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Some Analytical Methods for Solving Some Types of Nonlocal Problems

A Thesis
Submitted to the College of Science of Al-Nahrain University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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بِسْمِ اللهِ الرَّحْمَنِ الرَّحِيْمِ

وسارعوا إلى مغفرةٍ من ربكم وجنةٍ عرضها السموات والأرض أعدت للمتقيـن(*)

صدق الله العلي العظيم

سورة آل عمران الآية (133)

الإهداء

إلى...

من أحيا القلوب بعد مماتها وأنارها بعد ظلمتها وألف بينها بعد شتاتها الحبيب المصطفى محمد(صلى الله عليه وآله الغر الميامين).

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والــى فلــذة كبدي ولدي الحســـن

والى كل من يعتبر الرياضيات طريقاً في الحيـــاة

دنيـــا

Supervisors Certification

We certify that this thesis was prepared under our supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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In view of the available recommendations, I forward this thesis for debate by the examination committee.

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We certify that we have read this thesis entitled " ***Some Analytical Methods for Solving Some Types of Nonlocal Problems***" and as examining committee we examined its contents and what it connected with, and that is in our opinion it meets the standards of a thesis for the degree of Master of Science, in Mathematics.

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Before anything …

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Chapter Three: Real Life Applications For Nonlocal Problems Arising thermoelasticity

Abstract

This work concerns with the nonlocal problems and the main theme of it can be divide into three categories, which can be listed as follows:

First: **S**ome analytical methods namely, the separation of variables and the eigenfunction expansion method to solve some types of linear partial differential equations with nonlocal conditions, are presented.

Second: The varaitional iteration method is used to solve special types of the nonhomogenous partial differential equations and linear nonhomogenous parabolic integro-differential equations with nonlocal conditions.

Third : Some real life applications for the nonlocal problems arising in thermoelasticity problems and its solutions via the separation of variables, the eigenfunction expansion method and the varaitional iteration method with its convergence, are introduced.

Introduction

The nonlocal problems plays an important role in real life applications and they arise in various fields of mathematical physics (like string oscillation telegraph equations) [9], [13], biology and biotechnology (like evolution of dominant genes and propagation nerve pulses) [23] and in other fields, where certain problems of modern physics and technology can be efficiently described in terms of nonlocal problems for partial differential equations, [5].

The history of nonlocal problems with integral conditions for partial differential equations goes back to [7], where Cannon studied a problem for heat equation and this study was followed by many papers that were devoted to nonlocal problems arises with parametric and elliptic equations. Mixed problem with nonlocal integral conditions for one-dimensional hyperbolic equations were considered in [13] and [24].

The problems with integral nonlocal boundary conditions were discussed for parabolic equation in [19]. Numerical methods for solving problems with nonlocal conditions such as finite difference method were investigated in [8].

Development of the variational iteration method for solving linear, nonlinear, initial and boundary value problems given in [15]. It is worth mentioning that the method was first considered by [16], but the true potential of the variational iteration method was explored by He. In this method the solution is given in an infinite series usually converging to an accurate solutions, see [14], [17].

The variational iteration method which accurately computes the series solution is of great interest to applied science. The method provides the solution in rapidly convergent series with easily computable components the main advantage of the method is that it can be applied directly for all types of nonlinear differential and integral equations, homogenous or nonhomogeneous, with constant or variable coefficients. Moreover, this method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. [22]. [26] used variational iteration method to solve one dimensional nonlinear thermoelasiticity. [10] gave an approximate solution of differential equations arising in astrophysics using the variational iteration method.

This thesis consists of three chapters.

In chapter one, we present the method of separation of variables to solve some linear homogeneous partial differential equations with homogeneous nonlocal conditions and eigenfunction expansion to solve some differential equations with nonlocal homogenous and inhomogeneous conditions.

In chapter two, we solve the one-dimensional Heat equation with nonlocal condition via the variational iteration method and its convergence and present the solution of the parabolic partial differential equation with nonlocal condition via the variational iteration method.

In chapter three, we present the mathematical modeling of the thermoelasticity problem, and then solve the nonlocal problem arising in thermoelasticity problem using the separation of variables method and the eigenfunction expansion also we applied variational iteration method in solving the thermoelasticity problem after transforming the problem to a nonlocal condition problem and proved it is convergence under some conditions.

Chapter One
Solutions of Linear Partial Differential Equations with Nonlocal Conditions

1.1 Introduction

In modeling of many physical systems in various fields of physics, ecology, biology, etc, an integral term over the spatial domain is appeared in some part or in the whole boundary. Such boundary value problems are known as nonlocal problem. The integral term may appear in the boundary conditions. Non-local conditions appear when values of the function on the boundary are connected to values inside the domain, [4].

The aim of this chapter is to use the separation of the variables and the eigenfunction expansion method to solve special types of linear nonlocal problems.

This chapter consists of three sections.

In section two, we present the method of separation of variables to solve some types of linear homogeneous partial differential equations with homogeneous nonlocal conditions.

In section three, we present the eigenfunction expansion to solve some types of linear differential equations with nonlocal homogenous and nonhomogeneous conditions.

1.2 The Separation of Variables For Solving Partial Differential Equation With Nonlocal Conditions:

It is known that, the separation of variables is one of the oldest techniques for solving initial-boundary value problems and applied to problems where the partial differential equations and the boundary conditions are linear and homogeneous, [12, pp33].

In this section we use the separation of variables for solving special types of linear homogeneous partial differential equations together with non-local conditions. For this purpose, first consider the one dimensional heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 \le x \le \ell, \quad t \ge 0 \tag{1.1}$$

together with the initial condition:

$$u(x,0) = r(x), \qquad 0 \le x \le \ell, \tag{1.2}$$

the homogenous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad t \ge 0, \tag{1.3}$$

and the homogeneous nonlocal (integral) condition

$$\int_0^\ell u(x,t)dx = 0, \qquad t \ge 0 \tag{1.4}$$

where γ is a known nonzero constant, and r is a known function that must satisfy the following compatibility conditions:

$$r'(0) = 0$$

and

$$\int_0^\ell r(x)dx = 0.$$

By using the separation of variables, the solution of partial differential equation (1.1) can be expressed as:

$$u^*(x,t) = e^{-(\lambda\gamma)^2 t}\left[A\sin(\lambda x) + B\cos(\lambda x)\right] \tag{1.5}$$

where A, B, λ are arbitrary constants, [12, pp36]

Then this solution must satisfy the homogenous Neumann condition given by equation (1.3) and the homogenous nonlocal condition given by equation (1.4). To do this we differentiate the solution (1.5) with respect to x and by substituting $x = 0$ in the resulting equation one can have :

$$\left.\frac{\partial u^*(x,t)}{\partial x}\right|_{x=0} = \lambda A e^{-(\lambda\gamma)^2 t} = 0$$

Therefore either $\lambda = 0$ or $A = 0$.

If $\lambda = 0$, then the solution u^* becomes

$$u^*(x,t) = B.$$

Then this solution must satisfy the nonlocal condition given by equation (1.4), therefore

$$\int_0^{\ell} B dx = B\ell = 0$$

Since $\ell \neq 0$, then $B = 0$ and this implies that $u^*(x,t) = 0$. But this solution does not satisfy the initial condition given by equation (1.2). Therefore $A = 0$. In this case, equation (1.5) becomes:

$$u^*(x,t) = Be^{-(\lambda\gamma)^2 t}\cos(\lambda x).$$

This solution must satisfy the nonlocal condition given by equation (1.4), thus

$$Be^{-(\lambda\gamma)^2 t}\int_0^{\ell}\cos(\lambda x)dx = 0$$

therefore

$$\sin(\lambda\ell) = 0$$

and this implies that

$$\lambda = \mp\frac{n\pi}{\ell}, \qquad n = 1,2,\ldots.$$

Thus

$$u^*(x,t) = Be^{-\left(\frac{n\pi\gamma}{\ell}\right)^2 t}\cos\left(\frac{n\pi}{\ell}x\right)$$

and we have found an infinite number of functions:

$$u_n(x,t) = B_n e^{-\left(\frac{n\pi\gamma}{\ell}\right)^2 t}\cos\left(\frac{n\pi}{\ell}x\right), \qquad n = 1,2,\ldots.$$

each one satisfy the partial differential equation (1.1), the Neumann condition given by equation (1.3) and the nonlocal condition given by equation (1.4). The desired solution will be a certain sum of these simple functions and takes the form:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n e^{-\left(\frac{n\pi\gamma}{\ell}\right)^2 t}\cos\left(\frac{n\pi}{\ell}x\right) \qquad (1.6)$$

By substituting this sum into the initial condition given by equation (1.2) one can obtain:

$$r(x)=\sum_{n=1}^{\infty}B_n\cos\left(\frac{n\pi}{\ell}x\right).$$

But,

$$\int_0^{\ell}\cos\left(\frac{m\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)dx=\begin{cases}\frac{\ell}{2}, & n=m\\ 0, & n\neq m\end{cases} \qquad (1.7)$$

therefore

$$\sum_{n=1}^{\infty}\int_0^{\ell}B_n\cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)dx=\int_0^{\ell}r(x)\cos\left(\frac{m\pi x}{\ell}\right)dx \qquad m=1,2\ldots$$

gives

$$B_n=\frac{2}{\ell}\int_0^{\ell}r(x)\cos\left(\frac{n\pi}{\ell}x\right)dx, \qquad n=1,2,\ldots.$$

Thus equation (1.6) is solution of the nonlocal problem given by equations (1.1)-(1.4) .

To illustrate this method, consider the following example

Example (1.1)

It easy to check that the solution of the one-dimensional heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 \le x \le 1, \quad t > 0$$

together with the initial condition:

$$u(x,0) = \cos(2\pi x), \qquad 0 \le x \le 1,$$

the homogenous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad t \ge 0$$

and the homogeneous nonlocal condition:

$$\int_0^1 u(x,t)dx = 0, \qquad t \ge 0$$

takes the form

$$u(x,t) = e^{-4\pi^2 t}\cos(2\pi x), \qquad 0 \le x \le 1, \quad t \ge 0$$

Second, consider the one-dimensional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 \le x \le \ell, \quad 0 \le t \le T \tag{1.8}$$

together with the initial conditions:

$$u(x,0) = r(x), \qquad 0 \le x \le \ell, \tag{1.9}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = p(x), \qquad 0 \le x \le \ell, \tag{1.10}$$

the homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le T \tag{1.11}$$

and the homogeneous nonlocal condition:

$$\int_0^\ell u(x,t)dx = 0, \qquad 0 \le t \le T \tag{1.12}$$

where γ is a known nonzero constant and r, p are given functions that must satisfy the following compatibility conditions:

$$\int_0^{\ell} r(x)dx = 0$$

$$r'(0) = 0$$

and

$$\int_0^{\ell} p(x)dx = 0$$

By using the separation of variables, the solution of the partial differential equation (1.8) can be expressed as:

$$u^*(x,t) = [A\sin(\gamma\lambda t) + B\cos(\gamma\lambda t)][C\sin(\lambda x) + D\cos(\lambda x)]$$

where $A,\ B,\ C,\ D$ and λ are arbitrary constants, [12, pp155].

