Republic of Iraq
Ministry of Higher Education
and Scientific Research
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Approximation to the Mean and Variance of the Estimators Related to Gamma Distribution

A Thesis

Submitted to the College of Science/ Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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September 2014 Thee Alqedda 1435

بِسْمِ اللهِ الرَّحْمَنِ الرَّحِيْمِ

(فَأَمَّا الزبَدُ فَيَذْهَبُ جُفَاءً وَأَمَّا مَايَنْفَعُ النَّاسَ فَيَمْكُثُ فِي الأَرْضِ)

صَدَقَ اللّهُ العَظيم

(سورة الرعد، الآية 17)

Dedication

I dedicate this work to my family and my friends. A special feeling of gratitude to my loving father and mother, whose words of encouragement and push for tenacity ring in my ears. Also to my lovely brother and sister who never left me side and my gorgeous fiancé who supported me all the time. Also dedicate this thesis to my uncle who never left me side supported me throughout the process. I will always thanks my friends

Acknowledgments

I would like to express my thanks and gratitude to first and for most my **God** *and I would like to express my deep sense of gratitude to my supervisors* ***Dr. Akram M. Al-Abood*** *and* ***Dr. Alaudin N. Ahmad*** *they has been a constant source of encouragement and inspiration for me, they always found time to help me by allotting suitable time slots to discuss the problem. Their valuable suggestion and ingenious ideas have helped a lot in shaping this report.*

I am grateful to the staff of the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University for providing the necessary departmental facilities during the course of thesis work.

A special thanks is devoted to Mr. Fadi Adel.

Finally to my perfect family and all my friends I present my thanks.

Yamama Natheer

September, 2014

Summary

In this thesis, we study the gamma distribution because it has many applications in life – testing, survival and reliability investigation that appear in medical studies of chronic diseases and industrial life – testing. Approximation to the mean and variance of moments method estimators is made theoretically by using Taylor series expansion approximated up to second partial derivatives. The maximum likelihood estimators are derived and compared with several estimators that proposed in the literature. Where the practice show that the bias values of moment method estimators are adequate with the simulated bias values for moderate and large sample. While the variance values of the scale parameter are excellent in comparison with the simulated values.

A new bias corrected estimator based on the maximum likelihood estimator is suggested and show better performance in comparison with the other estimators proposed by McCullagh ,Nelder, Cardeiro, and Pearson.

The theoretical results are tested by using Monte – Carlo simulation and compared by utilizing the measurement of mean square error.

Notations and Abbreviations

r.v ≡ random variable.

r.s ≡ random sample.

U(a,b) ≡ Uniform distribution with parameters a and b.

p.d.f ≡ probability density function.

c.d.f≡ cumulative distribution function.

m.g.f ≡ moment generating function.

$G(\alpha, \beta)$ ≡ Gamma distribution with parameters α and β.

$\chi^2(r)$ ≡ Chi – square distribution with r degrees of freedom.

Exp (β) ≡ Exponential distribution with parameter β.

MM ≡ Moments Method.

MLE ≡ Maximum Likelihood Estimator.

MLM ≡ Maximum Likelihood Method.

$b(\hat{\theta})$ ≡ bias of $\hat{\theta}$.

$var(\hat{\alpha})_{asy}$ ≡ Asymptotic variance.

$cov(\hat{\alpha}, \hat{\beta})_{asy}$ ≡ Asymptotic covariance.

distn. ≡ distribution.

Eq. ≡ Equation.

Contents

Introduction:

The gamma distribution arise as a model from statistical studies of interval between events occurring in time or space, specifically when the interest in the waiting time from the occurrence of one event until r further events have occurred in a Poisson process with constant rate λ, [14]. This distribution sometimes referred to as a special Erlangian distribution after the Swedish scientist who used the distribution in early studies of queuing problem [2].

The gamma distribution has an important applications in the study of life time models, such as stops of a machine, failure or breakdowns of an equipment (e.g. electronic computer), air or road accidents [14], [12], coal mining disasters, telephone calls, daily rainfall [9], etc., are examples of such events that occur in a real time and have properties experted for gamma case [18].

Many authors and researchers concerned with the gamma distribution such as Minka, Thomas P. in 2002, [20] derives a fast algorithm for maximum-likelihood estimation of both parameters of a Gamma distribution or negative-binomial distribution, Gomes, O. Combes, C. Dussauchoy, A. in 2008, [7] focuses on the parameter estimation of the generalized gamma distribution. Because of many difficulties described in the literature to estimate the parameters, they propose here a new estimation method. The algorithm associated to this heuristic method is implemented in Splus. They validate the resulting routine on the particular cases of the generalized gamma distribution,

David E. Giles and Hui Feng in 2009, [5] considered the quality of the maximum likelihood estimators for the two-parameter gamma distribution in small samples. They show that the methodology suggested by Cox and Snell (1968) can be used very easily to bias-adjust these estimators. A simulation study shows that this analytic correction is frequently much more effective than bias-adjusting using the bootstrap – generally by an order of magnitude in percentage terms. The two bias-correction methods considered result in increased variability in small samples, and the original estimators and their bias-corrected counterparts all have similar percentage mean squared**.**

Apolloni, Bruno and Bassis, Simone in 2009, [3] provide an estimation procedure of the two-parameter Gamma distribution based on the Algorithmic Inference approach. As a key feature of this approach, they compute the joint probability distribution of these parameters without assuming any prior. To this end they propose a numerical algorithm which is often beneficial of a highly efficient speed up based on an approximate analytical expression of the probability distribution. They contrast the interval and point estimation with those recently obtained in Son and Oh (2006). They realize that the estimators are both unbiased and more accurate, albeit more dispersed than Bayesian methods.

H´ector M. Ramos, Antonio Peinado, Jorge Ollero and Mar´ıa G. Ramos in 2013, [10] analyse fertility curves from a novel viewpoint, that of inequality .Through sufficient conditions that can be easily verified, they compare inequality, in the Lorenz and Generalized Lorenz sense, in fertility curves fitted by gamma distributions, thus achieving a useful complementary instrument for demographic analysis. As a practical application, they examine inequality behavior in the distributions of specific fertility curves in Spain from 1975 to 2009.

In neuroscience, the gamma distribution is often used to describe the distribution of inter-spike intervals, [17]. Although in practice the gamma distribution often provides a good fit, there is no underlying biophysical motivation for using it.

In bacterial gene expression, the copy number of a constitutively expressed protein often follows the gamma distribution, where the scale and shape parameter are, respectively, the mean number of bursts per cell cycle and the mean number of protein molecules produced by a single mRNA during its lifetime.[6]

This thesis consists of three chapters. In chapter one we gave a brief summary of the important mathematical and statistical properties of gamma distribution. Where as in chapter two, we gave a full discussion on the approximation to the mean and variance of moments estimators by using Taylor series expansion and the maximum likelihood estimators which are derived and approximated by using Newton – Raphson method. In chapter three we introduce several alternative proposed estimators for the shape parameter of the maximum likelihood method with theoretical approximation to their biases and variances and a new estimator for the shape parameter is suggested which is based on bias correction for the maximum likelihood estimator.

-Chapter One-

Gamma Distribution

1.1 Introduction:

In this chapter, we shall introduce some definitions and mathematical forms related to gamma distribution. Moment properties of the distribution are illustrated, such as mean, variance, skewness, and kurtosis. Two methods of estimation are used, namely moments method and maximum likelihood method for estimating the distribution parameters. Some theorems are considered for generating random variates from gamma distribution by Monte – Carlo simulation.

1.2 Definition: [11]

A continuous $r.v\, X$ has a gamma distribution with parameters α and β, denoted by $X \sim G(\alpha,\beta)$ if and only if its $p.d.f.$ is given by:

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \quad ,0 < x < \infty$$
$$= 0 \quad ,e.w. \qquad (1.1)$$

where $\alpha > 0, \beta > 0$ and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$

is called a gamma function

To check the function of Eq. (1.1) is valid $p.d.f.$, we note that $f(x) > 0$ for all x and that

$$\int_0^\infty f(x)dx = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^\infty x^{\alpha-1}e^{-x/\beta}dx = 0 \qquad (1.2)$$

Making the transformation $t = x/\beta$ in Eq. (1.2) or equivalently $x = \beta t$ with $dx = \beta dt$, we have:

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-t}dt = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1$$

A special case of the gamma distribution is called the exponential distribution when $\alpha = 1$, where the $p.d.f.$ in Eq. (1.1) becomes:

$$f(x) = \frac{1}{\beta} e^{-x/\beta}, \; 0 \le x < \infty$$
$$= 0 \quad ,e.w. \qquad (1.3)$$

Also a special case of the gamma distribution is that play an extremely important role in both theoretical and applied statistics when the $r.v.$ $X \sim G(r/2, 2)$, the $p.d.f.$ in this case is:

$$f(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} x^{\frac{r}{2}-1} e^{-x/2}, 0 < x < \infty$$

$$= 0 \qquad , e.w. \qquad (1.4)$$

Where r is positive integer.

The $r.v\ X$ with $p.d.f.$ of Eq. (1.4) is said to have a *Chi-square* distribution with r degrees of freedom and denoted by $X \sim \chi^2(r)$.

The gamma distribution depends on the two parameters α and β which are often referred to as the shape and scale parameters. By varying the values of α and β, a wide range of distribution shape can be generated.

A computer program is made which gives a graphical representation of gamma $p.d.f.'s$ when α is an integer. Illustration in fig. (1.1) show some $p.d.f.'s$ for α fixed with β varying. The curve increasing to locate its maxima and then decreasing to have the x axis as an asymptote, furthermore there is a sever skeweness to the right as β increasing.

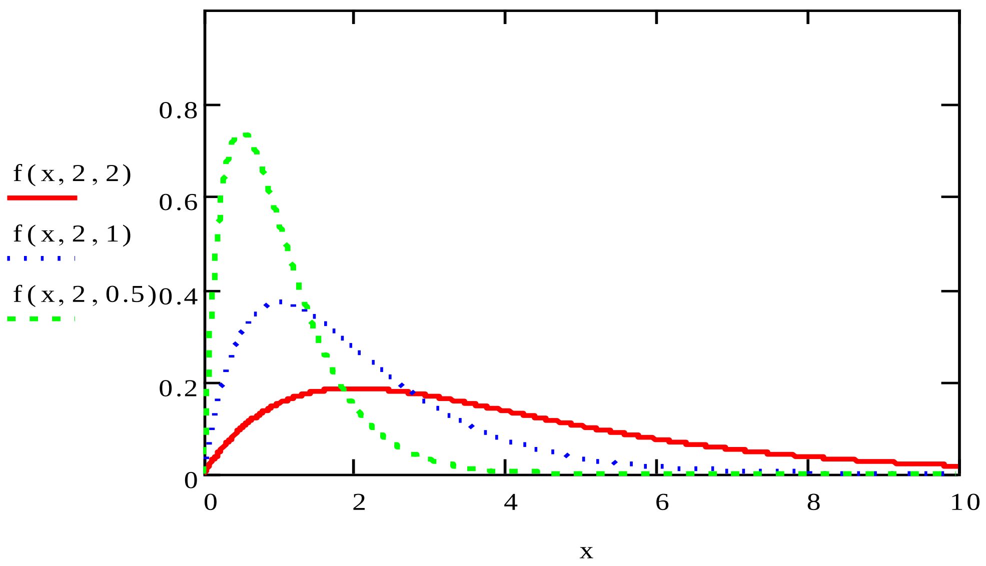

Fig. (1.1) Gamma distribution for fixed α and various values of β

In Fig. (1.2) show some $p.d.f.'s$ for β fixed with α varying. A well known exponential case arise when $\alpha = 1$, and for $\alpha > 1$ the curve arise similar behavior as given in the discussion of Fig. (1.1).

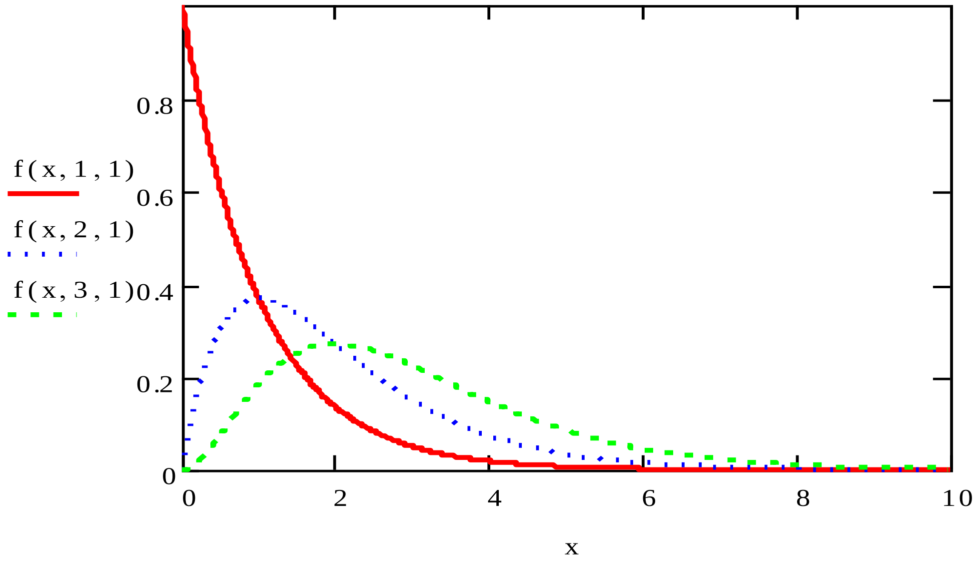

Fig. (1.2) Gamma distribution for fixed β and various values of α

1.3 The Cumulative Distribution Function: [11]

The $c.d.f.$ of gamma distribution is given by:

$$F(x) = pr(X \le x) = \int_0^x f(t)dt = \int_0^x \frac{1}{\Gamma(\alpha)\beta^{\alpha}} t^{\alpha-1} e^{-t/\beta} dt \qquad (1.5)$$

The right hand side of Eq. (1.5) is known as incomplete gamma function.

