المعادلات التي اعيدت نمذجتها الى فضاءات البعد الثاني باستخدام طريقة الاتجاه التكراري المتناوب ومناقشة النتائج التي حصلنا عليها ومدى مقدار استقرار النظام، ودقة النتائج في مراحل مختلفة من العلاج.

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طريقة الاتجاه التكراري المتناوب المطورة لحل المعادلات التفاضلية الجزئية مع تطبيق على الجروح المزمنة لمرضى السكر

رسالة
مقدمة الى كلية العلوم جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

مِن قِبَل

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بأشراف

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