Then this solution must satisfy the Neumann condition given by equation (1.11) and the homogenous nonlocal condition given by equation (1.12). So

$$\left.\frac{\partial u^*(x,t)}{\partial x}\right|_{x=0} = C\lambda[A\sin(\gamma\lambda t) + B\cos(\gamma\lambda t)] = 0$$

and this implies that either $C = 0$ or $\lambda = 0$. If $\lambda = 0$ then

$$u^*(x,t) = BD$$

this solution does not satisfy the initial condition given by equation (1.9). Therefore $C = 0$, and hence

$$u^*(x,t) = D\cos(\lambda x)[A\sin(\gamma\lambda t) + B\cos(\gamma\lambda t)].$$

In this case

$$\int_0^{\ell} u^*(x,t)dx = \frac{D}{\lambda}\sin(\lambda\ell)\left[A\sin(\gamma\lambda t) + B\cos(\gamma\lambda t)\right] = 0$$

and this implies that either $D = 0$ or $\sin\lambda\ell = 0$. If $D = 0$ then the zero solution does not satisfy the initial condition given by equation (1.9). Thus $\sin\lambda\ell = 0$ and this implies that $\lambda = \mp\frac{n\pi}{\ell} \qquad n = 1,2,\ldots.$

Hence

$$u^*(x,t) = D\cos\left(\frac{n\pi}{\ell}x\right)\left[A\sin\left(\frac{\gamma n\pi}{\ell}t\right) + B\cos\left(\frac{\gamma n\pi}{\ell}t\right)\right]$$

and we have found an infinite number of functions

$$u_n(x,t) = D_n\cos\left(\frac{n\pi}{\ell}x\right)\left[A_n\sin\left(\frac{\gamma n\pi}{\ell}t\right) + B_n\cos\left(\frac{\gamma n\pi}{\ell}t\right)\right], \qquad n = 1,2,\ldots$$

each one satisfy the partial differential equation (1.8), the homogeneous Neumann condition given by equation (1.11) and the homogenous nonlocal condition given by equation (1.12). The desired solution will be a certain sum of these simple functions and takes the form:

$$u(x,t) = \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{\ell}x\right)\left[A_n\sin\left(\frac{\gamma n\pi}{\ell}t\right) + B_n\cos\left(\frac{\gamma n\pi}{\ell}t\right)\right]$$

To satisfy the initial conditions given by equations (1.9)-(1.10), one must have:

$$\sum_{n=1}^{\infty} B_n\cos\left(\frac{n\pi}{\ell}x\right) = r(x)$$

and,

$$\sum_{n=1}^{\infty}\frac{\gamma n\pi}{\ell}A_n\cos\left(\frac{n\pi}{\ell}x\right) = p(x)$$

By using the orthogonality condition given by equation (1.7) and the above two equations, one can obtain:

$$A_n = \frac{2}{n\gamma\pi}\int_0^{\ell} p(x)\cos\left(\frac{n\pi}{\ell}x\right)dx \qquad n = 1,2,\ldots$$

and

$$B_n = \frac{2}{\ell}\int_0^{\ell} r(x)\cos\left(\frac{n\pi}{\ell}x\right)dx \qquad n = 1,2,\ldots$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{\ell}x\right)\left[\left\{\frac{2}{n\gamma\pi}\int_0^{\ell} p(x)\cos\left(\frac{n\pi}{\ell}x\right)dx\right\}\sin\left(\frac{\gamma n\pi}{\ell}t\right) + \left\{\frac{2}{\ell}\int_0^{\ell} r(x)\cos\left(\frac{n\pi}{\ell}x\right)dx\right\}\cos\left(\frac{\gamma n\pi}{\ell}\right)t\right]$$

is the solution of the nonlocal problem given by equations (1.8)-(1.12)

To illustrate this method, consider the following example:

Example (1.2)

It easy to check that the solution of the one-dimensional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < 1, \quad 0 \le t \le T$$

together with the initial conditions:

$$u(x,0) = \cos(2\pi x), \qquad 0 \le x \le 1$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \cos(3\pi x), \qquad 0 \le x \le 1,$$

the homogeneous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le T$$

and the homogeneous nonlocal condition:

$$\int_0^1 u(x,t)dx = 0, \qquad t \ge 0$$

takes the form

$$u(x,t) = \cos(2\pi x)\cos(2\pi t) + \frac{1}{3\pi}\cos(3\pi x)\sin(3\pi t).$$

Third, consider the one-dimensional Laplace equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad 0 < x < \ell, \quad 0 \le t \le T \tag{1.13}$$

together with the initial conditions, the Neumann condition and the nonlocal conditions given by equations (1.9)-(1.12). It is assumed that the previous compatibility conditions are satisfied.

By using the separation of variables, the solution of the partial differential equation (1.13) can be expressed as:

$$u^*(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)][Ce^{\lambda t} + De^{-\lambda t}]$$

where A, B, C, D and λ are arbitrary constants, [12, pp15].

Then this solution must satisfy the Neumann condition given by equation (1.11) and the homogenous nonlocal condition given by equation (1.12). So

$$\left.\frac{\partial u^*(x,t)}{\partial x}\right|_{x=0} = \lambda B[Ce^{\lambda t} + De^{-\lambda t}] = 0$$

and this implies that either $B = 0$ or $\lambda = 0$. If $\lambda = 0$ then

$$u^*(x,t) = A[C + D]$$

this solution does not satisfy the initial condition given by equation (1.9). Therefore $B = 0$. Hence

$$u^*(x,t) = A\cos(\lambda x)[Ce^{\lambda t} + De^{-\lambda t}]$$

In this case

$$\int_0^\ell u^*(x,t)dx = \frac{A}{\lambda}\sin(\lambda\ell)[Ce^{\lambda t} + De^{-\lambda t}] = 0$$

and this implies that either $A = 0$ or $\sin(\lambda\ell) = 0$. If $A = 0$ then the zero solution does not satisfy the initial condition given by equation (1.9). Thus $\sin(\lambda\ell) = 0$ and so $\lambda = \mp\frac{n\pi}{\ell}$, $n = 1,2,\ldots$. Thus

$$u_n^*(x,t) = A\cos\left(\frac{n\pi}{\ell}x\right)\left[Ce^{\left(\frac{n\pi}{\ell}t\right)} + De^{\left(-\frac{n\pi}{\ell}t\right)}\right]$$

and we have found an infinite number of functions

$$u_n(x,t) = A_n\cos\left(\frac{n\pi}{\ell}x\right)\left[C_n e^{\left(\frac{n\pi}{\ell}t\right)} + D_n e^{\left(-\frac{n\pi}{\ell}t\right)}\right], \qquad n = 1,2,\ldots$$

each one satisfy the partial differential equation (1.13), the Neumann condition given by equation (1.11) and the nonlocal condition given by equation (1.12). The desired solution will be a certain sum of these simple functions and takes the form:

$$u(x,t) = \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{\ell}x\right)\left[A_n e^{\left(\frac{n\pi}{\ell}t\right)} + B_n e^{\left(-\frac{n\pi}{\ell}t\right)}\right]$$

To satisfy the initial conditions given by equations (1.9)-(1.10), one must have:

$$\sum_{n=1}^{\infty}\left(A_n+B_n\right)\cos\left(\frac{n\pi}{\ell}x\right)=r(x)$$

and

$$\sum_{n=1}^{\infty}\left(A_n-B_n\right)\frac{n\pi}{\ell}\cos\left(\frac{n\pi}{\ell}x\right)=p(x)$$

By using the orthogonality condition given by equation (1.7) and the above two equations, one can obtain :

$$A_n+B_n=\frac{2}{\ell}\int_0^{\ell} r(x)\cos\left(\frac{n\pi}{\ell}x\right)dx, \qquad n=1,2,\ldots$$

and

$$A_n-B_n=\frac{2}{n\pi}\int_0^{\ell} p(x)\cos\left(\frac{n\pi}{\ell}x\right)dx, \qquad n=1,2,\ldots$$

Thus

$$A_n=\int_0^{\ell}\left[\frac{1}{\ell}r(x)+\frac{1}{n\pi}p(x)\right]\cos\left(\frac{n\pi}{\ell}x\right)dx \qquad n=1,2,\ldots$$

and

$$B_n=\int_0^{\ell}\left[\frac{1}{\ell}r(x)-\frac{1}{n\pi}p(x)\right]\cos\left(\frac{n\pi}{\ell}x\right)dx, \qquad n=1,2,\ldots$$

Hence

$$u(x,t)=\sum_{n=1}^{\infty}\left[\cos\left(\frac{n\pi}{\ell}x\right)\left(A_n e^{\left(\frac{n\pi}{\ell}t\right)}+B_n e^{-\left(\frac{n\pi}{\ell}t\right)}\right)\right]$$

is the solution of the nonlocal problem given by equations (1.13), (1.9)-(1.12)

To illustrate this method, consider the following example

Example (1.3)

Consider the one-dimensional homogeneous Laplace equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad 0 < x < \pi, \quad 0 \le t \le 1$$

together with the initial conditions:

$$u(x,0) = \cos(x), \qquad 0 \le x \le \pi$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = -\cos(x), \qquad 0 \le x \le \pi$$

the homogenous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le 1$$

and the homogeneous nonlocal condition:

$$\int_0^1 u(x,t)dx = 0, \qquad 0 \le t \le 1$$

In this case, $r(x) = \cos(x)$ and $p(x) = -\cos(x)$

therefore

$$A_n = \int_0^\pi \left[\frac{1}{\pi} r(x) + \frac{1}{n\pi} p(x)\right] \cos(nx) dx$$

$$= \int_0^\pi \left[\frac{1}{\pi} \cos(x) - \frac{1}{n\pi} \cos(x)\right] \cos(nx) dx$$

$$= 0, \qquad n = 1,2,\ldots$$

and

$$B_n = \int_0^\pi \left[\frac{1}{\pi} r(x) - \frac{1}{n\pi} p(x)\right] \cos(nx) dx$$

$$= \int_0^\pi \left[\frac{1}{\pi} \cos(x) + \frac{1}{n\pi} \cos(x)\right] \cos(nx) dx$$

$$= \begin{cases} 1 & n = 1 \\ 0 & n \ne 1 \end{cases}$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} \cos(nx) \left[A_n e^{nt} + B_n e^{-nt}\right]$$

$$= \cos(x)e^{-t}$$

is the exact solution of the above nonlocal problem.

1.3 The Eigenfunction Expansion Method for Solving Partial Differential Equations with Nonlocal Conditions:-

It is known that the eigenfunction expansion method is a technique for finding the solution of special types of linear partial differential equations as an infinite sum of eigenfunctions. These eigenfunctions are found by solving what is known as an eigenvalue problem corresponding to the original problem, [12, pp64].

It is used to solve the initial-boundary-value problem and applied to problems where the partial differential equation is linear and nonhomogeneous and the boundary conditions are linear and homogeneous, [12, pp70].

In this section we use the eigenfunction expansion method for solving special types of linear nonhomogeneous partial differential equations together with nonlocal conditions.

To do this, consider the initial-boundary value problem that consists of the nonhomogenous one-dimensional heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \le x \le \ell, \quad t > 0 \qquad (1.14)$$

together with the initial condition given by equation (1.2), the Neumann condition given by equation (1.3) and the nonlocal condition given by equation (1.4).

It is assumed that the compatibility conditions for this problem are satisfied. The basic idea of the eigenfunction expansion method is to decompose the known function f into the form

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) X_n(x)$$

where X_n are the eigenfunctions of the Sturm-Liouville system we get when solving the associated homogeneous problem to the original- problem by using the separation of variables to get:

$$X_n(x) = \cos\left(\frac{n\pi}{\ell}x\right), \qquad n = 1,2,\ldots.$$

Therefore

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\cos\left(\frac{n\pi}{\ell}x\right) \qquad n = 1,2\ldots.$$

By using the orthogonality condition given by equation (1.7), the above equation one can have:

$$f_n(t) = \frac{2}{\ell}\int_0^{\ell} f(x,t)\cos\left(\frac{n\pi}{\ell}x\right)dx \qquad n = 1,2,\ldots.$$

Hence by substituting

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\cos\left(\frac{n\pi}{\ell}x\right)$$

and

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\cos\left(\frac{n\pi}{\ell}x\right)$$

into equation (1.14) one can get the following equation

$$\sum_{n=}^{\infty}\left[T'(t_n) + \left(\frac{\gamma n\pi}{\ell}\right)^2 T_n(t) - f_n(t)\right]\cos\left(\frac{n\pi}{\ell}x\right) = 0$$

On the other hand, it is easy to check that this solution is automatically satisfy the Neumann condition given by equation (1.3) and the nonlocal condition (1.4). In this case the initial condition (1.2) become:

$$\sum_{n=1}^{\infty} T_n(0)\cos\left(\frac{n\pi}{\ell}x\right) = r(x)$$

By using the orthogonality condition given by equation (1.7), the above two equations become:

$$T_n'(t) + \left(\frac{\gamma n\pi}{\ell}\right)^2 T_n(t) = f_n(t), \qquad n = 1,2,\ldots$$

$$T_n(0) = \frac{2}{\ell}\int_0^{\ell} r(x)\cos\left(\frac{n\pi}{\ell}x\right)dx, \qquad n = 1,2,\dots$$

which has the solution

$$T_n(t) = e^{-\left(\frac{\gamma n\pi}{\ell}\right)^2 t} T_n(0) + \int_0^t e^{-\left(\frac{\gamma n\pi}{\ell}\right)^2 (t-\tau)} f_n(\tau)d\tau, \qquad n = 1,2,\dots.$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty}\left[e^{-\left(\frac{\gamma n\pi}{\ell}\right)^2 t} T_n(0) + \int_0^t e^{-\left(\frac{\gamma n\pi}{\ell}\right)^2 (t-\tau)} f_n(\tau)d\tau\right]\cos\left(\frac{n\pi}{\ell}x\right) \tag{1.15}$$

is the desired solution of the original problem.