Specific tables can be found for *F(x)* in most statistical books when the $r.v. X \sim G(r/2, 2) = \chi^2(r)$, where r is positive integer because of the importance of Chi-square distribution in statistical work.

1.4 Genesis of the Gamma Distribution: [11]

One of the widely quoted approach to the gamma distribution comes from a Poisson process with rate λ. To formally define a Poisson process, we consider events occurring randomly in time in the following sense.

There is a constant rate λ>0 called the rate of occurrence of the events and consider an interval of time $(x, x + \Delta x)$ involve all the time values that are greater than x and less than or equal to $x+\Delta x$. Furthermore let $u(\Delta x)$ be any function of Δx such that

$$\lim_{\Delta x \to 0} \frac{u(\Delta x)}{\Delta x} = 0$$

Then the Poisson postulates are the following:

(a) $\Pr[\text{no events during } (x,\Delta x)] = 1 - \lambda\,\Delta x + u(\Delta x)$:
(b) $\Pr[\text{one event during } (x,\Delta x)] = \lambda\,\Delta x + u(\Delta x)$;
(c) $\Pr[\text{two or more events during } (x,\Delta x)] = u(\Delta x)$.

If the number of events occurring during $(x,\Delta x)$ is independent of the occurring during $(0,x)$, a process of events satisfying the above conditions is called a Poisson process of rate λ.

It has been shown, by Hogg, [11] that if W_x is a $r.v.$ representing the number of events occurring in a fixed time, say $(0,x)$, then W_x has a Poisson distribution with $p.d.f.$

$$pr(W_x = w) = \frac{e^{-\lambda x}(\lambda x)^w}{w!}, w = 0,1,2,\ldots$$

$$= 0 \qquad , \text{ e.w.; where } \lambda > 0 \tag{1.6}$$

Now suppose that we are interested in the time X of the occurrence one event until K further events have occurred. Then the $c.d.f.$ of X is given by:

$$F(x) = pr(X \le x) = 1 - pr(X > x)$$

$$= 1 - pr\left[(k-1) or\ fewer\ events\ during\ (0,x)\right]$$

$$= 1 - \sum_{w=0}^{k-1} pr(W_x = w)$$

$$= 1 - \sum_{w=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^w}{w!} \tag{1.7}$$

Due to Thanon [19] the $p.d.f.$ of X when $x > 0$ is:

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left\{1 - \sum_{w=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^w}{w!}\right\}$$

$$= \lambda \frac{e^{-\lambda x}(\lambda x)^{k-1}}{(k-1)!} = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x} \tag{1.8}$$

With $f(x) = 0$ when $x \leq 0$. Where Eq.(1.8) is as Eq.(1.1) with α= k and $\beta = 1/\lambda$.

The $p.d.f.$ given by Eq. (1.8) is a special member of the gamma family of distribution. Eq. (1.7) that is interesting in showing that the $c.d.f.$ of the distribution can be expressed as a cumulative sum of Poisson probabilities.

1.5 Moments Generating Functions:

The moments are set of constants of a distribution which are used for measuring its properties and under certain circumstances they specify the distribution.

The moments of the $r.v\ X$, where defined in terms of the expected values of the powers of X when they exist. For instance $\mu_r{}' = E(X^r)$ is called the r^{th} moment of X about the origin and $\mu_r = E[(x-\mu)^r]$ is called the r^{th} central moment of X.

The generating function reflects certain properties of the distribution functions. They are often thought of as transforms of the density function (or probability function) defining the distribution. They could be used to generate moments and also have a particular usefulness in connection with sums of independent random variables. First we shall obtain a function of a real t called the moment generating function, denoted by $M(t)$, which can be used to find the moments of X as many as we wish.

For continuous $r.v.X$, the m.g.f is defined by:

$$M(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \tag{1.9}$$

Provided the integral converges absolutely.

When $X \sim G(\alpha, \beta)$ with p.d.f. given by Eq. (1.1), we have:

$$M(t) = \int_{0}^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} dx \tag{1.10}$$

$$= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \mathrm{e}^{\frac{-(1-\beta t)x}{\beta}}\, \mathrm{d}x$$

Putting $u = (1-\beta t)x$ implies $du = (1-\beta t)dx$

$$M(t)=\int_0^{\infty}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{u}{1-\beta t}\right)^{\alpha-1}e^{-\frac{u}{\beta}}\frac{du}{1-\beta t}$$

$$=\frac{1}{(1-\beta t)^{\alpha}}\int_0^{\infty}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}u^{\alpha-1}\mathrm{e}^{-\frac{u}{\beta}}du$$

$$=\frac{1}{(1-\beta t)^{\alpha}}\ ,t<\frac{1}{\beta} \tag{1.11}$$

From the theory of mathematical analysis, it has been shown that the existence of $M(t)$ for $t<1/\beta$ implies that the derivatives of $M(t)$ of all orders exist at t=0.

Thus the r^{th} moment of X about the origin $\mu_r{}'=E(X^r)$ can be found by finding the r^{th} derivative of $M(t)$ evaluated at t=0. That is:

$$\mu_r{}'=E\left(X^r\right)=\left.\frac{d^rM(t)}{dt^r}\right|_{t=0}$$

$$=\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1)\beta^r(1-\beta t)^{-(\alpha+r)}\Big|_{t=0}$$

$$=\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\beta^r \qquad ,r=1,2,3,\ldots \tag{1.12}$$

(i) Mean

$E(X)=\mu=\mu_1{}'$ is called the mean of the $r.v.X$. It is a measure of central tendency. Use of Eq. (1.12) with $r=1$ give:

$$\mu=\alpha\beta \tag{1.13}$$

(ii) Variance

$var(X)=\sigma^2=E[(X-\mu)^2]$ is called the variance of the $r.v.X$. It is a measure of dispersion. Use of Eq. (1.12) with $r=2$ and Eq. (1.13) give:

$$\sigma^2=\alpha\beta^2 \tag{1.14}$$

(iii) Coefficient of variation

The variational coefficient of the $r.v.X$ is defined by the ratio σ/μ. It is a measure of dispersion, it is independent of scale of measurement and is denoted by V. Now for gamma case

$$V=\frac{\sigma}{\mu}=\frac{\sqrt{\alpha}}{\beta}=\alpha^{-1/2} \tag{1.15}$$

Which is independent of the scale parameter β.

(iv) Coefficient of skewness

$\gamma_1 = \frac{E[(X-\mu)^3]}{(E[(X-\mu)^2])^{3/2}}$ is called the coefficient of skewness. It is a measure of departure of the frequency curve from symmetry. If $\gamma_1 = 0$, the curve is not skewed, $\gamma_1 > 0$, the curve is positively skewed, and if $\gamma_1 < 0$, the curve is negatively skewed, by Rahman [15].

Use of Eq. (1.12) with $r = 3$ give $E\left[(X-\mu)^3\right] = 2\alpha\beta^3$, and so:

$$\gamma_1 = \frac{2\alpha\beta^3}{(\alpha\beta^2)^{3/2}} = 2\alpha^{-1/2} > 0 \tag{1.16}$$

(v) Coefficient of Kurtosis

$\gamma_2 = \frac{E[(X-\mu)^4]}{(E[(X-\mu)^2])^2} - 3$ is called the coefficient of kurtosis. It is the measure of the degree of flattening of a frequency curve. If $\gamma_2 = 0$, the curve is called mesokurtic, if $\gamma_2 > 0$, the curve is called leptokurtic, and if $\gamma_2 < 0$, the curve is called platykurtic, by Rahman [15].

Use of Eq.(1.12) with $r = 4$ give $E\left[(X-\mu)^4\right] = 3\alpha(\alpha+2)\beta^4$

$$\gamma_2 = \frac{3\alpha(\alpha+2)\beta^4}{(\alpha\beta^2)^2} - 3 = 6\alpha^{-1} > 0 \tag{1.17}$$

1.6 Other Central Moments:

(i) Mode

A mode of distribution is defined to be the value of the $r.v.X$ which maximize the $p.d.f\ f(x)$. For continuous distribution the mode is the solution of:

$\left.\frac{df(x)}{dx}\right|_{x=mo} = 0\, and\, \left.\frac{d^2f(x)}{dx^2}\right|_{x=mo} < 0$ .The mode is a measure of location.

For gamma case with $p.d.f.$ given by Eq. (1.1). The logarithm of $f(x)$ is:

$$lnf(x) = -ln\Gamma(\alpha) - \alpha\, ln\,\beta + (\alpha-1)\,lnx - \frac{x}{\beta}$$

$$\frac{d\ln f(x)}{dx} = \frac{\alpha - 1}{x} - \frac{1}{\beta}$$

for maximum

$$\left.\frac{d\ln f(x)}{dx}\right|_{x=mo} = 0 \Rightarrow mo = (\alpha - 1)\beta, \alpha \geq 1 \qquad (1.18)$$

(ii) Median

A median of a distribution is defined to be the value me of the $r.v.X$ such that $1/2 = F(me) = pr(X \leq me)$.The median is a measure of location.

For gamma case, the $c.d.f.$ given by Eq.(1.5), we have:

$$F(me) = 1/2 = \int_0^{me} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} dx \qquad (1.19)$$

Where the right hand side of Eq. (1.19) is the incomplete gamma integral.

1.7 Point Estimation:

Point estimation is concerned with inference about the unknown parameters of a distribution from a sample. It provides a single value for each unknown parameter. Point estimation admits two problems.

First, developing methods of obtaining a statistic, say, $U = u(X_1 X_2, \dots, X_n)$ to represent or estimate the unknown parameter θ in the $p.d.f$ of $f(x;\theta)$ or some function of θ, say $\tau(\theta)$, such statistic is called point estimator.

Second, selecting criteria and technique to define and to find a best estimator among possible estimators.

(1.7.1) Definition (statistic):

A statistic is a function of one or more $r.v's$ that does not depend upon any unknown parameter. A statistic itself is a $r.v.$

(1.7.2) Definition (Sample mean and Sample variance):

Let $X_1, X_2, \dots, X_n$ be a $r.s$ of size n from a given distribution. The statistic

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad (1.20)$$

is called the sample mean, and the statistic

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 \tag{1.21}$$

is called the sample variance

(1.7.3) Definition (Unbiased Estimator):

An estimator (statistic) $\hat{\theta} = u(X_1, X_2, \dots, X_n)$ is defined to be an unbiased estimator of θ if and only if $E(\hat{\theta}) = \theta$, otherwise $\hat{\theta}$ is said to be biased estimator.

Note: The term $E\left(\theta\right) - \theta$ is called the bias of the estimator $\hat{\theta}$ and denoted by $b(\hat{\theta})$.

(1.7.4) Definition (Asymptotically Unbiased):

For a biased estimators, an estimator $\hat{\theta}$ is called asymptotically unbiased estimator of θ if and only if $\lim_{n \to \infty} E\left(\theta\right) = \theta$.

(1.7.5) Definition (Mean – Squared Error):

Let $\theta = u(X_1, X_2, \dots, X_n)$ be an estimator of θ. The mathematical expectation $E\left[\left\{\theta - \theta\right\}^2\right]$ is defined to be the mean – squared error of the estimator $\hat{\theta}$ and is denoted by $MSE[\hat{\theta}, \theta]$.

That is:

$$MSE\left[\theta, \theta\right] = E\left[\left\{\theta - \theta\right\}^2\right]$$

Propositions:

(i) The MSE is a measure of goodness or closeness of $\hat{\theta}$ to θ

(ii) $MSE\left[\theta, \theta\right] = var\left(\theta\right) + [E(\theta) - \theta]^2 = var\left(\theta\right) + \left\{b(\theta)\right\}^2$

Proof:

$$MSE\left[\theta,\theta\right]=E\left[\left\{\theta-\theta\right\}^2\right]=E\left[\left\{\left(\theta-E\left(\theta\right)\right)+\left(E\left(\theta\right)-\theta\right)\right\}^2\right]$$

$$=E\left[\left(\theta-E\left(\theta\right)\right)^2+2\left(\theta-E\left(\theta\right)\right)\left(E\left(\theta\right)-\theta\right)+\left(E\left(\theta\right)-\theta\right)^2\right]$$

$$=E\left[\left(\theta-E\left(\theta\right)\right)^2\right]+2\left(E\left(\theta\right)-\theta\right)E\left[\left(\theta-E\left(\theta\right)\right)\right]+E\left[\left(E\left(\theta\right)-\theta\right)^2\right]$$

$$=var\left(\theta\right)+2\left(E\left(\theta\right)-\theta\right)E\left[\left(\theta-E\left(\theta\right)\right)\right]+b^2(\theta)$$

$$=var\left(\theta\right)+0+b^2(\theta)=var\left(\theta\right)+\left\{b(\theta)\right\}^2$$

(1.7.6) Definition (Consistent Estimator):

Let the estimator $\theta_n=u(X_1,X_2,\dots,X_n)$ of θ based on the sample size n. Then $\hat{\theta}_n$ is said to be a mean – squared error consistent estimator of θ if and only if

$$\lim_{n\to\infty} MSE(\theta_n,\theta)=0$$

1.8 Methods of Estimation:

A variety of methods available for finding estimators for the distribution parameters have been proposed in the literature such as moments, maximum likelihood, minimum chi-square, minimum distance, least-square, and Bayesian method. These methods provide a single value for each unknown parameters of the distribution.

For gamma case, we shall discuss two methods, the method of moments and the maximum likelihood method, some general on the quality of estimators provided by these methods.

(1.8.1) Moment Method:

We will generalize the discussion by letting $X_1X_2,\dots,X_n$ represent a $r.s.$ of size n from a distribution whose, $p.d.f.$ $f(x;\theta_1,\theta_2,\dots,\theta_r)$.The expectation $\mu_r{}' = E(X^r)$ is known as the r^{th} distribution moment about origin, $r=1,2,3,\dots$ and $M_r=\frac{1}{n}\sum_{i=1}^{n}X_i^r$ is the r^{th} moment of the sample about origin, $r=1,2,3,\dots$.

The method of moments can be described as follows:

Equate μ_r' to M_r beginning with $r = 1$ and continuing until there is enough equations to provide a unique solution for $\theta_1, \theta_2, \dots, \theta_r$ say $\theta_1, \theta_2, \dots, \theta_r$.

For gamma case, we have two unknown parameters α and β and if a $r.s.$ of size n is taken then $M_r = \mu_r'$ at $\alpha = \alpha$ and $\beta = \beta$, $r = 1{,}2$. So we have first

$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$ and $\mu_1' = E(X) = \alpha\beta$, so equate M_1 with μ_1' with $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$ we get:

$$\bar{X} = \alpha\beta \tag{1.22}$$

and second

$M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ and $\mu_2' = E\left(X^2\right) = \alpha(\alpha+1)\beta^2$, so equate M_2 with μ_2' with $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$ we get:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = (\alpha\beta)^2 + \alpha\beta^2 \tag{1.23}$$

Solving Eq. (1.22) and Eq. (1.23) we get:

$$\alpha = \frac{n\bar{X}^2}{(n-1)S^2} \tag{1.24}$$

and

$$\beta = \frac{(n-1)S^2}{n\bar{X}}, \text{ where } \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 \tag{1.25}$$

Equations (1.24) and (1.25) represent the moments method estimators for α and β respectively.

(1.8.2) Maximum Likelihood Method:

The most important and widely used formal estimation technique is the method of maximum likelihood. Estimation by maximum likelihood is a general method that may be applied when the underlying distribution of observations is specified.

The principle of the method of maximum likelihood can be formulated as follows:

Let $\underset{\sim}{X} = (X_1, X_2, \dots, X_n)$ be a set of $r.v.\acute{s}$ that they may or may not be stochastically independent and let the joint $p.d.f.$ $f(\underset{\sim}{x};\underset{\sim}{\theta})$ depend on a vector of m parameters $\underset{\sim}{\theta} = (\theta_1, \theta_2, \dots, \theta_m)$. This joint $p.d.f.$ when regarded as a function of $\underset{\sim}{\theta}$ is called the likelihood function of the $r.v.\acute{s}$ and is denoted by $L(\underset{\sim}{\theta}, \underset{\sim}{x})$ if the $\underset{\sim}{\theta} = u_j(\underset{\sim}{x}), j = 1, 2, \dots, m$ that maximize this likelihood function with respect to $\theta_j, j = 1,2, \dots, m,$ respectively, then the $M.L.E.\acute{s}$ of the m parameters are:

$$\theta_j = u_j(\underset{\sim}{x}), j = 1, 2, \dots, m.$$

The most important case when $X_1, X_2, \dots, X_n$ represent a random sample of size n from a distribution whose $p.d.f.$ $f(x;\underset{\sim}{\theta})$, so that the likelihood function is

$$L(\underset{\sim}{\theta}; \underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i, \underset{\sim}{\theta}).$$

In practice, the most important comments on this method and the obtained estimators are:

1. Many likelihood functions satisfy the condition that the $M.L.E.\acute{s}$ are the solution of the likelihood equations $\frac{\partial L(\underset{\sim}{\theta}, \underset{\sim}{x})}{\partial \theta_j} = 0$ provide that $\underset{\sim}{\theta} = \underset{\sim}{\theta}, j = 1, 2, \dots, m$

2. Since $L(\underset{\sim}{\theta}; \underset{\sim}{x})$ and $logL(\underset{\sim}{\theta}; \underset{\sim}{x})$ have their maximum at the same value of $\underset{\sim}{\theta}$, so it is sometimes easier to find the maximum of the logarithm of the likelihood.

3. Due to Zack, S. [21], the $M.L.E.\acute{s}$ are not necessarily unique.

4. Due to Zack, S. [21], the $M.L.E.\acute{s}$ are not necessarily consistent.

5. The $M.L.E.\acute{s}$ in general are asymptotically unbiased.

6. Sometimes it is impossible to find the $M.L.E.\acute{s}$ in a convenient closed form.

7. The $M.L.E.\acute{s}$ not necessarily obtained by differentiation.

8. Due to Mood [2], the $M.L.E.\acute{s}$ has the invariance property. Let θ be a $M.L.E.\acute{s}$ of θ. If $\tau(\theta)$ is a function with single-valued inverse, then the $M.L.E.\acute{s}$ of $\tau(\theta)$ is $\tau(\hat{\theta})$.

For gamma case, let $X_1, X_2, \dots, X_n$ be a $r.s.$ of size n from $G(\alpha, \beta)$ where the distribution $p.d.f.$ is given by Eq. (1.1), so the likelihood function is:

$$L(\alpha.\beta;x) = \prod_{i=1}^{n} f(x_i;\alpha,\beta) = \left[\Gamma(\alpha)\right]^{-n} \beta^{-n\alpha} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{\frac{-1}{\beta}\sum_{i=1}^{n} x_i}$$

and

$$lnL(\alpha.\beta;x) = -nln\Gamma(\alpha) - n\alpha ln\beta + (\alpha-1)\sum_{i=1}^{n} lnx_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i$$

$$\frac{\partial lnL}{\partial\alpha} = -n\psi(\alpha) - nln\beta + \sum_{i=1}^{n} lnx_i \quad (1.26)$$

where $\psi(\alpha) = \frac{d}{d\alpha} ln\Gamma(\alpha)$

$$\frac{\partial lnL}{\partial\beta} = \frac{-n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i \quad (1.27)$$

We set $\frac{\partial lnL}{\partial\alpha} = 0$ and $\frac{\partial lnL}{\partial\beta} = 0$ at $\alpha = \alpha$ and $\beta = \beta$, we have:

$$-n\psi\left(\alpha\right) - nln\beta + \sum_{i=1}^{n} lnx_i = 0 \quad (1.28)$$

and $\frac{-n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i = 0$ that implies

$$\alpha\beta = \overline{X} \quad (1.29)$$

From Eqs. (1.28) and (1.29) we have:

$$ln\alpha - \psi\left(\alpha\right) = ln\overline{X} - \frac{1}{n}\sum_{i=1}^{n} lnx_i \quad (1.30)$$

Analytic solution for $\hat{\alpha}$ can not be found from Eq.(1.30) so that it is difficult to maximize $L(\alpha.\beta;x)$ with respect to α and β, owing to the presence of gamma function $\Gamma(\alpha)$. In such case numerical method must be used such as *Newton-Raphson* or *bisection* method etc.

1.9 Some Related Theorems:

Theorem (1.9.1) (Independent Sum Distribution):

If $X_1, X_2, \dots, X_\alpha$ is a $r.s.$ of size α from $Exp(\beta)$, then the $r.v.$ $Y = \sum_{i=1}^{\alpha} X_i \sim G(\alpha, \beta)$

For the proof, $Y = \sum_{i=1}^{\alpha} X_i \sim G(\alpha, \beta)$ we shall utilize the $m.g.f.$ technique as follows:

Since $X_i \sim Exp(\beta), i = 1, 2, \dots, \alpha,$ then the $m.g.f.$ of X_i is $M_{X_i}(t) = \frac{1}{1-\beta t}$, $1-\beta t \neq 0$

let $M_Y(t)$ be the $m.g.f.$ of Y, then

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t\sum_{i=1}^{\alpha} X_i}\right) = E\left(\prod_{i=1}^{\alpha} e^{tX_i}\right) = \prod_{i=1}^{\alpha} E\left(e^{tX_i}\right)$$

$= \prod_{i=1}^{\alpha} M_{X_i}(t) = \prod_{i=1}^{\alpha} \frac{1}{1-\beta t} = \frac{1}{(1-\beta t)^{\alpha}}$ which is the $m.g.f.$ of $G(\alpha,\beta)$ as given in Eq.(1.11).

Theorem (1.9.2):

As a consequence of theorem (1) of section (1.9.1). The sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim G\left(n\alpha, \frac{\beta}{n}\right)$ with $m.g.f.$ $M_{\bar{X}}(t) = \frac{1}{(1-\frac{\beta}{n}t)^{n\alpha}}$

Furthermore, using the relation (1.12) lead to the r^{th} moment $\bar{X}$ about origin is given by:

$$E\left(\bar{X}^r\right) = \left(\frac{\beta}{n}\right)^r \frac{\Gamma(n\alpha + r)}{\Gamma(n\alpha)} \tag{1.31}$$

Theorem (1.9.3):

Let $X_1, X_2, \dots, X_n$ be a $r.s.$ of size n from any distribution (discrete or continuous) having mean μ and variance σ^2, and let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ be the sample mean and the sample variance respectively, then

$$E(\bar{X}) = \mu \,, \quad var(\bar{X}) = \frac{\sigma^2}{n}$$

$$E(S^2) = \sigma^2 \,, \; var(S^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right), n > 1 \text{ where } \mu_r = E\left[(X-\mu)^r\right], r = 1,2,3,\ldots$$

For gamma case

$$\left.\begin{aligned} &E(\bar{X}) = \alpha\beta \,, \; var(\bar{X}) = \frac{\alpha\beta^2}{n} \\ &E(S^2) = \alpha\beta^2 \,, \quad var(S^2) = \frac{\alpha\beta^4(6n-6+2n\alpha)}{n(n-1)} \end{aligned}\right\} \qquad (1.32)$$

(1.9.4) Random Variate Generating:

We recall the properties of $c.d.f.$ $F(x) = pr(X \le x)$ of the $r.v.X$

1. $0 \le F(x) \le 1$

2. $F(-\infty) = 0$ and $F(\infty) = 1$

3. $F(x)$ is non – decreasing function of x.

4. $F(x)$ is continuous function to the right at each x.