To illustrate this method, consider the following example:

Example (1.4)

Consider the nonhomogeneous one-dimensional heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + t\cos(2\pi x)$$

together with the initial condition:

$$u(x,0) = \cos(3\pi x), \qquad 0 \le x \le 1$$

The homogeneous Neumann conditions:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad t \ge 0$$

and the homogenous nonlocal condition:

$$\int_0^1 u(x,t)dx = 0, \qquad t \ge 0$$

Then by using the eigenfunction expansion method, the solution of this nonlocal problem takes the form:

$$u(x,t) = \sum_{n=1}^{\infty}\left[e^{-(n\pi)^2 t} T_n(0) + \int_0^t e^{-(n\pi)^2 (t-\tau)} f_n(\tau)d\tau\right]\cos(n\pi x), \qquad 0 \le t \le T$$

where

$$f_n(t) = 2t\int_0^t \cos(2\pi x)\cos(n\pi x)dx = \begin{cases} t, & n = 2 \\ 0, & n \neq 2 \end{cases}$$

and

$$T_n(0) = 2\int_0^1 \cos(3\pi x)\cos(n\pi x)dx = \begin{cases} 1, & n = 3 \\ 0, & n \neq 3 \end{cases}$$

Therefore

$$u(x,t) = e^{-(3\pi)^2 t}\cos(3\pi x) + \left[\int_0^t \tau e^{-(2\pi)^2(t-\tau)}d\tau\right]\cos(2\pi x)$$

$$= e^{-9\pi^2 t}\cos(3\pi x) + e^{-4\pi^2 t}\cos(2\pi x)\left[\frac{1}{4\pi^2}te^{4\pi^2 t} - \frac{1}{16\pi^4}e^{4\pi^2 t} + \frac{1}{16\pi^4}\right]$$

$$= e^{-9\pi^2 t}\cos(3\pi x) + \frac{1}{4\pi^2}t\cos(2\pi x) - \frac{1}{16\pi^4}\cos(2\pi x) + \frac{1}{16\pi^4}e^{-4\pi^2 t}\cos(2\pi x)$$

$$= e^{-9\pi^2 t}\cos(3\pi x) + \frac{1}{4\pi^2}\left[t - \frac{1}{4\pi^4} + \frac{1}{4\pi^4}e^{-4\pi^2 t}\right]\cos(2\pi x)$$

is the solution of the above nonlocal problem.

Chapter Two
The Variational Iteration Method for Solving Partial Differential Equations with Nonlocal Conditions

2.1 Introduction

The variational iteration method, which is a modified general Lagrange multiplier has been shown to solve effectively, easily, and accurately a large class of linear and nonlinear problems with approximation converging rapidly to accurate solutions, [20].

[15] recently introduced variational iteration method which gives rapidly convergent successive approximations of the exact solution if such a solution exists. This method has proved successful in deriving analytical solutions of linear and nonlinear differential equations. In their paper, Jafari, Hossinzadeh and Salehpoor solved Gas Dynamics Equation using variational iteration method, [20]. Variational iteration method was used to solve some types of Volterra's integro-differential equations[1].

The aim of this chapter is to use the variational iteration method for solving some types of the one-dimensional linear nonhomogeneous partial differential equations.

This chapter consists of six sections.

In section two, we give the iteration formula that described the variatianal iteration method.

In section three, we solve the one-dimensional nonhomogeneous heat equation with nonlocal conditions via the variational iteration method. The convergence of this method is discussed.

In section four, we give the solutions of one dimensional nonhomogeneous heat equation with nonhomogeneous nonlocal condition via the variational iteration method and its convergence.

In sections five and six e, we use the variational iteration method to solve the one dimensional wave equation with homogenous and nonhomogeneous nonlocal conditions respectively.

2.2 The Variational Iteration Method, [27]

To illustrate the basic idea of this technique, we consider the following general nonlinear equation:

$$L[u(x,t)]+N[u(x,t)]=g(x,t) \qquad (2.1)$$

where L is a linear operator, N is a nonlinear operator, g is a given function of x and t and u is the unknown function that must be determined for $t \geq t_0$.

The basic character of the variational iteration method is to construct a correction function for equation (2.1) which reads

$$u_{i+1}(x,t)=u_i(x,t)+\int_{t_0}^{t} \lambda[Lu_i(x,s)+N\tilde{u}_i(x,s)-g(x,s)]ds, \qquad (2.2)$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory, u_i is the ith approximate solution, and $\tilde{u}_i$ denotes a restricted variation, i.e., $\delta\tilde{u}_i = 0$, [27]. Then we substitute λ into the following iteration formula:

$$u_{i+1}(x,t)=u_i(x,t)+\int_{0}^{t} \lambda[Lu_i(x,s)+Nu_i(x,s)-g(x,s)]ds, \qquad i=0,1,\cdots \qquad (2.3)$$

where u_0 is the initial approximation to the solution of equation (2.1).

2.3 The Variational Iteration Method for Solving Heat Equation with Homogeneous Nonlocal Conditions

In this section, we use the variatianal iteration method for solving the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t}=\gamma^2\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), \qquad 0\leq x\leq \ell, \qquad 0<t\leq T \qquad (2.4)$$

together with the initial condition

$$u(x,0) = r(x), \qquad 0 \le x \le \ell \tag{2.5}$$

the homogeneous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le T \tag{2.6}$$

and the homogeneous nonlocal condition

$$\int_0^{\ell} u(x,t)dx = 0, \qquad 0 \le t \le T \tag{2.7}$$

where γ is a nonzero constant, f is a known function of x and t, and r is a given function of x that must satisfy the following compatibility conditions

$$r'(0) = \int_0^{\ell} r(x)dx = 0$$

In order to use the variatianal iteration method to solve such type of nonlocal problems one must rewrite equation (2.4) as

$$L(u(x,t)) + N(u(x,t)) = f(x,t)$$

where $L = \frac{\partial}{\partial t}$ and $N = -\gamma^2 \frac{\partial^2}{\partial x^2}$.

Therefore equation (2.2) becomes:

$$u_{i+1}(x,t) = u_i(x,t) + \int_0^t \lambda(s,t)\left[\frac{\partial u_i(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds \tag{2.8}$$

where λ is the Lagrange multiplier. Thus by taking the variation of the above equation one can have:

$$\delta u_{i+1}(x,t) = \delta u_i(x,t) + \delta\int_0^t \lambda(s,t)\left[\frac{\partial u_i(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds$$

Then by using the integration by parts one can obtain

$$\delta u_{i+1}(x,t) = \delta u_i(x,t) + \lambda(s)\delta u_i(x,s)\big|_{s=t} - \delta\int_0^t \lambda'(s)u_i(x,s)ds + \delta\int_0^t\left[-\gamma^2\lambda(s)\frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - \lambda(s)f(x,s)\right]ds$$

$$= \delta u_i(x,t)[1+\lambda(s)]\big|_{s=t} - \int_0^t \lambda'(s)\delta u_i(x,s)ds + \delta\int_0^t\left[-\gamma^2\lambda(s)\frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - \lambda(s)f(x,s)\right]ds$$

The stationary conditions will be:

$$\left[1+\lambda(s)\right]_{s=t} = 0 \qquad (2.9)$$

and

$$\lambda'(s) = 0, \qquad 0 \le s \le t$$

The solution of the above differential equation is

$$\lambda(t) = A$$

where A is an arbitrary constant. To find the value of A, substitute λ into equation (2.9) to get:

$$1 + A\big|_{s=t} = 0$$

Therefore

$$\lambda(s) = A = -1$$

By substituting $\lambda = -1$ into equation (2.3) one can obtain the following iteration formula:

$$u_{i+1}(x,t) = u_i(x,t) - \int_0^t \left[\frac{\partial u_i(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 u_i(x,s)}{\partial x^2} - f(x,s) \right] ds \qquad (2.10)$$

For simplicity, let $u_0(x,t) = r(x)$, then

$$u_0(x,0) = r(x), \qquad 0 \le x \le \ell$$

$$\left.\frac{\partial u_0(x,t)}{\partial x}\right|_{x=0} = r'(x)\big|_{x=0} = r'(0) = 0, \qquad 0 \le t \le T$$

and

$$\int_0^\ell u_0(x,t)dx = \int_0^\ell r(x)dx = 0, \qquad 0 \le t \le T$$

Therefore $u_0(x,t) = r(x)$ is the initial approximation of the solution of equation (2.4) that satisfy the initial condition, the Neumann condition and the nonlocal condition given by equation (2.5)-(2.7).

Then by setting $i = 0$ into equation (2.10) one can have:

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,s)}{\partial s} - \gamma^2 \frac{\partial^2 u_0(x,s)}{\partial x^2} - f(x,s) \right] ds$$

$$= r(x) - \int_0^t \left[-\gamma^2 r''(x) - f(x,s) \right] ds$$

$$= r(x) + \gamma^2 r''(x) t + \int_0^t f(x,s) ds$$

By setting $i = 1$ in equation (2.10) and by substituting $u_1(x,t)$ in it, one can get $u_2(x,t)$. By continuing in this manner one can get:

$$u(x,t) = \lim_{i \to \infty} u_i(x,t)$$

is the solution of the nonlocal problem given by equations (2.4)-(2.7).

Next, to show the convergence of the variational iteration method for solving the nonlocal problem given by equations (2.4)-(2.7), we gives the following theorem. This theorem is a special case of theorem (1) that appeared in [2, pp17].

Theorem (2.1):

Let $u \in C^2(\Omega)$ be the exact solution of the nonlocal problem given by equations (2.4)-(2.7) and $u_i \in C^2(\Omega)$, where $\Omega = \{(x,t) | 0 \le x \le \ell, 0 \le t \le T\}$, be the obtained solution of the sequence defined by equation (2.10) with $u_0(x,t) = r(x)$. If

$$E_i(x,t) = u_i(x,t) - u(x,t) \qquad i = 0,1,\ldots$$

and

$$\left\| \frac{\partial^2 E_i(x,t)}{\partial x^2} \right\|_2 \le \| E_i(x,t) \|_2$$

where $\| E_i(x,t) \|_2 = \int_0^T \int_0^\ell | E_i(x,t) |^2 dx dt$,

then the sequence defined by equation (2.10) converges to u.