Theorem (1.9.5) (Inverse Transform):

The $r.v.$ $U = F(x) \sim U(0,1)$ if and only if $X = F^{-1}(U)$ has $c.d.f.$ $pr(X \le x) = F(x)$

Proof

$\Rightarrow$ Let the $r.v.$ $U \sim U(0,1)$, then U has $c.d.f.$

$$G(u) = pr(U \le u) = \begin{cases} 0 & ,u \le 0 \\ u & ,0 < u < 1 \\ 1 & ,u \ge 1 \end{cases}$$

Now $pr(X \le x) = pr\left[F^{-1}(u) \le x\right] = pr\left[u \le F(x)\right] = F(x)$

$\Leftarrow$ Conversely, let the $r.v.X$ has $c.d.f.$ $pr(X \le x) = F(x)$

$$G(u) = pr(U \le u) = pr(F(X) \le u) = pr\left(X \le F^{-1}(u)\right) = F\left[F^{-1}(u)\right] = u$$

The algorithm of generating $r.v.$ by inverse transform method can be described by the following steps:

IT – Algorithm

1. Generate U from U(0,1)

2. Set X=F^{-1}(U)

3. Deliver X as a $r.v.$ generated from the $p.d.f.$ $f(x)$

As an application of theorem (3), let us consider a generated $r.v.$ from $Exp(\beta)$ where the $p.d.f.$

$$f(x)=\frac{1}{\beta}e^{\frac{-x}{\beta}} \quad ,0<x<\infty$$

$$=0 \quad ,e.w.$$

with $c.d.f.$

$$F(x)=\int_0^x f(t)dt=\int_0^x \frac{1}{\beta}e^{\frac{-t}{\beta}}dt=-e^{\frac{-t}{\beta}}\Big|_x^0=\begin{cases}0 & ,x\le 0\\ 1-e^{\frac{-x}{\beta}} & ,0<x<\infty\\ 1 & ,x\to\infty\end{cases}$$

Set $u=F(x)\Rightarrow u=1-e^{\frac{-x}{\beta}}\Rightarrow e^{\frac{-x}{\beta}}=1-u\Rightarrow \frac{-x}{\beta}=\ln(1-u)$

$$\Rightarrow \quad x=-\beta\ln(1-\mathrm{u})$$

Apply **IT – Algorithm**

1. Read $\beta$and α (α is positive integer)
2. x=0
3. For I=1 to α
4. Generate U from U(0,1)
5. Set Y=$-\beta\ln(1-\mathrm{U})$
6. X=x+Y
7. Go to step(4)
8. Deliver X as a $r.v.$ generated from $G(\alpha,\beta)$.
9. End

-Chapter two-

Approximation to the Mean and Variance of Moment Method and Maximum Likelihood Method Estimators

2.1 Introduction:

In this chapter, we shall consider the approximation to the mean and variance of moments method and maximum likelihood method estimators due to gamma distribution. This approach showed that the estimators are asymptotically unbiased with mean square error approach zero as the sample size approach infinity. The theoretical approach assessed practically by using Monte – Carlo simulation.

2.2 Expectation of Quotient Function of Random Variables:

In general, there are no simple exact formulas for the mean and variance of the quotient of two random variables in terms of the moments of the two random variables; however, there are approximate formulas can be considered. One way of finding the approximate formula for $E(\frac{X}{Y})$ by considering Taylor series expansion of the function $g(x,y)=\frac{x}{y}$ expanded about the point $[E(X),E(Y)]$, where we drop all terms of order higher than 2, and then take the expectation of both sides. Furthermore the approximate formula for $var(\frac{X}{Y})$ is similarly obtained by expanding Taylor series and retaining only second-order terms as follows:

The Taylor series expansion of the function $g(x,y)=\frac{x}{y}$ about the point (μ_x,μ_y) is:

$$g(x,y)\approx g(\mu_x,\mu_y)+(x-\mu_x)\left.\frac{\partial g(x,y)}{\partial x}\right|_{\substack{\mu_x\\ \mu_y}}+(y-\mu_y)\left.\frac{\partial g(x,y)}{\partial y}\right|_{\substack{\mu_x\\ \mu_y}}+\frac{1}{2!}(x-\mu_x)^2\left.\frac{\partial^2 g(x,y)}{\partial x^2}\right|_{\substack{\mu_x\\ \mu_y}}$$

$$+\frac{1}{2!}(y-\mu_y)^2\left.\frac{\partial^2 g(x,y)}{\partial y^2}\right|_{\substack{\mu_x\\ \mu_y}}+(x-\mu_x)(y-\mu_y)\left.\frac{\partial^2 g(x,y)}{\partial x\,\partial y}\right|_{\substack{\mu_x\\ \mu_y}}+\ldots \qquad (2.1)$$

where $\mu_x=E(X)$ and $\mu_y=E(Y)$

Given $g(x,y)=\frac{x}{y}$, then

$$\left.\begin{aligned}
&\frac{\partial g(x,y)}{\partial x}=\frac{1}{y}\Rightarrow \left.\frac{\partial g(x,y)}{\partial x}\right|_{\substack{\mu_x\\ \mu_y}}=\frac{1}{\mu_y}\\
&\frac{\partial^2 g(x,y)}{\partial \mathrm{x}^2}=0\Rightarrow \left.\frac{\partial^2 g(x,y)}{\partial \mathrm{x}^2}\right|_{\substack{\mu_x\\ \mu_y}}=0\\
&\frac{\partial g(x,y)}{\partial y}=\frac{-x}{y^2}\Rightarrow \left.\frac{\partial g(x,y)}{\partial y}\right|_{\substack{\mu_x\\ \mu_y}}=\frac{-\mu_x}{\mu_y^{\,2}}\\
&\frac{\partial^2 g(x,y)}{\partial \mathrm{y}^2}=\frac{2\mathrm{x}}{\mathrm{y}^3}\Rightarrow \left.\frac{\partial^2 g(x,y)}{\partial \mathrm{y}^2}\right|_{\substack{\mu_x\\ \mu_y}}=\frac{2\mu_x}{\mu_y^{\,3}}\\
&\frac{\partial^2 g(x,y)}{\partial y\,\partial x}=\frac{\partial^2 g(x,y)}{\partial x\,\partial y}=\frac{-1}{y^2}\Rightarrow \left.\frac{\partial^2 g(x,y)}{\partial y\,\partial x}\right|_{\substack{\mu_x\\ \mu_y}}=\frac{-1}{\mu_y^{\,2}}
\end{aligned}\right\} \qquad (2.2)$$

Take the expectation of both sides of Eq. (2.1) with substitution the results of Eq. (2.2), we have:

$$E\left(\frac{X}{Y}\right)\approx\frac{E(X)}{E(Y)}+\frac{\mathrm{E}(\mathrm{X})}{\{\mathrm{E}(\mathrm{Y})\}^3}\mathrm{var}(\mathrm{Y})-\frac{1}{\{\mathrm{E}(\mathrm{Y})\}^2}\mathrm{cov}(\mathrm{X},\mathrm{Y})$$

$$=\frac{E(X)}{\mathrm{E}(\mathrm{Y})}\left[1+\frac{\mathrm{var}(\mathrm{Y})}{\{\mathrm{E}(\mathrm{Y})\}^2}-\frac{\mathrm{E}(\mathrm{XY})-\mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y})}{\mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y})}\right]$$

$$=\frac{E(X)}{E(Y)}\left[2+\frac{\mathrm{var}(\mathrm{Y})}{\{\mathrm{E}(\mathrm{Y})\}^2}-\frac{\mathrm{E}(\mathrm{XY})}{\mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y})}\right] \qquad (2.3)$$

Take the variance of both sides of Eq. (2.1) with substitution the results of Eq. (2.2), we have:

$$var\left(\frac{X}{Y}\right)\approx\frac{var(X)}{\{E(Y)\}^2}+var(y)\frac{\{E(X)\}^2}{\{E(Y)\}^4}-2cov(X,Y)\frac{E(X)}{\{E(Y)\}^3}$$

$$=\left[\frac{E(X)}{E(Y)}\right]^2\left[\frac{var(X)}{\{E(X)\}^2}+\frac{var(Y)}{\{E(Y)\}^2}-\frac{2cov(X,Y)}{E(X)E(Y)}\right]$$

$$=\left[\frac{E(X)}{E(Y)}\right]^2\left[\frac{var(X)}{\{E(X)\}^2}+\frac{var(Y)}{\{E(Y)\}^2}-\frac{2\{E(XY)-E(X)E(Y)\}}{E(X)E(Y)}\right]$$

$$=\left[\frac{E(X)}{E(Y)}\right]^2\left[2+\frac{var(X)}{\{E(X)\}^2}+\frac{var(Y)}{\{E(Y)\}^2}-\frac{2E(XY)}{E(X)E(Y)}\right] \qquad (2.4)$$

2.3 Approximation to the Mean and Variance of Moments Method Estimators:

In this section, we shall consider the approximation to the mean and variance of moments method estimators by using equations (2.3) and (2.4).

(2.3.1) Approximation to the mean of $\hat{\alpha}$:

Consider the expectation of $\hat{\alpha}$ given by Eq. (1.24)

$$E(\alpha)=E\left[\frac{n\bar{X}^2}{(n-1)S^2}\right]=\left(\frac{n}{n-1}\right)E\left[\frac{\bar{X}^2}{S^2}\right]$$

Use of Eq. (2.3), with $X=\bar{X}^2$ and $Y=S^2$, we have:

$$E(\alpha)\approx\left(\frac{n}{n-1}\right)\left[\frac{E(\bar{X}^2)}{E(S^2)}\right]\left[2+\frac{var(S^2)}{\{E(S^2)\}^2}-\frac{E(\bar{X}^2S^2)}{E(\bar{X}^2)E(S^2)}\right] \qquad (2.5)$$

consider

$$E(\bar{X}^2S^2)=E\left[\bar{X}^2\frac{1}{(n-1)}\left\{\sum_{i=1}^{n}X_i^2-n\bar{X}^2\right\}\right]$$

$$=\frac{1}{(n-1)}E\left[\bar{X}^2\sum_{i=1}^{n}X_i^2-n\bar{X}^4\right]$$

$$=\frac{1}{n^2(n-1)}E\left[\left(\sum_{i=1}^{n}X_i\right)^2\left(\sum_{i=1}^{n}X_i^2\right)\right]-\frac{n}{(n-1)}E(\bar{X}^4)$$

$$=A-B$$

where $A=\frac{1}{n^2(n-1)}E\left[\left(\sum_{i=1}^{n}X_i\right)^2\left(\sum_{i=1}^{n}X_i^2\right)\right]$ and $B=\frac{n}{(n-1)}E(\bar{X}^4)$

consider

$$A=\frac{1}{n^2(n-1)}E\left[\left\{\sum_{i=1}^{n}X_i^2+2\sum_{i=1}^{n-1}\sum_{j=2}^{n}X_iX_j\right\}\left(\sum_{i=1}^{n}X_i^2\right)\right]$$

$$= \frac{1}{n^2(n-1)} E\left[\left(\sum_{i=1}^{n} X_i^2\right)^2 + 2\left(\sum_{i=1}^{n} X_i^2\right)\left(\sum_{i=1}^{n-1}\sum_{j=2}^{n} X_i X_j\right)\right]$$

$$= \frac{1}{n^2(n-1)} E\left[\sum_{i=1}^{n} X_i^4 + 2\sum_{i-1}^{n-1}\sum_{j=2}^{n} X_i^2 X_j^2 + 2\sum_{i-1}^{n-1}\sum_{j=2}^{n} X_i^3 X_j + 2\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} X_i^2 X_j X_k\right]$$

$$= \frac{1}{n^2(n-1)}\left[\begin{array}{l}\sum_{i=1}^{n} E\left(X_i^4\right) + 2\sum_{i=1}^{n-1}\sum_{j=2}^{n} E\left(X_i^2\right)E\left(X_j^2\right) + 2\sum_{i=1}^{n-1}\sum_{j=2}^{n} E\left(X_i^3\right)E\left(X_j\right)\\ +2\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} E\left(X_i^2\right)E\left(X_j\right)E\left(X_k\right)\end{array}\right]$$

Use of Eq. (1.12), with r=1, 2, 3, 4, we have:

$$A = \frac{1}{n^2(n-1)}\left[\begin{array}{l}\sum_{i=1}^{n}\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4 + 2\sum_{i=1}^{n-1}\sum_{j=2}^{n}\alpha^2(\alpha+1)^2\beta^4\\ +2\sum_{i=1}^{n-1}\sum_{j=2}^{n}\alpha(\alpha+1)(\alpha+2)\beta^3\alpha\beta + 2\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n}\alpha(\alpha+1)\beta^2(\alpha\beta)^2\end{array}\right]$$

$$= \frac{1}{n^2(n-1)}\left[\begin{array}{l}n\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4 + n(n-1)\alpha^2(\alpha+1)^2\beta^4\\ +2n(n-1)\alpha^2(\alpha+1)(\alpha+2)\beta^4 + \frac{2\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)}{2}\alpha^3(\alpha+1)\beta^4\end{array}\right]$$

$$= \frac{n\alpha(\alpha+1)\beta^4}{n^2(n-1)}[(n\alpha+2)(n\alpha+3)]$$

and

$$B = \frac{n}{(n-1)} E\left(\bar{X}^4\right)$$

Use of Eq. (1.31), with r=4, we have:

$$B = \frac{n}{(n-1)}\; n\alpha(n\alpha+1)(n\alpha+2)(n\alpha+3)\left(\frac{\beta}{n}\right)^4$$

$$= \frac{\alpha\beta^4}{n^2(n-1)}(n\alpha+1)(n\alpha+2)(n\alpha+3)$$

so

$$E\left(\bar{X}^2 S^2\right) = \frac{n\alpha(\alpha+1)\beta^4}{n^2(n-1)}\left[(n\alpha+2)(n\alpha+3)\right] - \frac{\alpha\beta^4}{n^2(n-1)}(n\alpha+1)(n\alpha+2)(n\alpha+3)$$

$$= \frac{\alpha\beta^4}{n^2}(n\alpha+2)(n\alpha+3)$$

Now

$$\frac{E\left(\bar{X}^2\right)}{E\left(S^2\right)} = \frac{n\alpha\left(n\alpha+1\right)\frac{\beta^2}{n^2}}{\alpha\beta^2} = \frac{n\alpha+1}{n}$$

$$E(\bar{X}^2)E(S^2) = n\alpha(n\alpha+1)\frac{\beta^2}{n^2}\alpha\beta^2 = \frac{\alpha^2(n\alpha+1)}{n}\beta^4$$

$$var\left(S^2\right) = \frac{\alpha\beta^4\left(6n-6+2n\alpha\right)}{n(n-1)}$$

$$\frac{\operatorname{var} S^2}{\left[E\left(S^2\right)\right]^2} = \frac{6n-6+2n\alpha}{n(n-1)\alpha} = \frac{6}{n\alpha} + \frac{2}{n-1}$$