Proof

Since u is the exact solution of equation (2.4), then

$$u_{i+1}(x,t)-u(x,t)=u_i(x,t)-u(x,t)-\int_0^t\left[\frac{\partial u_i(x,s)}{\partial s}-\gamma^2\frac{\partial^2 u_i(x,s)}{\partial x^2}+f(x,s)\right]ds$$

$$+\int_0^t\left[\frac{\partial u(x,s)}{\partial s}-\gamma^2\frac{\partial^2 u(x,s)}{\partial x^2}-f(x,s)\right]ds$$

$$=u_i(x,t)-u(x,t)-\int_0^t\left[\frac{\partial}{\partial s}\{u_i(x,s)-u(x,s)\}-\gamma^2\frac{\partial^2}{\partial x^2}\{u_i(x,s)-u(x,s)\}\right]ds$$

But

$$E_i(x,t)=u_i(x,t)-u(x,t) \qquad i=0,1,\ldots.$$

Therefore

$$E_i(x,s)=u_i(x,s)-u(x,s) \qquad i=0,1,\ldots.$$

Hence

$$E_{i+1}(x,s)=E_i(x,t)-\int_0^t\left[\frac{\partial E_i(x,s)}{\partial s}-\gamma^2\frac{\partial^2 E_i(x,s)}{\partial x^2}\right]ds$$

$$=E_i(x,t)-\left[E_i(x,t)-E_i(x,0)\right]+\gamma^2\int_0^t\left[\frac{\partial^2 E_i(x,s)}{\partial x^2}\right]ds$$

$$=E_i(x,0)+\gamma^2\int_0^t\left[\frac{\partial^2 E_i(x,s)}{\partial x^2}\right]ds$$

But

$$E_i(x,0)=u_i(x,0)-u(x,0) \qquad i=0,1,\ldots$$

then

$$E_0(x,0)=u_0(x,0)-u(x,0)=r(x)-r(x)=0$$

and from equation (2.10), one can have

$$u_{i+1}(x,0)=u_i(x,0), \qquad i=0,1,\ldots$$

therefore

$$u_i(x,0)=u_0(x,0), \qquad i=0,1,\ldots.$$

Hence

$$E_i(x,0) = u_i(x,0) - u(x,0) = 0, \qquad i = 0,1,\ldots$$

and this implies that

$$E_{i+1}(x,t) = \gamma^2 \int_0^t \frac{\partial^2 E_i(x,s)}{\partial x^2} ds$$

Thus, according to norm properties, we have

$$\|E_{i+1}(x,t)\|_2 = \gamma^2 \left\| \int_0^t \frac{\partial^2 E_i(x,s)}{\partial x^2} ds \right\|_2 \le \gamma^2 \int_0^t \left\| \frac{\partial^2 E_i(x,s)}{\partial x^2} \right\|_2 ds$$

hence

$$\|E_{i+1}(x,t)\|_2 \le \gamma^2 \int_0^t \|E_i(x,s)\|_2 \, ds$$

For $i = 0$ one can have:

$$\begin{aligned}
\|E_1(x,t)\|_2 &\le \gamma^2 \int_0^t (\|E_0(x,s)\|_2) ds \\
&\le \gamma^2 \max_{(x,s)\in\Omega} \|E_0(x,s)\|_2 \int_0^t ds \\
&= \gamma^2 \max_{(x,s)\in\Omega} \|E_0(x,t)\|_2 \, t
\end{aligned}$$

For $i = 1$,

$$\begin{aligned}
\|E_2(x,t)\|_2 &\le \gamma^2 \int_0^t (\|E_1(x,s)\|_2) ds \\
&\le \gamma^4 \int_0^t (\max_{(x,s)\in\Omega} \|E_0(x,s)\|_2 \, s) ds \\
&= \gamma^4 \max_{(x,s)\in\Omega} \|E_0(x,t)\|_2 \frac{t^2}{2!}
\end{aligned}$$

By continuing in this manner one can have:

$$\|E_i(x,t)\|_2 \le \gamma^{2i} \max_{(x,s)\in\Omega} \|E_0(x,t)\|_2 \frac{t^i}{i!}$$

By letting $i \to \infty$ one can obtain:

$$\|E_i(x,s)\|_2 \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty$$

and this implies that

$$E_i(x,s) \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty$$

Therefore

$$\lim_{i \to \infty} E_i(x,t) = 0$$

which gives

$$\lim_{i \to \infty} u_i(x,t) = u(x,t)$$

To illustrate this method, consider the following example

Example (2.1):

Consider the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + (t^3 + 3t^2)\cos(x), \qquad 0 \le x \le \pi, \qquad 0 < t \le 1$$

together with the initial condition

$$u(x,0) = 0 \qquad 0 \le x \le \pi,$$

the homogenous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 < t \le 1.$$

and the homogenous nonlocal condition

$$\int_0^{\pi} u(x,t)dx = 0, \qquad 0 \le t \le 1$$

It is easy to check that the compatibility conditions are satisfied.

To solve this example by using the variational iteration method, we consider the iteration formula given by equation (2.10):

$$u_{i+1}(x,t) = u_i(x,t) - \int_0^t \left[\frac{\partial u_i(x,s)}{\partial s} - \frac{\partial^2 u_i(x,s)}{\partial x^2} - f(x,s) \right] ds$$

$$= u_i(x,t) - \int_0^t \left[\frac{\partial u_i(x,s)}{\partial s} - \frac{\partial^2 u_i(x,s)}{\partial x^2} - (s^3 + 3s^2)\cos(x) \right] ds$$

Let $u_0(x,t) = r(x) = 0$, then

$$u_1(x,t) = \int_0^t \left[(s^3 + 3s^2)\cos(x) \right] ds$$

$$= \left(t^3 + \frac{t^4}{4} \right)\cos(x),$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,s)}{\partial s} - \frac{\partial^2 u_1(x,s)}{\partial x^2} - f(x,s) \right] ds$$

$$= \left(t^3 + \frac{t^4}{4} \right)\cos(x) - \int_0^t \left[\left(3s^2 + s^3\right)\cos(x) + \left(s^3 + \frac{s^4}{4} \right)\cos(x) - \left(s^3 + 3s^2\right)\cos(x) \right] ds$$

$$= \left(t^3 - \frac{t^5}{20} \right)\cos(x),$$

and

$$u_3(x,t) = u_2(x,t) - \int_0^t \left[\frac{\partial u_2(x,s)}{\partial s} - \frac{\partial^2 u_2(x,s)}{\partial x^2} - \left(s^3 + 3s^2\right)\cos(x) \right] ds$$

$$= \left(t^3 - \frac{t^5}{20} \right)\cos(x) - \int_0^t \left[\left(3s^2 - \frac{s^4}{4} \right)\cos(x) + \left(s^3 - \frac{s^5}{20} \right)\cos(x) - \left(s^3 + 3s^2\right)\cos(x) \right] ds$$

$$= \left(t^3 + \frac{t^6}{120} \right)\cos(x).$$

By continuing in this manner one can get:

$$u_i(x,t) = \left[t^3 + (-1)^{i+1} \frac{1}{(4)(5)\cdots(3+i)} t^{3+i} \right]\cos(x)$$

$$= \left[t^3 + (-1)^{i+1} \frac{6}{(3+i)!} t^{3+i} \right]\cos(x), \qquad i = 1,2,\ldots.$$

Then

$$u(x,t) = \lim_{i\to\infty} u_i(x,t) = t^3\cos(x) + 6\lim_{i\to\infty}(-1)^i \frac{t^{3+i}}{(3+i)!}\cos(x)$$

$$= t^3\cos(x)$$

is the exact solution of the above nonlocal problem.

Example (2.2):

Consider the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+x^3-\frac{3}{4}x^2-6tx+\frac{3}{2}t+6x-\frac{3}{2}, \qquad 0\le x\le 1, \quad 0<t\le 1$$

together with the initial condition

$$u(x,0)=-x^3+\frac{3}{4}x^2, \qquad 0\le x\le 1$$

the homogenous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}=0, \qquad 0<t\le 1.$$

and the homogenous nonlocal condition

$$\int_0^1 u(x,t)dx=0, \qquad 0\le t\le 1$$

It is easy to check that the compatibility conditions are satisfied.

To solve This example by using the variational iteration method, we consider the iteration formula given by equation (2.10)

$$u_{i+1}(x,t)=u_i(x,t)-\int_0^t\left[\frac{\partial u_i(x,s)}{\partial s}-\frac{\partial^2 u_i(x,s)}{\partial x^2}-f(x,s)\right]ds$$

$$=u_i(x,t)-\int_0^t\left[\frac{\partial u_i(x,s)}{\partial s}-\frac{\partial^2 u_i(x,s)}{\partial x^2}-\left[x^3-\frac{3}{4}x^2-6sx+\frac{3}{2}s+6x-\frac{3}{2}\right]\right]ds$$

Let $u_0(x,t)=r(x)=-x^3+\frac{3}{4}x^2$

then

$$u_1(x,t)=-x^3+\frac{3}{4}x^2-\int_0^t\left[-x^3+\frac{3}{4}x^2+6sx-\frac{3}{2}s\right]ds$$

$$=-x^3+\frac{3}{4}x^2+tx^3-\frac{3}{4}tx^2-3t^2x+\frac{3}{4}t^2$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,s)}{\partial t} - \frac{\partial^2 u_1(x,s)}{\partial x^2} - f(x,s) \right] ds$$

$$= -x^3 + \frac{3}{4}x^2 + tx^3 - \frac{3}{4}tx^2 - 3t^2 x + \frac{3}{4}t^2 - \int_0^t \left[-6sx + \frac{3}{2}s \right] ds$$

$$= -x^3 + \frac{3}{4}x^2 + tx^3 - \frac{3}{4}tx^2$$

$$u_3(x,t) = u_2(x,t) - \int_0^t \left[\frac{\partial u_2(x,s)}{\partial s} - \frac{\partial^2 u_2(x,s)}{\partial x^2} - f(x,s) \right] ds$$

$$= -x^3 + \frac{3}{4}x^2 + tx^3 - \frac{3}{4}tx^2$$

Therefore

$$u_i(x,t) = u_2(x,t) = -x^3 + \frac{3}{4}x^2 + tx^3 - \frac{3}{4}tx^2, \qquad i = 1,2,\ldots$$

and this implies that

$$u(x,t) = \lim_{i \to \infty} u_i(x,t) = -x^3 + \frac{3}{4}x^2 + tx^3 - \frac{3}{4}tx^2$$

is the exact solution of the above nonlocal problem.

2.4 The Variational Iteration Method for Solving Heat Equation with Nonhomogeneous Nonlocal condition:

In this section we use the variatianal iteration method foe solving the one dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \le x \le \ell, \qquad 0 < t \le T \tag{2.11}$$

together with the initial condition:

$$u(x,0) = r(x) \qquad 0 \le x \le \ell, \tag{2.12}$$

the nonhomogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \alpha(t), \qquad 0 \le t \le T \tag{2.13}$$

and the nonhomogeneous nonlocal condition:

$$\int_0^{\ell} u(x,t)dx = \beta(t), \qquad 0 \le t \le T \tag{2.14}$$

where γ is a nonzero constant, f is a known function of x and t, and r, α, β are given functions that must satisfy the following compatibility conditions

$$r'(0) = \alpha(0)$$

and

$$\int_0^{\ell} r(x)dx = \beta(0).$$

For the sake of simplicity transform the nonlocal problem given by equations (2.11)-(2.14) to an equivalent one, but with homogeneous Neumann condition and homogeneous nonlocal condition. To do this, we use the transformation that appeared in [12].

$$v(x,t) = u(x,t) - z(x,t) \tag{2.15}$$

where $z(x,t) = \alpha(t)\left[x - \frac{\ell}{2}\right] + \frac{\beta(t)}{\ell}$.