Therefore

$$E\left(\alpha\right) = \left(\frac{n}{n-1}\right)\left(\frac{n\alpha+1}{n}\right)\left[2 + \frac{6}{n\alpha} + \frac{2}{n-1} - \frac{\frac{\alpha\beta^4}{n^2}\left(n\alpha+2\right)\left(n\alpha+3\right)}{\frac{\alpha^2\left(n\alpha+1\right)\beta^4}{\text{n}}}\right]$$

$$= \left(\frac{\alpha+\frac{1}{n}}{1-\frac{1}{n}}\right)\left(2 + \frac{6}{n\alpha} + \frac{2}{n-1} - \frac{\left(\alpha+\frac{2}{n}\right)\left(\alpha+\frac{3}{n}\right)}{\alpha\left(\alpha+\frac{1}{n}\right)}\right) \qquad (2.6)$$

By taking the limit of Eq. (2.6) as $n \to \infty$,we get:

$$\lim_{n\to\infty} E\left(\alpha\right) = \left(\frac{\alpha+0}{1-0}\right)\left(2+0+0-\frac{\left(\alpha+0\right)\left(\alpha+0\right)}{\alpha\left(\alpha+0\right)}\right) = \alpha$$

Therefore, according to definition (1.7.4) $\hat{\alpha}$ is asymptotically unbiased estimator for α, where the bias of $\hat{\alpha}$ is:

$$b\left(\alpha\right) = E\left(\alpha\right) - \alpha \qquad (2.7)$$

(2.3.2) Approximation to the variance of $\hat{\alpha}$:

Use of Eq. (2.4) with $X = \bar{X}^2$ and $Y = S^2$, we have:

$$var(\alpha) = var\left[\frac{n\bar{X}^2}{(n-1)S^2}\right] = \left(\frac{n}{n-1}\right)^2 var\left(\frac{\bar{X}^2}{S^2}\right)$$

$$\approx \left(\frac{n}{n-1}\right)^2 \left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 \left[2 + \frac{var(\bar{X}^2)}{\{E(\bar{X}^2)\}^2} + \frac{var(S^2)}{\{E(S^2)\}^2} - \frac{2E(\bar{X}^2 S^2)}{E(\bar{X}^2)E(S^2)}\right]$$

$$= \left(\frac{n}{n-1}\right)^2 \left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 \left[2 + \frac{E(\bar{X}^4) - \{E(\bar{X}^2)\}^2}{\{E(\bar{X}^2)\}^2} + \frac{var(S^2)}{\{E(S^2)\}^2} - \frac{2E(\bar{X}^2 S^2)}{E(\bar{X}^2)E(S^2)}\right]$$

$$= \left(\frac{n}{n-1}\right)^2 \left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 \left[1 + \frac{E(\bar{X}^4)}{\{E(\bar{X}^2)\}^2} + \frac{var(S^2)}{\{E(S^2)\}^2} - \frac{2E(\bar{X}^2 S^2)}{E(\bar{X}^2)E(S^2)}\right]$$

where

$$\left[\frac{E(\bar{X}^2)}{E(S^2)}\right]^2 = \left[\frac{n\alpha(n\alpha+1)(\frac{\beta}{n})^2}{\alpha\beta^2}\right]^2 = \left(\frac{n\alpha+1}{n}\right)^2$$

$$\frac{E(\bar{X}^4)}{\{E(\bar{X}^2)\}^2} = \frac{n\alpha(n\alpha+1)(n\alpha+2)(n\alpha+3)(\frac{\beta}{n})^4}{\left[n\alpha(n\alpha+1)\left(\frac{\beta}{n}\right)^2\right]^2} = \frac{(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)}$$

$$\frac{var(S^2)}{\{E(S^2)\}^2} = \frac{6n - 6 + 2n\alpha}{n\alpha(n-1)}$$

$$\frac{2E(\bar{X}^2 S^2)}{E(\bar{X}^2)E(S^2)} = \frac{\frac{2\alpha\beta^4}{n^2}(n\alpha+2)(n\alpha+3)}{\alpha^2(n\alpha+1)\frac{\beta^4}{n}} = \frac{2(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)}$$

Therefore

$$var(\alpha) = \left(\frac{n}{n-1}\right)^2 \left(\frac{n\alpha+1}{n}\right)^2 \left[1 + \frac{(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)} + \frac{6n-6+2n\alpha}{n\alpha(n-1)} - \frac{2(n\alpha+2)(n\alpha+3)}{n\alpha(n\alpha+1)}\right]$$

$$= (\alpha+1)\left[\frac{\frac{2\alpha}{n}+\frac{2}{n^2}}{\left(1-\frac{1}{n}\right)^3}\right] = \frac{2n\,(n\alpha+1)(\alpha+1)}{(n-1)^3} \qquad (2.8)$$

(2.3.3) Approximation to the mean of $\widehat{\beta}$:

Consider the expectation of $\widehat{\beta}$ given by Eq. (1.25)

$$E(\beta) = E\left[\frac{(n-1)S^2}{n\bar{X}}\right] = \left(\frac{n-1}{n}\right)E\left(\frac{S^2}{\bar{X}}\right)$$

Use of Eq. (2.3), with $X = S^2$ and $Y = \bar{X}$, we have:

$$E(\beta) \approx \left(\frac{n-1}{n}\right)\left[\frac{E(S^2)}{E(\bar{X})}\right]\left[2+\frac{var(\bar{X})}{\{E(\bar{X})\}^2}-\frac{E(\bar{X}S^2)}{E(\bar{X})E(S^2)}\right]$$

From Eq. (1.31), we have:

$$,var(\bar{X}) = \frac{\alpha\beta^2}{n} \quad E(S^2) = \alpha\beta^2 \quad \text{and} \quad E(\bar{X}) = \alpha\beta$$

$$var(S^2) = \frac{\alpha\beta^4(6n-6+2n\alpha)}{n(n-1)}$$

consider

$$E(\bar{X}S^2) = \mathrm{E}\left[\bar{X}\,\frac{1}{n-1}\left\{\sum_{i=1}^{n}X_i^2 - n\bar{X}^2\right\}\right]$$

$$= \frac{1}{n-1}\mathrm{E}\left(\bar{X}\sum_{i=1}^{n}X_i^2\right) - \frac{n}{n-1}E(\bar{X}^3)$$

$$= \frac{1}{n(n-1)}E\left[\left\{\sum_{i=1}^{n}X_i\right\}\left\{\sum_{i=1}^{n}X_i^2\right\}\right] - \frac{n}{n-1}E(\bar{X}^3)$$

$$= A - B$$

where

$$A = \frac{1}{n(n-1)}E\left[\sum_{i=1}^{n}X_i^3 + \sum_{i=1}^{n-1}\sum_{j=2}^{n}X_i^2X_j\right], \text{ and } B = \frac{n}{n-1}E(\bar{X}^3)$$

consider

$$A=\frac{1}{n(n-1)}\left[\sum_{i=1}^{n}E\left(X_i^{3}\right)+\sum_{i=1}^{n-1}\sum_{j=2}^{n}E\left(X_i^{2}\right)E\left(X_j\right)\right]$$

Use of Eq. (1.12) with r=2, 3 we have:

$$A=\frac{1}{n(n-1)}\left[\sum_{i=1}^{n}\alpha(\alpha+1)(\alpha+2)\beta^{3}+\sum_{i=1}^{n-1}\sum_{j=2}^{n}\alpha(\alpha+1)\beta^{2}\alpha\beta\right]$$

$$=\frac{1}{n(n-1)}\left[n\alpha(\alpha+1)(\alpha+2)\beta^{3}+n(n-1)\alpha^{2}(\alpha+1)\beta^{3}\right]$$

$$=\frac{n\alpha(\alpha+1)(n\alpha+2)\beta^{3}}{n(n-1)}$$

Use of Eq. (1.31), with r=3, we have:

$$B=\frac{n}{n-1}n\alpha(n\alpha+1)(n\alpha+2)\left(\frac{\beta}{n}\right)^{3}=\frac{\alpha(n\alpha+1)(n\alpha+2)\beta^{3}}{n(n-1)}$$

Therefore

$$E\left(\bar{X}S^{2}\right)=\frac{n\alpha(\alpha+1)(n\alpha+2)\beta^{3}}{n(n-1)}-\frac{\alpha(n\alpha+1)(n\alpha+2)\beta^{3}}{n(n-1)}=\frac{\alpha(n\alpha+2)\beta^{3}}{n}$$

so

$$E(\beta)\approx\left(\frac{n-1}{n}\right)\left(\frac{\alpha\beta^{2}}{\alpha\beta}\right)\left[2+\frac{\left(\frac{\alpha\beta^{2}}{n}\right)}{(\alpha\beta)^{2}}-\frac{\frac{\alpha(n\alpha+2)\beta^{3}}{n}}{(\alpha\beta)(\alpha\beta^{2})}\right]=\left(\frac{n-1}{n}\right)\beta\left[1-\frac{1}{n\alpha}\right]$$

$$=\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n\alpha}\right)\beta \tag{2.9}$$

By taking the limit of Eq. (2.9) as $n\to\infty$, we get :

$$\lim_{n\to\infty}E(\beta)=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n\alpha}\right)\beta=\beta$$

Therefore $\hat{\beta}$ is asymptotically unbiased estimator for β, where the bias of $\hat{\beta}$ is

$$b(\beta)=E(\beta)-\beta \tag{2.10}$$

(2.3.4) Approximation to the variance of $\hat{\boldsymbol{\beta}}$:

Use of Eq. (2.4) with $X = S^2$ *and* $Y = \bar{X}$, we have:

$$var(\beta) = var\left[\frac{(n-1)S^2}{n\bar{X}}\right]$$

$$= \left(\frac{n-1}{n}\right)^2 \mathrm{var}\left(\frac{S^2}{\bar{X}}\right)$$

$$\approx \left(\frac{n-1}{n}\right)^2 \left[\frac{E(S^2)}{E(\bar{X})}\right]^2 \left[2 + \frac{var(S^2)}{\{E(S^2)\}^2} + \frac{var(\bar{X})}{\{E(\bar{X})\}^2} - \frac{2E(\bar{X}S^2)}{E(\bar{X})E(S^2)}\right]$$

$$= \left(\frac{n-1}{n}\right)^2 \left(\frac{\alpha\beta^2}{\alpha\beta}\right)^2 \left[2 + \frac{\frac{\alpha\beta^4(6n-6+2n\alpha)}{n(n-1)}}{(\alpha\beta^2)^2} + \frac{\frac{\alpha\beta^2}{n}}{(\alpha\beta)^2} - \frac{\frac{2\alpha(n\alpha+2)\beta^3}{\mathrm{n}}}{(\alpha\beta)(\alpha\beta^2)}\right]$$

$$= \left(\frac{n-1}{n}\right)^2 \left(\frac{3}{n\alpha} + \frac{2}{n-1}\right)\beta^2 \tag{2.11}$$

(2.3.5) Mean Square Error of the MM Estimators:

Using the definition (1.7.5), we have:

$$MSE(\alpha,\alpha) = E\left[(\alpha-\alpha)^2\right] = \mathrm{var}(\alpha) + b^2(\alpha) = \mathrm{var}(\alpha) + \left[E(\alpha) - \alpha\right]^2$$

Where $var(\hat{\alpha})$ and $E(\hat{\alpha})$ are given by Eq.'s (2.8) and (2.6) respectively.

so

$$MSE(\alpha,\alpha) = (\alpha+1)\left[\frac{\frac{2\alpha}{n} + \frac{2}{n^2}}{\left(1-\frac{1}{n}\right)^3}\right] + \left[\left(\frac{\alpha+\frac{1}{n}}{1-\frac{1}{n}}\right)\left(2 + \frac{6}{n\alpha} + \frac{2}{n-1} - \frac{\left(\alpha+\frac{2}{n}\right)\left(\alpha+\frac{3}{n}\right)}{\alpha\left(\alpha+\frac{1}{n}\right)}\right) - \alpha\right]^2 \tag{2.12}$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} MSE(\alpha,\alpha) = 0$$