Therefore

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial v(x,t)}{\partial t} + \frac{\partial z(x,t)}{\partial t}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 v(x,t)}{\partial x^2}.$$

Then the nonlocal problem given by equations (2.11)-(2.14) reduces to the nonlocal problem that consists of the nonhomogeneous one-dimensional heat equation:

$$\frac{\partial v(x,t)}{\partial t} = \gamma^2 \frac{\partial^2 v(x,t)}{\partial x^2} + g(x,t), \qquad 0 \le x \le \ell, \qquad 0 < t \le T \tag{2.16}$$

together with the initial condition

$$v(x,0) = m(x), \qquad 0 \le x \le \ell \tag{2.17}$$

the homogeneous Neumann condition

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 < t \le T \tag{2.18}$$

and the homogenous nonlocal condition:

$$\int_0^{\ell} v(x,t)dx = 0, \qquad 0 < t \le T \tag{2.19}$$

$$m(x) = r(x) - z(x,0)$$

$$g(x,t) = f(x,t) - \frac{\partial z(x,t)}{\partial t}$$

In this case, equation (2.10) takes the form:

$$v_{i+1}(x,t) = v_i(x,t) - \int_0^t \left[\frac{\partial v_i(x,s)}{\partial s} - \frac{\partial^2 v_i(x,s)}{\partial x^2} - g(x,s) \right] ds, \qquad i = 0,1,\ldots. \tag{2.20}$$

For simplicity, let $v_0(x,t) = m(x)$, then

$$v_0(x,0) = m(x), \qquad 0 \le x \le \ell$$

$$\left.\frac{\partial v_0(x,t)}{\partial x}\right|_{x=0} = m'(x)\big|_{x=0} = m'(0) = r'(0) - \left.\frac{\partial z(x,0)}{\partial x}\right|_{x=0}$$

$$= r'(0) - \alpha(0)$$

$$= 0$$

and

$$\int_0^{\ell} v_0(x,t)dx = \int_0^{\ell} m(x)dx = \int_0^{\ell} r(x)dx - \int_0^{\ell} z(x,0)dx$$

$$= \beta(0) - \int_0^{\ell} \left\{ \alpha(0)\left[x - \frac{\ell}{2} \right] + \frac{r(0)}{\ell} \right\} dx$$

$$= \beta(0) - \frac{\ell^2}{2}\alpha(0) + \frac{\ell^2}{2}\alpha(0) - \beta(0)$$

$$= 0$$

Therefore $v_0(x,t) = m(x)$ is the initial approximation that satisfy the initial condition, the Neumann condition and the nonlocal condition given by equation (2.16)-(2.19).

By setting $i = 0$ into equation (2.20) one can have

$$v_1(x,t) = m(x) + m''(x)t + \int_0^t g(x,s)ds$$

By setting $i = 1$ into equation (2.20) one can have

$$v_2(x,t) = m(x) + m''(x)t + \int_0^t g(x,s)ds - \int_0^t \left[m'''(x)\frac{t^2}{2} + \int_0^t \int_0^s \frac{\partial^2 g(x,r)}{\partial x^2} dr d\tau \right] ds$$

By continuing in this manner one can get:

$$v(x,t) = \lim_{i \to \infty} v_i(x,t)$$

is the solution of the nonlocal problem given by (2.16)-(2.19).

Thus

$$u(x,t) = v(x,t) + z(x,t)$$

is the solution of the nonlocal problem given by equations (2.11)-(2.14)

To illustrate this method, consider the following example

Example (2.3):

Consider the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 \le x \le 1, \quad 0 < t \le 1$$

together with the initial condition:

$$u(x,0) = x^3, \qquad 0 \le x \le 1$$

the nonhomogenous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 6t, \qquad 0 \le t \le 1$$

and the nonhomogenous nonlocal condition

$$\int_0^1 u(x,t)dx = \frac{1}{4} + 3t, \qquad 0 \le t \le 1$$

In this example, we have

$r(x) = x^3, \quad \alpha(t) = 6t$ and $\beta(t) = \frac{1}{4} + 3t$

It is noted that, the compatibility condition are satisfied. That is

$$r'(0) = 0 = \alpha(0)$$

and

$$\int_0^1 r(x)dx = \int_0^1 x^3 dx = \frac{1}{4} = \beta(0)$$

This example can be solved by using the variational iteration method. To do this, we transform the above nonlocal problem with nonhomogeneous Neumann condition and nonlocal condition to an equivalence one with homogeneous Neumann condition and nonlocal condition by using the transformation given by equation (2.15) to obtain the following one dimensional heat equation

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - 6x, \qquad 0 \le x \le 1, \qquad 0 \le t \le 1$$

together with the initial condition

$$v(x,0) = u(x,0) - z(x,0) = x^3 - \frac{1}{4}, \qquad 0 \le x \le 1$$

the homogenous Neumann condition

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 < t \le 1.$$

and the homogeneous nonlocal condition

$$\int_0^1 v(x,t)dx = 0 \qquad 0 \le t \le 1$$

In this case,

$m(x) = x^3 - \frac{1}{4}$ and $z(x,t) = 6tx + \frac{1}{4}$

Next, consider the iteration formula given by equation(2.20).

By setting $i = 0$, $v_0(x,t) = m(x) = x^3 - \frac{1}{4}$

and $g(x,s) = \frac{-\partial z(x,s)}{\partial s} = -6x$ in it, one can have:

$$v_1(x,t) = x^3 - \frac{1}{4} - \int_0^t [-6x + 6x] ds$$

$$= x^3 - \frac{1}{4} = v_0$$

Then by setting $i = 2$ and $v_1(x,t) = x^3 - \frac{1}{4}$ into equation (2.20) one can have:

$$v_2(x,t) = m(x) + m''(x)t + \int_0^t g(x,s)ds - \int_0^t \left[m'''(x)\frac{t^2}{2} + \int_0^t \int_0^s \frac{\partial^2 g(x,r)}{\partial x^2} drd\tau \right] ds$$

$$= x^3 - \frac{1}{4} - \int_0^t [-6x - 6x] ds$$

$$= x^3 - \frac{1}{4}$$

$$= x^3 - \frac{1}{4}.$$

So

$$v_i(x,t) = v_1(x,t) == x^3 - \frac{1}{4}, \qquad i = 1,2,\cdots.$$

Hence

$$v(x,t) = \lim_{i \to \infty} v_i(x,t) = v_1(x,t) = x^3 - \frac{1}{4}$$

is the exact solution of the above nonlocal problem.

Therefore

$$u(x,t) = v(x,t) + z(x,t)$$

$$= x^3 - \frac{1}{4} + 6tx + \frac{1}{4}$$

$$= x^3 + 6tx$$

is the exact solution of the original nonlocal problem.

Example (2.4):

Consider the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = xe^t - 6x + \sin(x) + t\sin(x), \qquad 0 \le x \le 1, \quad 0 \le t \le 1$$

together with the initial condition

$$u(x,0) = x^3 + x \qquad 0 \le x \le 1$$

the nonhomogeneous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = e^t + t, \qquad 0 \le t \le 1$$

and the nonhomogeneous nonlocal condition

$$\int_0^1 u(x,t)dx = \frac{1}{4} + \frac{1}{2}e^t - t\cos(1) + t \qquad 0 \le t \le 1$$

In this example, we have

$$r(x) = x^3 + x, \qquad 0 \le x \le 1$$

$$\alpha(t) = e^t + t, \qquad 0 \le t \le 1$$

and

$$\beta(t) = \frac{1}{2}e^t - t\cos(1) + t + \frac{1}{4}$$

It is noted that, the compatibility condition are satisfied. That is

$$r'(0) = 1 = \alpha(0)$$

and

$$\int_0^1 r(x)dx = \int_0^1 \left[x^3 + x\right]dx = \frac{1}{4}x^4 + \frac{1}{2}x^2\Big|_0^1 = \frac{3}{4} = \beta(0)$$

This example can be solved by using variational iteration method. To do this, we transform the above problem with nonhomogeneous Neumann and nonlocal conditions to an equivalence one with homogeneous Neumann and nonlocal conditions by using the transform given by equation (2.15) to give the following equation:

$$\frac{\partial v(x,t)}{\partial t}-\frac{\partial^2 v(x,t)}{\partial x^2}=t\sin(x)+\sin(x)-7x-\frac{1}{2}+\cos(1), \qquad 0\le x\le 1, \qquad ,0\le t\le 1$$

together with initial condition

$$v(x,0)=r(x)-z(x,0)$$

$$=x^3-\frac{1}{4}, \qquad 0\le x\le 1$$

The homogeneous Neumann condition

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0}=0, \qquad 0\le t\le 1$$

and the homogeneous nonlocal condition

$$\int_0^1 v(x,t)dx=0, \qquad t\ge 0$$

In this case

$$m(x)=v(x,0)=x^3-\frac{1}{4}, \quad z(x,t)=e^t x+tx+\frac{1}{2}t-t\cos(1)+\frac{1}{4}$$

and $g(x,t)=t\sin(x)+\sin(x)-7x-\frac{1}{2}+\cos(1)$

$$v_1(x,t)=m(x)+m''(x)t+\int_0^t g(x,s)ds$$

$$v_1(x,t)=x^3-\int_0^t\left[\frac{\partial}{\partial s}\left[-6x+t\sin(x)-\cos(x)+7x+\frac{1}{2}-\cos(1)\right]\right]ds$$

$$=x^3-\frac{1}{4}+\frac{t^2}{2}\sin(x)-t\sin(x)-tx+t\cos(1)-\frac{1}{2}t$$

Then by setting $i=2$ and $v_1(x,t)=x^3-\frac{1}{4}+\frac{t^2}{2}\sin(x)-t\sin(x)-tx+t\cos(1)-\frac{1}{2}t$ into equation (2.20) one can have:

$$v_2(x,t)=m(x)+m''(x)t+\int_0^t g(x,s)ds-\int_0^t\left[m'''(x)\frac{t^2}{2}+\int_0^t\int_0^s\frac{\partial^2 g(x,r)}{\partial x^2}drd\tau\right]ds$$

$$v_2(x,t)=x^3-\frac{1}{4}+\frac{t^3}{3!}\sin(x)+t\sin(x)-tx+t\cos(1)-\frac{1}{2}t$$

By continuing in this manner one can get:

$$v(x,t) = \lim_{i\to\infty} v_i(x,t)$$

Hence

$$v(x,t) = x^3 - \frac{1}{4} - tx + t\sin(x) + t\cos(1) - \frac{1}{2}t$$

Thus we obtain

$$u(x,t) = v(x,t) + z(x,t)$$

$$= x^3 + xe^t + t\sin(x)$$

is the exact solution of the above nonlocal problem.

2.5 The Variational Iteration Method for Solving Wave Equation with Homogeneous Nonlocal Condition

In this section, we use the variational iteration method for solving the one-dimensional nonhomogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \le x \le \ell, \qquad 0 \le t \le T \qquad (2.21)$$

together with the initial conditions

$$u(x,0) = r(x), \qquad 0 \le x \le \ell \qquad (2.22)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = p(x), \qquad 0 \le x \le \ell \qquad (2.23)$$

the homogeneous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le T \qquad (2.24)$$

and the homogeneous nonlocal condition

$$\int_0^\ell u(x,t)dx = 0, \qquad 0 \le t \le T \qquad (2.25)$$

where γ is a nonzero constant, f is known function of x, t and r, p are given functions of x that must satisfy the following compatible conditions:

$$r'(0) = p'(0) = 0)$$

and

$$\int_0^{\ell} r(x)dx = \int_0^{\ell} p(x)dx = 0$$

In order to use the variational iteration method to solve such type of nonlocal problem one must rewrite equation (2.21) as

$$L[u(x,t)] + N[u(x,t)] = g(x,t)$$

where $L = \frac{\partial^2}{\partial t^2}$ and $N = -\gamma^2 \frac{\partial^2}{\partial x^2}$.

Therefore equation (2.8) becomes:

$$u_{i+1}(x,t) = u_i(x,t) + \int_0^t \lambda(s)\left[\frac{\partial^2 u_i(x,s)}{\partial s^2} - \gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds \qquad i = 0,1,\ldots \qquad (2.26)$$

where λ is the Lagrange multiplier. Thus by taking the variation of the above equation one can have:

$$\delta u_{i+1}(x,t) = \delta u_i(x,t) + \delta\int_0^t \lambda(s)\left[\frac{\partial^2 u_i(x,s)}{\partial s^2} - \gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} + f(x,s)\right]ds$$

Then by using the integration by parts one can obtain

$$\delta u_{i+1}(x,t) = \delta u_i(x,t) + \delta\left[\lambda(s)\frac{\partial u_i(x,s)}{\partial s}\right]_{s=0}^{s=t} - \delta\int_0^t \lambda'(s)\frac{\partial u_i(x,s)}{\partial s}ds + \delta\int_0^t \lambda(s)\left[-\gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds$$

$$= \delta u_i(x,t) + \lambda(s)\delta\frac{\partial u_i(x,s)}{\partial s}\bigg|_{s=t} - \delta[\lambda'(s)\delta u_i\ (x,s)]_{s=0}^{s=t} + \delta\int_0^t \lambda''(s)u_i(x,s)ds +$$

$$\delta\int_0^t \lambda(s)\left[-\gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds$$

$$= [1-\lambda'(s)]\delta u_i(x,t) + \lambda(s)\delta\frac{\partial u_i(x,s)}{\partial s}\bigg|_{s=t} + \int_0^t \lambda''(s)\delta u_i(x,s)ds + \delta\int_0^t \lambda(s)\left[-\gamma^2 \frac{\partial^2 \tilde{u}_i(x,s)}{\partial x^2} - f(x,s)\right]ds$$

The stationary conditions will be:

$$[1-\lambda'(s)]_{s=t} = 0 \qquad (2.27)$$

$$\lambda(s)\big|_{s=t} = 0 \tag{2.28}$$

and

$$\lambda''(s) = 0 \tag{2.29}$$

The solution of the above differential equation is

$$\lambda(s) = A + Bs$$

where A and B are arbitrary constants. To find the value of A and B substitute λ into equations (2.27)-(2.28) to get:

$$[1 - B]_{s=t} = 0$$

$$A + Bt = 0$$

Therefore $B = 1$ and $A = -t$. Hence

$$\lambda(s,t) = s - t.$$

By substituting λ into equation (2.3) one can obtain the following iteration formula:

$$u_{i+1}(x,t) = u_i(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 u_i(x,s)}{\partial s^2} - \gamma^2 \frac{\partial^2 u_i(x,s)}{\partial x^2} - f(x,s)\right] ds \tag{2.30}$$

For simplicity, let $u_0(x,t) = r(x) + tp(x)$, then

$$u_0(x,0) = r(x), \qquad 0 \le x \le \ell$$

$$\left.\frac{\partial u_0(x,t)}{\partial t}\right|_{t=0} = p(x), \qquad 0 \le x \le \ell$$

$$\left.\frac{\partial u_0(x,t)}{\partial x}\right|_{x=0} = r'(0) + p'(0)t = 0, \qquad 0 \le t \le T$$

and

$$\int_0^\ell u_0(x,t)dx = \int_0^\ell r(x)dx + t\int_0^\ell p(x)dx = 0, \qquad 0 \le t \le T$$

Therefore u_0 satisfy the initial condition, the Neumann condition and the nonlocal condition given by equation (2.22)-(2.25).

Then by setting $i = 0$ into equation (2.30) one can have:

$$u_1(x,t) = r(x) + p(x)t + \int_0^t (s-t)\left[-\gamma^2 r''(x) - \gamma^2 p''(x) - f(x,s)\right]ds \qquad (2.30\text{-a})$$

$$= r(x) + p(x)t - \frac{\gamma^2}{2} r''(x)t^2 - \frac{\gamma^2}{3} p''(x)t^3 + \gamma^2 t^2 r''(x) + \frac{\gamma^2}{2} t^3 p''(x) + t\int_0^t f(x,s)ds - \int_0^t sf(x,s)ds$$

By setting $i = 1$ and substituting $u_1(x,t)$ of equation (2.30-a) in equation (2.30), one can get $u_2(x,t)$. By continuing in this manner one can have:

$$u(x,t) = \lim_{i\to\infty} u_i(x,t)$$

is the solution of the nonlocal problem given by equations (2.21)-(2.25).

To illustrate this method, consider the following example:

Example (2.5):

Consider the one dimensional nonhomogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2e^{-t}\cos(x), \qquad 0 \le x \le \pi, \qquad 0 \le t \le 1$$

together with initial condition:

$$u(x,0) = \cos(x), \qquad 0 \le x \le \pi$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = -\cos(x), \qquad 0 \le x \le \pi$$

the homogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le 1$$

and the homogenous nonlocal condition:

$$\int_0^\pi u(x,t)dx = 0, \qquad 0 \le t \le 1$$

It is easy to check that the compatibility conditions are satisfied.

To solve this example by using the variational iteration method, we consider the iteration formula given by equation (2.30):

$$u_{i+1}(x,t) = u_i(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 u_i(x,s)}{\partial s^2} - \gamma^2 \frac{\partial^2 u_i(x,s)}{\partial x^2} - f(x,s)\right]ds$$

$$= u_i(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 u_i(x,s)}{\partial s^2} - \frac{\partial^2 u_i(x,s)}{\partial x^2} - 2e^{-s}\cos(x)\right]ds$$

Let $u_0(x,t) = r(x) + p(x)t$

$$= \cos(x) - \cos(x)t, \quad 0 \le x \le \pi, \quad 0 \le t \le 1$$

then

$$u_1(x,t) = u_0(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 u_0(x,s)}{\partial s^2} - \frac{\partial^2 u_0(x,s)}{\partial x^2} - 2e^{-s}\cos(x)\right]ds$$

$$= \cos(x) - \cos(x)t + \int_0^t (s-t)\left[-(-\cos(x) + s\cos(x) - 2e^{-s}\cos(x)\right]ds$$

$$= \left[-1 + t - \frac{t^2}{2!} + \frac{t^3}{3!}\right]\cos(x) - 2e^{-t}\cos(x)$$

$$u_2(x,t) = u_1(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 u_1(x,s)}{\partial s^2} - \frac{\partial^2 u_1(x,s)}{\partial x^2} - 2e^{-s}\cos(x)\right]ds$$

$$= \left[\left(-1 + t - \frac{t^2}{2!} + \frac{t^3}{3!}\right) + 2e^{-t}\right]\cos(x) + 2e^{-t}\cos(x) + \int_0^t (s-t)\left[\left(-1 + 2s + 2e^{-s}\right)\cos(x) + \left(-1 + s - \frac{s^2}{2!} + \frac{s^3}{3!} + 2e^{-s}\right)\cos(x) - 2e^{-s}\cos(x)\right]ds$$

$$u_2(x,t) = \left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}\right]\cos(x)$$

and

$$u_3(x,t) = u_2(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 u_2(x,s)}{\partial s^2} - \frac{\partial^2 u_2(x,s)}{\partial x^2} - 2e^{-s}\cos(x)\right]ds$$

$$= \sum_{i=0}^{5}\frac{(-t)^i}{i!}\cos(x) + \int_0^t (s-t)\left[\sum_{i=0}^{3}\frac{(-s)^i}{i!}\cos(x) + \sum_{i=0}^{5}\frac{(-s)^i}{i!}\cos(x) - 2e^{-s}\cos(x)\right]ds$$

$$= -\sum_{i=0}^{7}\frac{(-t)^i}{i!}\cos(x) + 2e^{-t}\cos(x)$$

By continuing in this manner one can get:

$$u_i(x,t) = \begin{cases} \left(-\sum_{i=0}^{2i+1} \frac{(-t)^i}{i!} + 2e^{-t} \right) \cos(x) & i \quad \text{odd integer} \\ \\ \sum_{i=0}^{2i+1} \frac{(-t)^i}{i!} \cos(x) & i \quad \text{even integer} \end{cases}$$

Then

$$u(x,t) = \lim_{i \to \infty} u_i(x,t) = e^{-t} \cos(x)$$

is the exact solution of the above nonlocal problem.

2.6 The Variational Iteration Method for Solving Wave Equation with Nonhomogeneous Nonlocal Condition

In this section, we use the variational iteration method for solving the one-dimensional nonhomogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \le x \le \ell, \qquad 0 \le t \le T \tag{2.31}$$

together with the initial conditions

$$u(x,0) = r(x), \qquad 0 \le x \le \ell \tag{2.32}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = p(x), \qquad 0 \le x \le \ell \tag{2.33}$$

the nonhomogeneous Neumann condition

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \alpha(t), \qquad 0 \le t \le T \tag{2.34}$$

and the nonhomogeneous nonlocal condition

$$\int_0^\ell u(x,t)dx = \beta(t), \qquad 0 \le t \le T \tag{2.35}$$

where γ is a nonzero constant, f a known function of x, t and r, p, α, β are given functions of x that must satisfy the following completely conditions:

$$r'(0) = \alpha(0)$$
$$p'(0) = \alpha'(0)$$

and

$$\int_0^\ell r(x)dx = \beta(0)$$
$$\int_0^\ell p(x)dx = \beta'(0)$$

In order to use the variational iteration method to solve such type of nonlocal problem, we first transform this nonlocal problem into other nonlocal problem, but with homogenous Neumann condition and homogeneous nonlocal condition. To do this we use the transformation given by equation (2.15). then

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2 z(x,t)}{\partial t^2}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 v(x,t)}{\partial x^2}$$

Therefore the nonlocal problem given by equations (2.31)-(2.35) is transformed to the one-dimensional nonhomogeneous wave equation:

$$\frac{\partial^2 v(x,t)}{\partial t^2} = \gamma^2 \frac{\partial^2 v(x,t)}{\partial x^2} + g(x,t), \qquad 0 \le x \le \ell, \quad 0 \le t \le T \tag{2.36}$$

together with the initial conditions

$$v(x,0) = q_1(x), \qquad 0 \le x \le \ell \tag{2.37}$$

$$\left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0} = q_2(x), \qquad 0 \le x \le \ell \tag{2.38}$$

the homogeneous Neumann condition

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le T \tag{2.39}$$

and the homogeneous nonlocal condition

$$\int_0^\ell v(x,t)dx = 0, \qquad 0 \le t \le T \tag{2.40}$$

where $g(x,t) = f(x,t) - \frac{\partial^2 z(x,t)}{\partial t^2}$, $q_1(x) = r(x) - z(x,0)$

and $q_2(x) = p(x) - \left.\frac{\partial z(x,t)}{\partial t}\right|_{t=0}$.

To solve this nonlocal problem by the variatianal iteration method, consider the iteration formula

$$v_{i+1}(x,t) = v_i(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 v_i(x,s)}{\partial s^2} - \gamma^2 \frac{\partial^2 v_i(x,s)}{\partial x^2} - g(x,s)\right]ds \qquad i = 0,1,\ldots \tag{2.41}$$

For simplicity, let $v_0(x,t) = q_1(x) + q_2(x)t$, then

$$v_0(x,0) = q_1(x), \qquad 0 \le x \le \ell$$

$$\left.\frac{\partial v_0(x,t)}{\partial t}\right|_{t=0} = q_2(x), \qquad 0 \le x \le \ell$$

$$\begin{aligned}\left.\frac{\partial v_0(x,t)}{\partial x}\right|_{x=0} &= q_1'(0) + q_2'(0)t = r'(0) - \left.\frac{\partial z}{\partial x}\right|_{x,t=0} + \left[p'(0) - \left.\frac{\partial^2 z(x,t)}{\partial t \partial x}\right|_{x,t=0}\right] \\ &= r'(0) - \alpha(0) + [p'(0) - \alpha'(0)]t \\ &= 0\end{aligned}$$

and

$$\int_0^\ell v_0(x,t)dx = \int_0^\ell [q_1(x) + q_2(x)t]dx$$

$$= \int_0^{\ell} r(x)dx - \int_0^{\ell} z(x,0)dx + t\int_0^{\ell} p(x)dx - t\int_0^{\ell} \left.\frac{\partial z(x,t)}{\partial t}\right|_{t=0} dx$$

$$= \beta(0) - \int\left[\alpha(0)\left(x - \frac{\ell}{2}\right) + \frac{\beta'(0)}{\ell}\right]dx + t\beta'(0) - t\int\left[\alpha'(0)\left(x - \frac{\ell}{2}\right) + \frac{\beta'(0)}{\ell}\right]dx$$

$$= \beta(0) - \beta(0) + t\beta'(0) - t\beta'(0)$$
$$= 0$$

Therefore v_0 satisfy the initial approximation to the solution of the differential equation(2.36) that satisfies the initial conditions, the Neumann condition and the nonlocal condition given by equation (2.37)-(2.40).