Then according to the definition (1.7.6), $\hat{\alpha}$ is mean square error consistent estimator of α, similarly:

$$MSE\left(\beta,\beta\right)=var\left(\beta\right)+\left(E\left(\beta\right)-\beta\right)^{2}$$

Where $var(\hat{\beta})$ and $E(\hat{\beta})$ are given by Eq.'s (2.11) and (2.9) respectively.

so

$$MSE\left(\beta,\beta\right)=\left(1-\frac{1}{n}\right)^{2}\left(\frac{3}{n\alpha}+\frac{2}{n-1}\right)\beta^{2}+\left[\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n\alpha}\right)\beta-\beta\right]^{2} \tag{2.13}$$

Taking the limit as $n \to \infty$, we have:

$$\lim_{n\to\infty} MSE\left(\beta,\beta\right)=0$$

Accordingly, $\hat{\beta}$ is mean squared error consistent estimator of β

2.4 Estimation of Parameters by Maximum Likelihood Method:

For gamma case, let $X_1, X_2, \ldots, X_n$ be a $r.s.$ of size n from $G(\alpha,\beta)$ where the distribution $p.d.f.$ is given by Eq. (1.1). So the likelihood function is:

$$L\left(\alpha,\beta;\underset{\sim}{x}\right)=\prod_{i=1}^{n} f\left(x_i;\alpha,\beta\right)=\left\{\Gamma\left(\alpha\right)\right\}^{-n}\beta^{-n\alpha}\left\{\prod_{i=1}^{n}x_i\right\}^{\alpha-1}e^{-\frac{1}{\beta}\sum_{i=1}^{n}x_i}$$

The log likelihood function is:

$$lnL=-nln\Gamma\left(\alpha\right)-n\alpha ln\beta+(\alpha-1)\sum_{i=1}^{n}ln\left(x_i\right)-\frac{1}{\beta}\sum_{i=1}^{n}x_i$$

$$\frac{\partial lnL}{\partial\alpha}=-n\psi\left(\alpha\right)-nln\beta+\sum_{i=1}^{n}ln\left(x_i\right)$$

$$\frac{\partial lnL}{\partial\beta}=-\frac{n\alpha}{\beta}+\frac{1}{\beta^{2}}\sum_{i=1}^{n}x_i$$

$$\frac{\partial^{2}lnL}{\partial\alpha^{2}}=-n\psi'\left(\alpha\right) \tag{2.14}$$

$$\frac{\partial^{2}lnL}{\partial\beta^{2}}=\frac{n\alpha}{\beta^{2}}-\frac{2}{\beta^{3}}\sum_{i=1}^{n}x_i \tag{2.15}$$

$$\frac{\partial^{2}lnL}{\partial\alpha\partial\beta}=\frac{\partial^{2}lnL}{\partial\beta\partial\alpha}=\frac{-n}{\beta} \tag{2.16}$$

Where $\psi(\alpha) = \frac{d}{d\alpha} ln\Gamma(\alpha)$ and $\psi'(\alpha) = \frac{d^2}{d\alpha^2} ln\Gamma(\alpha)$ are known as the diagamma and trigamma respectively.

The MLE of α and β are therefore given by setting:

$\frac{\partial lnL}{\partial\alpha} = \frac{\partial lnL}{\partial\beta} = 0$ at $\alpha = \alpha$ and $\beta = \beta$ That is

$$-n\psi(\alpha) - n ln\,\beta + \sum_{i=1}^{n} ln(x_i) = 0 \qquad (2.17)$$

$$-\frac{n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i = 0 \qquad (2.18)$$

From Eq. (2.17) and (2.18), we have:

$$\beta = \frac{\bar{X}}{\alpha} \qquad (2.19)$$

$$ln\,\alpha - \psi(\alpha) = ln\bar{X} - \frac{1}{n}\sum_{i=1}^{n} ln(x_i) \qquad (2.20)$$

and the large sample information variance – covariance matrix is:

$$I = -\begin{pmatrix} E(\frac{\partial^2 lnL}{\partial\alpha^2}) & E\left(\frac{\partial^2 lnL}{\partial\alpha\partial\beta}\right) \\ E(\frac{\partial^2 lnL}{\partial\beta\partial\alpha}) & E(\frac{\partial^2 lnL}{\partial\beta^2}) \end{pmatrix}^{-1}$$

Where the expectations of Esq.'s (2.14), (2.15) and (2.16) are:

$$E\left(\frac{\partial^2 lnL}{\partial\alpha^2}\right) = -n\psi'(\alpha),\; E\left(\frac{\partial^2 lnL}{\partial\alpha\partial\beta}\right) = E\left(\frac{\partial^2 lnL}{\partial\beta\partial\alpha}\right) = \frac{-n}{\beta} \quad and$$

$$E\left(\frac{\partial^2 lnL}{\partial\beta^2}\right) = \frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}E\left(\sum_{i=1}^{n} X_i\right) = \frac{n\alpha}{\beta^2} - \frac{2n\alpha\beta}{\beta^3} = -\frac{n\alpha}{\beta^2}$$

Therefore

$$I = -\begin{pmatrix} -n\psi'(\alpha) & \frac{-n}{\beta} \\ \frac{-n}{\beta} & -\frac{n\alpha}{\beta^2} \end{pmatrix}^{-1} = \begin{pmatrix} n\psi'(\alpha) & \frac{n}{\beta} \\ \frac{n}{\beta} & \frac{n\alpha}{\beta^2} \end{pmatrix}^{-1} = \frac{\beta^2}{n^2(\alpha\psi'(\alpha)-1)}\begin{pmatrix} \frac{n\alpha}{\beta^2} & \frac{-n}{\beta} \\ \frac{-n}{\beta} & n\psi'(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\alpha}{n(\alpha\psi'(\alpha)-1)} & \dfrac{-\beta}{n(\alpha\psi'(\alpha)-1)} \\ \dfrac{-\beta}{n(\alpha\psi'(\alpha)-1)} & \dfrac{\beta^2\psi'(\alpha)}{n(\alpha\psi'(\alpha)-1)} \end{pmatrix} \quad (2.21)$$

The asymptotic variances and covariance of MLE are:

$$\left.\begin{aligned} var(\alpha)_{asy.} &= \frac{\alpha}{n(\alpha\psi'(\alpha)-1)} \text{ , } var(\beta)_{asy.} = \frac{\beta^2\psi'(\alpha)}{n(\alpha\psi'(\alpha)-1)} \text{ and} \\ cov(\alpha,\beta)_{asy.} &= -\frac{\beta}{n(\alpha\psi'(\alpha)-1)} \end{aligned}\right\} \quad (2.22)$$

Analytic solution of $\hat{\alpha}$ and $\hat{\beta}$ can not be obtained from the non – linear equations (2.19) and (2.20), so iterative (numerical) method to the likelihood is required such as Newton – Raphson which can be made as follows:

Let

$$f(\alpha) = ln\,\alpha - \psi(\alpha) - ln\bar{X} + \frac{1}{n}\sum_{i=1}^{n} ln(x_i) = 0 \quad (2.23)$$

$$f'(\alpha) = \frac{1}{\alpha} - \psi'(\alpha) \quad (2.24)$$

The Newton – Raphson approximation for $\hat{\alpha}$ can be found by using repeatedly the following recurrence formula:

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha)}{f'(\alpha)} \quad (2.25)$$

If tables of the digamma and trigamma functions are not available, an excellent approximation is given by [1]:

$$\psi(x) \approx lnx - (2x - \frac{1}{3} + \frac{1}{16x})^{-1} \quad (2.26)$$

$$\psi'(x) = (x - \frac{1}{2} + \frac{1}{10x})^{-1} \quad (2.27)$$

Use the approximation of equations (2.26) and (2.27), then equation (2.25) becomes:

$$\alpha_{i+1} = \alpha_i - \frac{\frac{1}{n}\sum_{i=1}^{n} ln\left(x_i\right) - ln\overline{X} + (2\alpha_i - \frac{1}{3} + \frac{1}{16\alpha_i})^{-1}}{\frac{1}{\alpha_i} - (\alpha_i - \frac{1}{2} + \frac{1}{10\alpha_i})^{-1}} \tag{2.28}$$

The right hand side of Eq. (2.28) yields a new trial value for $\hat{\alpha}$, where the process is repeated until successive $\hat{\alpha}$ estimates agree to a given specified tolerance. While the value of the estimator $\hat{\beta}$ is obtained by using (2.19) which $\beta = \frac{\overline{X}}{\alpha}$.

-Chapter Three-

Improve Estimators for Maximum Likelihood Method

3.1 Introduction:

In this chapter, we shall introduce several alternative estimators for the shape parameter of MLM that are proposed in literatures that are concerned with theoretical approximation to their biases and variances were developed and used to compare the variance and mean square error properties of the estimators. A new estimator for the shape parameter based on bias correction for the maximum likelihood estimator is derived and evaluated by simulation.

3.2 Improved Estimators of MLM:

Several estimators are proposed in literatures for improvement of MLE and some of their properties are developed and investigated.

McCullagh and Nelder [12] consider a number of estimators for the shape parameter α. The first is the ML estimator $\hat{\alpha}$ which is the solution of

$$2n\left[ln\,\alpha - \psi\left(\alpha\right)\right] = D \tag{3.1}$$

Where $\psi(\alpha) = \frac{d}{d\alpha} ln\Gamma(\alpha)$ is the digamma function and D is the deviance statistic given by:

$$D = 2n\left[n\,ln\bar{X} - \sum lnx_i\right] \tag{3.2}$$

The deviance statistic D is proportional to the twice difference between the maximum attained value of the log-likelihood and log-likelihood when α is treated as known.

An exact solution for $\hat{\alpha}$ satisfying Eq. (3.1) has to be found iteratively.

Green Wood and Durand [8] give the approximation:

$$\alpha = \begin{cases} D_1^{-1}\left(0.500876 + 0.1648852 D_1 - 0.0544274 D_1^2\right), 0 < D_1 < 0.5772 \\ \dfrac{8.898919 + 9.05995 D_1 + 0.9775373 D_1^2}{D_1(17.79728 + 11.968477 D_1 + D_1^2)}, D_1 > 0.5772 \end{cases}$$

where $D_1 = \frac{D}{2n}$.

The maximum errors in these approximations are claimed to be 0.0088% and 0.0054% respectively.

Using the asymptotic formula [16]

$$\psi(\alpha) = ln\,\alpha - \frac{1}{2\alpha} - \frac{1}{12\alpha^2} + \frac{1}{120\alpha^4} + o\left(\alpha^{-6}\right) \tag{3.3}$$

and usage of the approximation $\psi(\alpha)$ given by the equation [1]:

$$\psi(\alpha) = ln\,\alpha - (2\alpha - \frac{1}{3} + \frac{1}{16\alpha})^{-1} \tag{3.4}$$

We propose that, if α is sufficient large and terms of order α^{-2} and α^{-1} are ignored in equations (3.3) and (3.4) respectively, we have:

$$\psi(\alpha) \approx ln\,\alpha - \frac{1}{2\alpha} \tag{3.5}$$

and

$$\psi(\alpha) \approx ln\,\alpha - (2\alpha - \frac{1}{3})^{-1} \tag{3.6}$$

Equations (3.5) and (3.6) provide the following simple approximation to $\hat{\alpha}$

$$\alpha_1 = nD^{-1} \tag{3.7}$$

and

$$\alpha_2 = nD^{-1} + \frac{1}{6} \tag{3.8}$$

Also, Cordeiro, G.M [4] show that the expectation of the deviance statistic is:

$$E(D) = 2n\left[ln\,\alpha - \psi(\alpha)\right] - \frac{2}{\alpha} + o\left(n^{-1}\right) \tag{3.9}$$

Equating D to its expectation corrected to $o(n^{-1})$, McCullage and Nelder suggest that the improvement to the ML procedure is to use the estimator $\hat{\alpha}_3$ which is the solution of:

$$2n\left[ln\,\alpha_3 - \psi\left(\alpha_3\right)\right] - 2\alpha_3^{-1} = D \tag{3.10}$$

If terms of $o(\alpha^{-2})$ are ignored, the estimators

$$\alpha_4 = \left(n-2\right)D^{-1} \tag{3.11}$$

Provides an approximation to $\hat{\alpha}_3$.

The final estimator proposed by McCullagh and Nelder is the moment estimator :

$$\alpha_5 = (n-2)T^{-1} \qquad (3.12)$$

Where $T = \sum_{i=1}^{n}(x_i - 1)^2$ is the Pearson statistic for the gamma shape parameter α. This estimator has the advantage of being much less sensitive to very small samples of gamma distn than the estimators based on the deviance statistic which is infinite if any observation is zero.