Therefore by substituting $i = 0$ into the iterative formula (2.41) one can have:

$$v_1(x,t) = v_0(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 v_0(x,s)}{\partial s^2} - \gamma^2 \frac{\partial^2 v_0(x,s)}{\partial x^2} - g(x,s)\right]ds$$

$$= q_1(x) + q_2(x)t + \int_0^t (s-t)\left[-\gamma^2\left(q_1''(x) + q_2''(x)s\right) - g(x,s)\right]ds$$

$$= q_1(x) + q_2(x)t + \left[-\gamma^2 q_1''(x)\frac{t^2}{2} - \gamma^2 q_2''(x)\frac{t^3}{3} + t^2\gamma^2 q_1''(x) + \gamma^2\frac{t^3}{2}q_2''(x)\right] - \int_0^t (s-t)[g(x,s)]ds$$

$$= q_1(x) + q_2(x)t + \gamma^2\left[q_1''(x)\frac{t^2}{2} + q_2''(x)\frac{t^3}{6}\right] - \int_0^t (s-t)[g(x,s)]ds$$

By setting $i = 1$ in equation (2.41) and by substituting $v_1(x,t)$ in it, one can get $v_2(x,t)$. By continuing in this manner one can get:

$$v(x,t) = \lim_{i\to\infty} v_i(x,t)$$

is the solution of the nonlocal problem given by equations (2.36)-(2.40).

Therefore

$$u(x,t) = v(x,t) + z(x,t)$$

is the solution of the original nonlocal problem.

To illustrate this method, consider the following example:

<u>Example (2.5):</u>

Consider the one dimensional nonhomogeneous wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = e^{-t}x + \frac{1}{(x+1)^2}, \qquad 0 \le x \le 1, \quad 0 \le t \le 1 \qquad (2.42)$$

together with initial condition:

$$u(x,0) = \ln(x+1) + x, \qquad 0 \le x \le 1 \qquad (2.43)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = -x, \qquad 0 \le x \le 1 \qquad (2.44)$$

The nonhomogeneous Neumann condition:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 1 + e^{-t}, \qquad 0 \le t \le 1 \qquad (2.45)$$

and the nonhomogeneous nonlocal condition:

$$\int_0^1 u(x,t)dx = 2\ln(2) - 1 + \frac{1}{2}e^{-t}, \qquad 0 \le t \le 1 \qquad (2.46)$$

In this example, we have

$r(x) = \ln(x+1) + x$, $p(x) = -x$, $\alpha(t) = 1 + e^{-t}$

and $\beta(t) = 2\ln(2) - 1 + \frac{1}{2}e^{-t}$.

Therefore

$$r'(x) = \frac{1}{x+1} + 1, \qquad 0 \le x \le 1$$
$$p'(x) = -1, \qquad 0 \le x \le 1$$
$$\alpha'(x) = -e^{-t}, \qquad 0 \le t \le 1$$

and

$$\beta'(t) = -\frac{1}{2}e^{-t}, \qquad 0 \le t \le 1$$

Hence

$$r'(0) = 2 = \alpha(0)$$
$$p'(0) = -1 = \alpha'(0)$$

$$\int_0^1 r(x)dx = \int_0^1 [\ln(x+1)+x]dx = 2\ln(2) - \frac{1}{2} = \beta(0)$$

and

$$\int_0^1 -x dx = -\frac{1}{2} = \beta'(0)$$

This implies that the compatibility conditions are satisfied.

To solve the example by variational iteration method, we use the transformation:

$$\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial^2 v(x,t)}{\partial x^2} + \frac{1}{(x+1)^2}, \qquad 0 \le x \le 1, \quad 0 \le t \le 1 \tag{2.47}$$

together with initial condition:

$$v(x,0) = \ln(x+1) - x + \frac{3}{2} - 2\ln(2), \qquad 0 \le x \le 1 \tag{2.48}$$

$$\left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0} = 0, \qquad 0 \le x \le 1 \tag{2.49}$$

The homogeneous Neumann condition:

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=0} = 0, \qquad 0 \le t \le 1 \tag{2.50}$$

and the homogenous nonlocal condition:

$$\int_0^1 v(x,t)dx = 0, \qquad 0 \le t \le 1. \tag{2.51}$$

Let $v_0(x,t) = q_1(x) + q_2(x)t = \ln(x+1) - x + \frac{3}{2} - 2\ln(2)$

Therefore by substituting $i = 0$ into the iterative formula (2.41) one can have:

$$v_1(x,t) = v_0(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 v_0(x,s)}{\partial s^2} - \gamma^2 \frac{\partial^2 v_0(x,s)}{\partial x^2} - g(x,s)\right]ds$$

$$= \ln(x+1) - x + \frac{3}{2} - 2\ln(2) + \int_0^t (s-t)\left[\frac{1}{(x+1)^2} - \frac{1}{(x+1)^2}\right]ds$$

$$= \ln(x+1) - x + \frac{3}{2} - 2\ln(2)$$

Therefore

$$v_i(x,t) = v_0(x,t), \qquad i = 1,2,\ldots$$

and this implies that

$$v(x,t) = \lim_{i\to\infty} v_i(x,t) = \lim_{i\to\infty} v_0(x,t) = v_0(x,t)$$

$$= \ln(x+1) - x + \frac{3}{2} - 2\ln(2)$$

is the exact solution of the nonlocal problem given by equations (2.47)-(2.51). Therefore

$$u(x,t) = v(x,t) + z(x,t)$$

$$= \ln(x+1) - x + \frac{3}{2} - 2\ln(2) + x - \frac{3}{2} + e^{-t}x + 2\ln(2)$$

$$= \ln(x+1) + e^{-t}x$$

is the exact solution of the original nonlocal problem given by equations (2.42)-(2.46).

Chapter Three
Real Life Application For Nonlocal Problems Arising in Thermoelasticity

3.1 Introduction

In recent years, problems with integral conditions have received an increasing attention. The physical significance of integral conditions (mean, total flux, total energy, total mass, moments,…) has served as a fundamental reason for the interest carried to this type of problem [25].

The aim of this chapter is to solve the real applications for nonlocal problem arising thermoelasticity and use the separation of the variables, the eigenfunction expansion method and the variational iteration method for solving such types of nonlocal problems.

This chapter consists of five sections

In section two, we give the mathematical modeling of the thermoelasticity problem.

In section three, we use the separation of variables to solve special types of homogenous thermoelasticity problem.

In section four, we use the eigenfunction expansion method to solve special types of nonhomogeneous nonlocal problems.

In section five, we use the variational iteration method to solve a nonlocal problems arising in thermoelasticity.

3.2 The Mathematical Modeling of the thermoelasticity Problem, [6]:

In this section we describe the mathematical modeling for a thermoelasticity rod problem. Let us consider a rod $0 \le x \le 1$, the temperature $v = v(x,t)$ and the transverse displacement $z = z(x,t)$. The thermoelasticity rod problem can be described by the coupled partial differential equations

$$\mu \frac{\partial^2 v(x,t)}{\partial x^2} = k \frac{\partial v(x,t)}{\partial t} + v_0 \beta \frac{\partial^3 z(x,t)}{\partial x^2 \partial t}, \tag{3.1}$$

$$\alpha \frac{\partial^4 z(x,t)}{\partial x^4} = \beta \frac{\partial^2 v(x,t)}{\partial x^2} \tag{3.2}$$

where μ is the thermal conductivity, k is the specific heat at constant strain, α is the flexural rigidity, β is a measure of the cross-coupling between thermal and mechanical efforts, v_0 is a uniform reference temperature.

If we suppose that the initial temperature of the rod is $r(x)$, and the initial displacement is $f(x)$; the ends $x = 0$ and $x = 1$ are clamped. Then

$$v(x,0) = r(x) \tag{3.3}$$

$$z(x,0) = f(x) \tag{3.4}$$

$$z(0,t) = \left. \frac{\partial z(x,t)}{\partial x} \right|_{x=0} = z(1,t) = \left. \frac{\partial z(x,t)}{\partial x} \right|_{x=1} = 0 \tag{3.5}$$

Moreover if we assume that the average temperature in the rod $0 \le x \le 1$ is equal to $g_1(t)$. That is

$$\int_0^1 v(x,t)dx = g_1(t) \tag{3.6}$$

and the difference between the heat exchange of the atmosphere on the end $x = 0$ and the temperature on the end $x = 1$ is equal to $g_2(t)$, then by using Newton's law one can have:

$$\left. \frac{\partial v(x,t)}{\partial x} \right|_{x=0} + v(0,t) - v(1,t) = g_2(t) \tag{3.7}$$

We reformulate the problem given by equation (3.1)-(3.7) into an equivalent form where the coupled partial differential equations (3.1)-(3.2) is reduced to one partial differential equation. To do this we introduce a new unknown function u defined as follows:

$$u(x,t)=\frac{k}{v_0}\left[v(x,t)-v_0(x,t)\right]+\beta\frac{\partial^2 z(x,t)}{\partial x^2} \tag{3.8}$$

where u is the entropy. Then

$$v_0\frac{\partial u(x,t)}{\partial t}=k\frac{\partial v(x,t)}{\partial t}+v_0\beta\frac{\partial^3 z(x,t)}{\partial x^2\partial t} \tag{3.9}$$

$$\mu\frac{\partial^2 v(x,t)}{\partial x^2}=v_0\frac{\partial u(x,t)}{\partial t} \tag{3.10}$$

By using equation (3.2), (3.8)-(3.10), one can get:

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{k}{v_0}\frac{\partial^2 v(x,t)}{\partial x^2}+\beta\frac{\partial^4 z(x,t)}{\partial x^4}$$

$$=\frac{k}{v_0}\frac{v_0}{\mu}\frac{\partial u(x,t)}{\partial t}+\frac{\beta^2}{\alpha}\frac{\partial^2 v(x,t)}{\partial x^2}$$

$$=\frac{k}{\mu}\frac{\partial u(x,t)}{\partial t}+\frac{\beta^2}{\alpha}\frac{v_0}{\mu}\frac{\partial u(x,t)}{\partial t}$$

$$=\left[\frac{k}{\mu}+v_0\frac{\beta^2}{\alpha\mu}\right]\frac{\partial u(x,t)}{\partial t}$$

Therefore, the entropy u is a solution of the heat equation:

$$\mu\frac{\partial^2 u(x,t)}{\partial x^2}=\left[k+v_0\frac{\beta^2}{\alpha}\right]\frac{\partial u(x,t)}{\partial t} \tag{3.11}$$

To deduce the initial condition on the entropy u, we use the conditions given by equations (3.3)-(3.4) to get:

$$u_0(x)=\frac{k}{v_0}\left[r(x)-v_0\right]+\beta f''(x) \tag{3.12}$$

Then

$$u(x,0)=u_0(x) \tag{3.13}$$

To deduce the first boundary condition on the entropy u, we integrate u with respect to x from $x=0$ to $x=1$ to get:

$$\int_0^1 u(x,t)dx = \frac{k}{v_0}\left[\int_0^1 v(x,t)dx - v_0\right] + \beta\left[\left.\frac{\partial z(x,t)}{\partial x}\right|_{x=1} - \left.\frac{\partial z(x,t)}{\partial x}\right|_{x=0}\right]$$

By using equation (3.5)-(3.6) one can have:

$$\int_0^1 u(x,t)dx = \frac{k}{v_0}[g_1(t) - v_0] \qquad (3.14)$$

Let $\theta_1(t) = \frac{k}{v_0}[g_1(t) - v_0]$

$$\int_0^1 u(x,t)dx = \theta_1(t) \qquad (3.15)$$

which is the average entropy. To conclude the second boundary condition, we multiply equation (3.9) by the weight $(1-x)$ and we integrate the result over $[0,1]$ with respect to x to obtain

$$\int_0^1 (1-x)u(x,t)dx = \frac{k}{v_0}\int_0^1 (1-x)v(x,t)dx - k, \qquad (3.16)$$

which is the weight average entropy. Then, instead of searching for a pair of function (v,z), a solution of the problem given by equation (3.1)-(3.7), is made by searching for the function u, solution of problem given by equation (3.10)-(3.13), then the solution will be $v = u + z$.