3.3 Bias Corrected Estimators for the Shape Parameter:

The bias of the MLE for the shape parameter α of gamma distn was developed theoretically to order n^{-1} by AL-Abood, A.M. [1] as:

$$bias(\alpha) = \frac{n^{-1}}{2[\alpha\psi'(\alpha)-1]}\left[1 - \frac{\alpha^2\psi''(\alpha)+1}{\alpha\psi'(\alpha)-1}\right] \qquad (3.13)$$

by using the expansions:

$$\alpha\psi'(\alpha) - 1 = \frac{1}{2\alpha} + \frac{1}{6\alpha^2} + o(\alpha^{-4}) \qquad (3.14)$$

$$\alpha^2\psi''(\alpha) + 1 = -\frac{1}{\alpha} - \frac{1}{2\alpha^2} + o(\alpha^{-4}) \qquad (3.15)$$

and neglecting terms of $o(\alpha^{-2})$, we obtain from Eq.(3.13)

$$bias(\alpha) = \frac{3\alpha}{n} \qquad (3.16)$$

Making a direct correction for the bias of $\hat{\alpha}$, and terms of $o(\alpha^{-2})$ are ignored, we lead to a new biased estimator

$$\alpha_6 = \alpha(1 - \frac{3}{n})$$

Where $\hat{\alpha}$ is the MLE for α.

3.4 Monte-Carlo Investigation:

In this section, we report the results of a large scale Monte – Carlo investigation to assess the approximation to the biases and variances of the MM and MLM estimators and to make comparisons by using their mean square error.

In order to assess the adequacy of the theoretical approximations, a large Monte – Carlo study was made by generating random sample of size n=5 (1) 10 (2) 20 (5) 30 according to theorems (1.9.5) and (1.9.3), where initial values of $\alpha = 5,7,9$ and $\beta = 1$. A simulation run size of 500 was used.

Tables (1) and (2) show the values of the biases of the MM estimators $(\hat{\alpha}, \hat{\beta})$ as obtained by simulation due to Eq.[s] (1.24) and (1.25) and by approximation that obtained due to Eq.['s] (2.6), (2.7), (2.9), and (2.10).

Table (1): Values of Bias($\hat{\alpha}$) for MM estimators

$\beta = 1$						
	α=5		α=7		α=9	
n	Approximation	Simulation	Approximation	Simulation	Approximation	Simulation
5	5.25	2.37083	7	2.98716	8.75	3.32808
6	4.08	2.40943	5.44	2.2744	6.8	2.7626
7	3.33333	1.42182	4.44444	1.72905	5.55556	3.00455
8	2.81633	1.68867	3.7551	1.98075	4.69388	1.94989
9	2.4375	1.93541	3.25	1.15999	4.0625	2.40418
10	2.14815	2.25317	2.8642	1.90294	3.58025	2.89508
12	1.73554	1.2234	2.31405	1.36811	2.89256	1.73801
14	1.45562	1.35262	1.94083	0.98249	2.42604	1.07712
16	1.25333	1.1387	1.67111	1.45167	2.08889	1.92193
18	1.10035	0.97917	1.46713	1.47563	1.83391	1.68708
20	0.98061	0.9987	1.30748	0.99699	1.63435	1.39289
25	0.77083	0.9141	1.02778	0.9413	1.28472	1.21671
30	0.63496	0.55858	0.84661	0.83417	1.05826	0.7874

Table (2): Values of Bias($\hat{\beta}$) for MM estimators

$\beta = 1$						
	$\alpha = 5$		$\alpha = 7$		$\alpha = 9$	
n	Approximation	Simulation	Approximation	Simulation	Approximation	Simulation
5	-0.232	-0.15378	-0.22286	-0.11832	-0.21778	-0.11915
6	-0.19444	-0.11074	-0.18651	-0.08912	-0.1821	-0.06205
7	-0.16735	0.0127	-0.16035	-0.06793	-0.15646	-0.12339
8	-0.14687	-0.10286	-0.14063	-0.10512	-0.13715	-0.04692
9	-0.13086	-0.14353	-0.12522	0.0018	-0.12209	-0.11695
10	-0.118	-0.18514	-0.11286	-0.06488	-0.11	-0.14579
12	-0.09861	-0.06449	-0.09425	-0.02788	-0.09182	-0.04229
14	-0.08469	-0.13184	-0.0809	-0.00416	-0.0788	0.03444
16	-0.07422	-0.06029	-0.07087	-0.04212	-0.06901	-0.05979
18	-0.06605	-0.06149	-0.06305	-0.07001	-0.06139	-0.04682
20	-0.0595	-0.07074	-0.05679	-0.02775	-0.05528	-0.03871
25	-0.04768	-0.07811	-0.04549	-0.04598	-0.04427	-0.04803
30	-0.03978	-0.0299	-0.03794	-0.04317	-0.03691	-0.01658

Tables (3) and (4) show the values of the variances of the MM estimators $(\hat{\alpha}, \hat{\beta})$ as obtained by simulation by using the sample variance and by approximation that obtained due to Eq.'s (2.8) and (2.11).

Table (3): Values of Variance($\hat{\alpha}$) for MM estimators

	$\alpha = 5, \beta = 1$		$\alpha = 7, \beta = 1$		$\alpha = 9, \beta = 1$	
n	Approximation	Simulation	Approximation	Simulation	Approximation	Simulation
5	24.375	12.12293	45	23.72805	71.875	24.86459
6	17.856	12.18914	33.024	12.81018	52.8	24.38329
7	14	8.71919	25.92593	10.8948	41.48148	21.70034
8	11.47522	7.82786	21.27114	11.7356	34.05248	16.22109
9	9.70313	9.14601	18	9.59002	28.82813	15.59192
10	8.39506	10.26788	15.58299	14.77234	24.96571	19.54907
12	6.59955	9.11409	12.26146	14.34252	19.6544	18.6218
14	5.42922	4.80818	10.09376	7.35904	16.18571	14.63454
16	4.608	5.97148	8.57126	10.85605	13.74815	17.83718
18	4.0081	5.00374	7.44474	9.29434	11.94382	15.56929
20	3.53404	4.59067	6.57822	7.27794	10.55547	11.64068
25	2.73438	3.4305	5.09259	5.30994	8.17419	8.52178
30	2.22887	2.3261	4.15269	4.52283	6.66694	6.87471

Table (4): Values of Variance($\hat{\beta}$) for MM estimators

	$\alpha=5,\beta=1$		$\alpha=7,\beta=1$		$\alpha=9,\beta=1$	
n	Approximation	Simulation	Approximation	Simulation	Approximation	Simulation
5	0.3968	0.20328	0.37486	0.18724	0.36267	0.21977
6	0.34722	0.28505	0.32738	0.23484	0.31636	0.20874
7	0.30787	0.30435	0.28988	0.17256	0.27988	0.15019
8	0.27617	0.15397	0.25977	0.13748	0.25065	0.13354
9	0.25021	0.16917	0.23516	0.16267	0.22679	0.11304
10	0.2286	0.13217	0.21471	0.17134	0.207	0.10064
12	0.19479	0.18355	0.18279	0.14324	0.17612	0.14211
14	0.16961	0.10034	0.15905	0.12591	0.15318	0.16734
16	0.15015	0.14042	0.14073	0.12996	0.1355	0.12219
18	0.13467	0.11983	0.12618	0.11163	0.12146	0.11806
20	0.12207	0.10687	0.11434	0.10645	0.11004	0.09459
25	0.09892	0.08947	0.0926	0.07083	0.08909	0.06979
30	0.08313	0.07528	0.07779	0.06948	0.07483	0.06695

Tables (5) and (6) show the values of the mean square error of the MM estimators $(\hat{\alpha},\hat{\beta})$ as obtained by simulation by using the sample mean square error and by approximation that obtained due to Eq.'s (2.12) and (2.13).

Table (5): Values of Mean Square Error($\hat{\alpha}$) for MM estimators

	$\alpha=5,\beta=1$		$\alpha=7,\beta=1$		$\alpha=9,\beta=1$	
n	Approximation	Simulation	Approximation	Simulation	Approximation	Simulation
5	51.9375	17.74376	94	32.65115	148.4375	35.94073
6	34.5024	17.9945	62.6176	17.98306	99.04	32.01525
7	25.11111	10.74075	45.67901	13.88441	72.34568	30.72767
8	19.40691	10.67946	35.37193	15.65898	56.08496	20.02317
9	15.64453	12.89183	28.5625	10.93559	45.33203	21.37202
10	13.0096	15.34466	23.78662	18.39353	37.78387	27.93055
12	9.61164	7.54232	17.61628	16.21425	28.02131	21.64248
14	7.54806	6.63777	13.86058	8.32432	22.07136	15.79473
16	6.17884	7.26812	11.36387	11.36387	18.1116	21.53101
18	5.21158	5.96251	9.5972	11.47182	15.30705	18.41552
20	3.53404	5.58808	8.28772	8.27193	13.22657	13.58082
25	3.32856	4.26608	6.14892	6.19598	9.8247	10.00215
30	2.63205	2.63812	4.86944	5.21867	7.78686	7.49471

Table (6): Values of Mean Square Error($\hat{\beta}$) for MM estimators

	$\alpha=5,\beta=1$		$\alpha=7,\beta=1$		$\alpha=9,\beta=1$	
n	Approximation	Simulation	Approximation	Simulation	Approximation	Simulation
5	0.45062	0.22692	0.42452	0.20124	0.41009	0.23397
6	0.38503	0.29732	0.36217	0.24278	0.34952	0.21259
7	0.33588	0.30451	0.31559	0.17717	0.30436	0.16541
8	0.29774	0.16455	0.27954	0.14853	0.26946	0.13574
9	0.26733	0.18977	0.25084	0.16268	0.2417	0.12671
10	0.24252	0.16645	0.22745	0.17555	0.2191	0.1219
12	0.20452	0.18771	0.19167	0.14401	0.18455	0.1439
14	0.17678	0.11772	0.16559	0.12593	0.15939	0.16853
16	0.15565	0.14405	0.14575	0.13174	0.14026	0.12576
18	0.13903	0.12361	0.13015	0.11653	0.12522	0.12025
20	0.12562	0.11187	0.11756	0.10722	0.1131	0.09609
25	0.10119	0.09558	0.09467	0.07294	0.09105	0.07209
30	0.08472	0.07617	0.07923	0.07134	0.07619	0.06723

Tables (7), (8) and (9) show the values of the biases, variances and mean square error of the MLE $(\hat{\alpha},\hat{\beta})$ as obtained by simulation and by theoretical that obtained due to Eq.'s (3.16) and (2.22) respectively.

While tables (10), (11) and (12) show the values of biases of MLE $\hat{\alpha}$ together with the values of the improved biases $\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_4,\hat{\alpha}_5$ and the values of the corrected bias estimator $\hat{\alpha}_6$.

Table (7) Values of Bias($\hat{\alpha}$) and Bias($\hat{\beta}$) for MLM estimators

	$\alpha = 5, \beta = 1$			$\alpha = 7, \beta = 1$			$\alpha = 9, \beta = 1$		
	$b(\hat{\alpha})$		$b(\hat{\beta})$	$b(\hat{\alpha})$		$b(\hat{\beta})$	$b(\hat{\alpha})$		$b(\hat{\beta})$
n	Theoretical	Simulation	Simulation	Theoretical	Simulation	Simulation	Theoretical	Simulation	Simulation
5	3	2.04111	-0.10814	4.2	2.70891	-0.10844	5.4	3.02904	-0.10162
6	2.5	2.1025	-0.10642	3.5	2.09933	-0.08322	4.5	2.40645	-0.05511
7	2.14286	1.31711	-0.00551	3	1.60727	-0.07551	3.85714	2.5565	-0.09605
8	1.875	1.31309	-0.03448	2.625	1.83353	-0.10689	3.375	1.61974	-0.01569
9	1.66667	1.61363	-0.13241	2.33333	0.81528	0.01817	3	2.12565	-0.10182
10	1.5	2.04787	-0.17605	2.1	1.67138	-0.03664	2.7	2.71648	-0.13586
12	1.25	0.95664	-0.04089	1.75	1.06105	0.00314	2.25	1.55872	-0.03055
14	1.07143	1.08533	-0.10668	1.5	0.97105	-0.0043	1.92857	0.90267	0.03876
16	0.9375	0.98106	-0.05228	1.3125	1.31881	-0.04178	1.6875	1.64802	-0.05027
18	0.83333	0.84131	-0.05429	1.16667	1.21819	-0.05314	1.5	1.51209	-0.04039
20	0.75	0.84259	-0.05756	1.05	0.87179	-0.02577	1.35	1.2887	-0.03513
25	0.6	0.80161	-0.07397	0.84	0.83423	-0.03704	1.08	1.10255	-0.04064
30	0.5	0.42132	-0.01699	0.7	0.68198	-0.03341	0.9	0.68123	-0.01359

Table (8) Values of Variance($\hat{\alpha}$) and Variance($\hat{\beta}$) for MLM estimators

	$\alpha = 5, \beta = 1$			$\alpha = 7, \beta = 1$			$\alpha = 9, \beta = 1$		
	$var(\hat{\alpha})$		$var(\hat{\beta})$	$var(\hat{\alpha})$		$var(\hat{\beta})$	$var(\hat{\alpha})$		$var(\hat{\beta})$
n	Theoretical	Simulation	Simulation	Theoretical	Simulation	Simulation	Theoretical	Simulation	Simulation
5	9.41667	11.11392	0.23477	18.77647	22.6066	0.16325	31.33636	22.80789	0.21954
6	7.84722	10.15468	0.19921	15.64706	12.71802	0.18108	26.11364	21.11633	0.17912
7	6.72619	7.92019	0.22353	13.41176	9.32675	0.14289	22.38312	19.15875	0.14975
8	5.88542	8.10156	0.18817	11.73529	10.1473	0.11633	19.58523	16.00244	0.12485
9	5.23148	7.6699	0.13585	10.43137	7.71405	0.13101	17.40909	14.18832	0.1058
10	4.70833	9.20799	0.12061	9.38824	14.91269	0.1672	15.66818	18.40463	0.10182
12	3.92361	5.0632	0.16887	7.82353	12.75732	0.14965	13.05682	18.41361	0.13632
14	3.3631	3.98227	0.08729	6.70588	7.44462	0.11956	11.19156	13.45246	0.14605
16	2.94271	5.22839	0.11526	5.86765	9.62167	0.10916	9.79261	15.47995	0.10377
18	2.61574	4.29514	0.1018	5.21569	8.20768	0.09548	8.70455	14.14481	0.10624
20	2.35417	4.02474	0.09339	4.69412	6.23494	0.09011	7.83409	10.84131	0.08903
25	1.88333	2.78473	0.07494	3.75529	4.92104	0.06834	6.26727	7.99315	0.06645
30	1.56944	1.97305	0.06202	3.12941	3.86593	0.0596	5.22273	5.96691	0.05844

Table (9) Values of Mean Square Error($\hat{\alpha}$) and Mean Square Error($\hat{\beta}$) for MLM estimators

$\alpha = 5, \beta = 1$				$\alpha = 7, \beta = 1$			$\alpha = 9, \beta = 1$		
$MSE(\hat{\alpha})$			$MSE(\hat{\beta})$	$MSE(\hat{\alpha})$		$MSE(\hat{\beta})$	$MSE(\hat{\alpha})$		$b(\hat{\beta})$
n	Theoretical	Simulation	Simulation	Theoretical	Simulation	Simulation	Theoretical	Simulation	Simulation
5	18.41667	15.28005	0.24646	36.41647	29.94481	0.17501	60.49636	31.98299	0.22987
6	14.09722	14.57519	0.21054	27.89706	17.1252	0.18801	46.36364	26.90734	0.18216
7	11.31804	9.65496	0.22356	22.41176	11.91008	0.14859	37.26065	25.69446	0.15898
8	9.40105	9.82577	0.18936	18.62592	13.50912	0.12775	30.97586	18.62599	0.1251
9	8.00927	10.2737	0.15338	15.87579	8.37873	0.13134	26.40909	18.70671	0.11617
10	6.95833	13.40175	0.15161	13.79824	17.70619	0.16854	22.95818	25.78391	0.12027
12	5.48611	5.97837	0.17054	10.88603	13.88315	0.14966	18.11932	20.8432	0.13725
14	4.51106	5.16022	0.09867	8.95588	8.38757	0.11958	14.91094	14.26727	0.14755
16	3.82162	6.19087	0.118	7.59031	11.36092	0.11091	12.64027	18.19592	0.10629
18	3.31018	5.00294	0.10474	6.57681	9.69166	0.09831	10.95455	16.43123	0.10787
20	2.91667	4.7347	0.0967	5.79662	6.99497	0.09077	9.65659	12.50205	0.09026
25	2.24333	3.4273	0.08041	4.46089	5.61698	0.06971	7.43367	9.20876	0.0681
30	1.81194	2.15056	0.06231	3.61941	4.33102	0.06072	6.03273	6.43099	0.05862

Table (10) Values of Improved Bias Estimator for MLM with $\alpha = 5$

$\alpha = 5, \beta = 1$							
Theoretical		Simulation					
n	$b(\hat{\alpha})$	$b(\hat{\alpha})$	$b(\hat{\alpha}_1)$	$b(\hat{\alpha}_2)$	$b(\hat{\alpha}_4)$	$b(\hat{\alpha}_5)$	$\hat{\alpha}_6$
5	3	2.04111	-3.62401	-3.45734	-4.17441	-4.96307	2.81644
6	2.5	2.1025	-3.84315	-3.67648	-4.22877	-4.96502	3.55125
7	2.14286	1.31711	-4.12051	-3.95384	-4.37179	-4.96422	3.60977
8	1.875	1.31309	-4.23096	-4.06429	-4.42322	-4.95987	3.94568
9	1.66667	1.61363	-4.28306	-4.11639	-4.44238	-4.95538	4.40908
10	1.5	2.04787	-4.31137	-4.1447	-4.4491	-4.95391	4.93350
12	1.25	0.95664	-4.517	-4.35033	-4.5975	-4.9533	4.46748
14	1.07143	1.08533	-4.57683	-4.41017	-4.63728	-4.95418	4.78133
16	0.9375	0.98106	-4.63623	-4.46957	-4.6817	-4.95527	4.85961
18	0.83333	0.84131	-4.68441	-4.51774	-4.71948	-4.95368	4.86775
20	0.75	0.84259	-4.71591	-4.54924	-4.74432	-4.95386	4.96620
25	0.6	0.80161	-4.77437	-4.6077	-4.79242	-4.95285	5.10541
30	0.5	0.42132	-4.82464	-4.65797	-4.83633	-4.95415	4.87918

Table (11) Values of Improved Bias Estimator for MLM with $\alpha = 7$

$\alpha = 7, \beta = 1$							
Theoretical		Simulation					
n	$b(\hat{\alpha})$	$b(\hat{\alpha})$	$b(\hat{\alpha}_1)$	$b(\hat{\alpha}_2)$	$b(\hat{\alpha}_4)$	$b(\hat{\alpha}_5)$	$\hat{\alpha}_6$
5	4.2	2.70891	-5.09075	-4.92409	-5.85445	-6.98424	3.88356
6	3.5	2.09933	-5.51055	-5.34388	-6.00703	-6.98287	4.54967
7	3	1.60727	-5.79361	-5.62694	-6.13829	-6.98177	4.91844
8	2.625	1.83353	-5.91614	-5.74947	-6.1871	-6.98063	5.52096
9	2.33333	0.81528	-6.14965	-5.98298	-6.33862	-6.98122	5.21018
10	2.1	1.67138	-6.1491	-5.98243	-6.31928	-6.9799	6.06997
12	1.75	1.06105	-6.34176	-6.17509	-6.45146	-6.97902	6.04579
14	1.5	0.97105	-6.44223	-6.27557	-6.52191	-6.97953	6.26297
16	1.3125	1.31881	-6.49023	-6.32356	-6.55395	-6.97925	6.75903
18	1.16667	1.21819	-6.55246	-6.38579	-6.60219	-6.97847	6.84845
20	1.05	0.87179	-6.61453	-6.44786	-6.65307	-6.97838	6.69102
25	0.84	0.83423	-6.69313	-6.52646	-6.71768	-6.97809	6.89412
30	0.7	0.68198	-6.74935	-6.58268	-6.76606	-6.97765	6.91378

Table (12) Values of Improved Bias Estimator for MLM with $\alpha = 9$

$\alpha = 9, \beta = 1$							
Theoretical		Simulation					
n	$b(\hat{\alpha})$	$b(\hat{\alpha})$	$b(\hat{\alpha}_1)$	$b(\hat{\alpha}_2)$	$b(\hat{\alpha}_4)$	$b(\hat{\alpha}_5)$	$\hat{\alpha}_6$
5	5.4	2.61479	-6.70971	-6.54305	-7.62583	-8.99102	4.64592
6	4.5	2.40645	-7.12617	-6.9595	-7.75078	-8.99045	5.70323
7	3.85714	2.5565	-7.37243	-7.20577	-7.83745	-8.98929	4.95277
8	3.375	1.61974	-7.69295	-7.52628	-8.01971	-8.98972	6.63734
9	3	2.12565	-7.78199	-7.61532	-8.05266	-8.98842	7.4171
10	2.7	2.71648	-7.84471	-7.67805	-8.07577	-8.98815	8.20154
12	2.25	1.55872	-8.13372	-7.96705	-8.2781	-8.98791	7.91904
14	1.92857	0.90267	-8.30432	-8.13765	-8.4037	-8.98821	7.78067
16	1.6875	1.64802	-8.34471	-8.17804	-8.42662	-8.98759	8.65152
18	1.5	1.51209	-8.42507	-8.2584	-8.48895	-8.98742	8.76008
20	1.35	1.2887	-8.49373	-8.32707	-8.54436	-8.98747	8.74539
25	1.08	1.10255	-8.60243	-8.43577	-8.63424	-8.98719	8.89024
30	0.9	0.68123	-8.68273	-8.51607	-8.70388	-8.98702	8.71311

Conclusions and Recommendations

(1) The biases of moment method estimators $\hat{\alpha}$ and $\hat{\beta}$ as given in tables (1) and (2) show that the approximation values is over estimate than the simulation values for small sample sizes and it is rapidly become adequate for moderate and large samples.

(2) Tables (3) and (4) show that the values of variance $\hat{\alpha}$ as given by approximation equation (2.8) approach slowly to the simulated values as α and the sample size n are increases. While the values of variance $\hat{\beta}$ as given by the approximation equation (2.11) are excellent in comparison with the simulated values, and that reflect on the values of the mean square error of $\hat{\alpha}$ and $\hat{\beta}$ as a consequences behavior of the variances.

(3) Table (7) show an excellent closeness between the values of the biases of maximum likelihood method estimators $\hat{\alpha}$ and $\hat{\beta}$ as obtained theoretically by equation (3.16) and by simulation. These biases show that they are superior than those of moment method.

(4) Tables (8) and (9) show that the variances and mean square errors values of maximum likelihood method of $\hat{\alpha}$ and $\hat{\beta}$ as obtained theoretically by equation (2.22) and by simulation are good for all sample sizes and approach zero as the sample size n increase . Furthermore these results are better than the results as given by tables (3), (4), (5) and (6).

(5) Tables (10), (11) and (12) show the bias values of maximum likelihood method $b(\hat{\alpha})$ with the values of the McCllage and Nelder bias estimators $b(\hat{\alpha}_1), b(\hat{\alpha}_2), b(\hat{\alpha}_4)$ and $b(\hat{\alpha}_5)$ together with the value of a new bias estimator $\hat{\alpha}_6$. We disagree with the proposed estimators of McCllage and Nelder that give an improvement while the new corrected bias that we propose gives results better than the results of moment method and maximum likelihood method.

For future works, the following problems may be recommended:

(1) Other numerical method could be used for an approximate solution of equation (2.24) such as bisection method, fixed point method, secant method ,..,etc, which might be compared with the used Newton – Raphson method in this thesis.

(2) The approximate biases and variances values of moment method as given in tables (1) and (2) become more closer to the simulation biases and variances values if we approximate by the Taylor series expansion up to 3^{rd} order of partial derivatives.

(3) This work can be used to generalize gamma distribution of three parameters and other life distribution.

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الملخص

في هذه الرسالة تطرقنا الى توزيع كاما لاهميته في تطبيقات الاختبارات الحياتيه وبحوث البقاء والتي تظهر في الدراسات الطبيه للامراض المزمنه وحياه المركبات الصناعيه.

تقريب لمعدل وتباين مخمنات طريقة العزوم اشتقت نظرياً باستخدام مفكوك تايلر مقرب لغايه المشتقة الجزئية الثانية. مخمنات الترجيح الاعظم اشتقت نظريا وقورنت مع مخمنات مقترحه في الموسوعه العلميه . حيث اظهر التطبيق عند حجوم العينات المتوسطة والكبيرة أن قيم التحيز لمخمنات طريقة العزوم متفقه مع قيم التحيز لطريقة المحاكاة. بينما كانت قيم التباين لمعلمة القياس متفقه تماماً بالطرقتين.

أفترضنا تحيز لمخمن معدل جديد يعتمد على مخمن الترجيح الاعظم والذي اظهر تحسن اكبر بالمقارنة مع المخمنات الاخرى المقترحة من قبل Cordeiro,Nelder, McCullagh وPearson. النتائج النظرية اختبرت باستخدام طرائق مونت – كارلو وقورنت باستخدام مقياس معدل مربع الخطأ.

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تقريب المعدل والتباين للمخمنات المرتبطة بتوزيع كاما

رسالة

مقدمه الى كلية العلوم / جامعة النهرين

كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل

يمامه نذير محمود

بكالوريوس 2012

إشراف

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ذو القعدة | أيلول
1435 هـ | 2014 م