3.3 The Separation of Variables for Solving A Nonlocal Problem Arising in Thermoelasticity:

In this section we try to use the separation of variables to solve special types of the nonlocal problems arising in thermoelasticity problem. To do this, consider the one-dimensional homogeneous heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 \le x \le 1, \qquad 0 < t \le T \qquad (3.17)$$

together with the initial condition:

$$u(x,0) = u_0(x) \qquad 0 \le x \le 1 \qquad (3.18)$$

and the homogeneous nonlocal conditions:

$$\int_0^1 u(x,t)dx = 0, \qquad 0 \le t \le T \qquad (3.19)$$

and

$$\int_0^1 xu(x,t)dx = 0, \qquad 0 \le t \le T \qquad (3.20)$$

where u_0 is a given function that must satisfy the following compatibility conditions:

$$\int_0^1 u_0(x)dx = 0$$

and

$$\int_0^1 xu_0(x)dx = 0.$$

By using the separation of variables, the solution of the partial differential equation (3.17) can be expressed as:

$$u^*(x,t) = e^{-\lambda^2 t}\left[A\sin(\lambda x) + B\cos(\lambda x)\right] \qquad (3.21)$$

where A, B, λ are arbitrary constants. Then this solution must satisfy the nonlocal conditions given by equations (3.19)-(3.20).

Therefore

$$\int_0^1 e^{-\lambda^2 t}\left[A\sin(\lambda x)+B\cos(\lambda x)\right]dx=0$$

and

$$\int_0^1 xe^{-\lambda^2 t}\left[A\sin(\lambda x)+B\cos(\lambda x)\right]dx=0$$

These equations becomes:

$$A(1-\cos\lambda)+B\sin\lambda=0 \tag{3.22}$$

$$A(\sin\lambda-\lambda\cos\lambda)+B(\lambda\sin\lambda+\cos\lambda-1)=0 \tag{3.23}$$

Assume that $1-\cos\lambda\neq 0$

$$A=-\frac{\sin\lambda}{1-\cos\lambda}B$$

By substituting the above equation into equation (3.23) one can get

$$B\left[-2+\lambda\sin\lambda+2\cos\lambda\right]=0$$

This equation gives $B=0$ and hence $A=0$. But this is contradiction, since the zero solution does not satisfy the initial condition given by equation (3.18). Therefore $1=\cos\lambda$ and this implies that

$$\lambda=\mp 2n\pi,\qquad n=1,2,\ldots.$$

Thus,

$$u_n^{|*}(x,t)=Be^{-(2n\pi)^2 t}\cos(2n\pi x),\qquad n=1,2,\ldots\qquad 0\le x\le 1,\qquad 0\le t\le T$$

and we have found an infinite number of functions:

$$u_n(x,t)=B_n e^{-(2n\pi)^2 t}\cos(2n\pi x),\qquad n=1,2\ldots$$

each one satisfy the partial differential equation (3.17) and the nonlocal conditions (3.19)-(3.20). The desired solution will be a certain sum of these simple functions and takes the form:

$$u(x,t)=\sum_{n=1}^{\infty}B_n e^{-(2n\pi)^2 t}\cos(2n\pi x) \tag{3.24}$$

With the initial condition (3.18), one can used the orthogonality condition given by (1.7) to get

$$B_n = 2\int_0^1 r(x)\cos(2n\pi x)dx, \qquad n = 1,2\ldots.$$

To illustrate this method, consider the following example

Example (3.1)

Consider the one-dimensional homogenous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 \le x \le 1, \quad 0 \le t \le 1$$

together with the initial condition:

$$u(x,0) = \cos(2\pi x)$$

and the homogenous nonlocal conditions:

$$\int_0^1 u(x,t)dx = 0, \qquad 0 \le t \le 1$$

and

$$\int_0^1 xu(x,t)dx = 0, \qquad 0 \le t \le 1.$$

It is easy to check that the compatibility conditions are satisfied. That is

$$\int_0^1 u_0(x)dx = \int_0^1 \cos(2\pi x)dx = 0$$

$$\int_0^1 xu_0(x)dx = \int_0^1 x\cos(2\pi x)dx = 0$$

.

Therefore the solution of this nonlocal problem is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-(2n\pi)^2 t}\cos(2n\pi x)$$

But

$$B_n = 2\int_0^1 u_0(x)\cos(2n\pi x)dx = \begin{cases} 1, & n = 1 \\ 0, & n \ne 1 \end{cases}$$

Thus the solution of the above nonlocal problems is:

$$u(x,t) = e^{-4\pi^2 t}\cos(2\pi x), \qquad 0 \le x \le 1, \qquad 0 \le t \le T$$

3.4 The Eigenfunction Expansion Method for Solving Nonlocal Problems Arising in Thermoeslasticty Problems:

In this section, we try to use the eigenfunction expansion method as a technique for solving special types of nonlocal problems arising in thermoelasticity. To do this, consider the one dimensional homogeneous heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 \le x \le 1, \qquad 0 \le t \le T \tag{3.25}$$

together with the initial condition:

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1 \tag{3.26}$$

with the nonhomogeneous nonlocal conditions:

$$\int_0^1 u(x,t)dx = \theta_1(t), \qquad 0 \le t \le T \tag{3.27}$$

and

$$\int_0^1 xu(x,t)dx = \theta_2(t), \qquad 0 \le t \le T \tag{3.28}$$

It is assumed that the compatibility conditions are satisfied. That is

$$\int_0^1 u_0(x)dx = \theta_1(0).$$

$$\int_0^1 xu_0(x)dx = \theta_2(0).$$

It is convenient at the beginning to reduce the nonlocal problem given by equation by equations (3.25)-(3.28) with nonhomogeneous nonlocal conditions given by equations (3.27)-(3.28) to an equivalent one with homogeneous nonlocal conditions. For this purpose, we introduce a new unknown function z by setting :

$$v(x,t) = u(x,t) - z(x,t) \qquad 0 \le x \le 1, \qquad 0 \le t \le T$$

where

$$z(x,t)=6[2\theta_2(t)-\theta_1(t)]x-2[3\theta_2(t)-2\theta_1(t)]$$ [21]

Since $\frac{\partial^2 z(x,t)}{\partial x^2}=0,$ then the function v is seen to be the solution of the one dimensional nonhomogeneous heat equation:

$$\frac{\partial v(x,t)}{\partial t}=\frac{\partial^2 v(x,t)}{\partial x^2}+g(x,t), \qquad 0\le x\le 1, \quad 0\le t\le T \tag{3.29}$$

together with the initial condition

$$v(x,0)=m(x), \qquad 0\le x\le 1 \tag{3.30}$$

and the homogeneous non-local conditions:

$$\int_0^1 v(x,t)dx=0, \qquad 0\le t\le T \tag{3.31}$$

and

$$\int_0^1 xv(x,t)dx=0, \qquad 0\le t\le T \tag{3.32}$$

where $g(x,t)=-\frac{\partial z(x,t)}{\partial t}$ and $m(x)=u_0(x)-z(x,0)$

Hence instead of looking for the function u, we search for the function v. The solution of the nonlocal problem given by equations (3.25)-(3.28) will be simply given by the formula:

$$u(x,t)=v(x,t)+z(x,t)$$

The basic idea of the eigenfunction method is to decompose g into the form

$$g(x,t)=\sum_{n=1}^{\infty} g_n(t)X_n(x)$$

where X_n are the eigenvectors of the Sturm-Lioville problem we get when solving the associated homogeneous problem to the original- problem by using the separation of variables.

$$X_n(x)=\cos(2n\pi x), \qquad n=1,2\ldots$$

Therefore

$$g(x,t)=\sum_{n=1}^{\infty} g_n(t)\cos(2n\pi x)$$

By using the orthogonality condition given by equation (1.7) and the above equation one can have:

$$g_n(t)=2\int_0^1 g(x,t)\cos(2n\pi x)dx \qquad n=1,2,\ldots$$

Therefore by substituting

$$v(x,t)=\sum_{n=1}^{\infty} T_n(t)\cos(2n\pi x)$$

and

$$g(x,t)=\sum_{n=1}^{\infty} g_n(t)\cos(2n\pi x)$$

into equation (3.29) one can get the following ordinary differential equation:

$$\sum_{n=1}^{\infty}\left[T'(t_n)+(2n\pi)^2 T_n(t)-g_n(t)\right]\cos(2n\pi x)=0$$

On the other hand, it is easy to check that this solution v is automatically satisfy the nonlocal conditions given by equations (3.31)-(3.32). In this case, the initial condition given by equation (3.30) becomes:

$$\sum_{n=1}^{\infty} T_n(0)\cos(2n\pi x)=m(x) \qquad 0\le x\le 1$$

By using the orthogonality condition given by equation (1.7) and the above two equations becomes:

$$T_n'(t)+(2n\pi)^2 T_n(t)=g_n(t), \qquad n=1,2,\ldots$$

$$T_n(0)=2\int_0^1 m(x)\cos(2n\pi x)dx, \qquad n=1,2,\ldots$$

which has a solution

$$T_n(t)=e^{-(2n\pi)^2 t}T_n(0)+\int_0^t e^{-(2n\pi)^2(t-\tau)}g_n(\tau)d\tau. \qquad n=1,2,\ldots.$$

Therefore

$$v(x,t)=\sum_{n=1}^{\infty}\left[e^{-(2n\pi)^2 t}T_n(0)+\int_0^t e^{-(2n\pi)^2(t-\tau)}g_n(\tau)d\tau\right]\cos(2n\pi x)$$

and hence,

$$u(x,t)=\sum_{n=1}^{\infty}\left[e^{-(2n\pi)^2 t}T_n(0)+\int_0^t e^{-(2n\pi)^2(t-\tau)}g_n(\tau)d\tau\right]\cos(2n\pi x)+z(x,t)$$

is the desired solution of the nonlocal problem given by equations (3.25)-(3.28).

3.5 The Variational Iteration Method for Solving A Nonlocal Problem Arising in Thermoelasticity

In this section we use the variational iteration method to solve the nonlocal problem arising in thermoelasticity.

To do this consider the one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), \quad 0\le x\le 1, \quad 0\le t\le 1 \qquad (3.33)$$

together with initial condition

$$u(x,0)=u_0(x), \quad 0\le x\le 1 \qquad (3.33)$$

and the homogeneous nonlocal conditions:

$$\int_0^1 u(x,t)dx=\theta_1(t), \quad 0\le t\le T \qquad (3.34)$$

and

$$\int_0^1 xu(x,t)dx=\theta_2(t), \quad 0\le t\le T \qquad (3.35)$$

As mentioned above, this nonlocal problem is transformed to an equivalent one with homogeneous nonlocal conditions by using the transformation :

$$v(x,t) = u(x,t) - z(x,t), \quad 0 \le x \le 1, \quad 0 \le t \le T$$

where z is defined previously.

Then the function v is seen to be the solution of the partial differential equation :

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + g(x,t), \quad 0 \le x \le 1, \quad 0 \le t \le T \qquad (3.36)$$

together with initial condition

$$v(x,0) = m(x), \quad 0 \le x \le 1 \qquad (3.37)$$

and the homogeneous nonlocal conditions:

$$\int_0^1 v(x,t)dx = 0, \quad 0 \le t \le T \qquad (3.38)$$

and

$$\int_0^1 xv(x,t)dx = 0, \quad 0 \le t \le T \qquad (3.39)$$

where

$$g(x,t) = f(x,t) - \frac{\partial z(x,t)}{\partial t}$$

$$m(x) = u_0(x) - z(x,0)$$

According to the variational iteration method, we consider the correction functional in t direction for equation (3.36) in the following form:

$$v_{i+1}(x,t) = v_i(x,t) + \int_0^t \lambda(s)\left[\frac{\partial v_i(x,s)}{\partial s} - \frac{\partial^2 \tilde{v}_i(x,s)}{\partial x^2} - g(x,s)\right]ds \quad i = 0,1,\ldots \qquad (3.40)$$

where λ is the generalized Lagrange multiplier. Thus by taking the variation of above equation one can have:

$$\delta v_{i+1}(x,t) = \delta v_i(x,t) + \delta\int_0^t \lambda(s)\left[\frac{\partial v_i(x,s)}{\partial s} - \frac{\partial^2 v_i(x,s)}{\partial x^2} - g(x,s)\right]ds$$

Thus by using the integration by part, the above equation becomes:

$$\delta v_{i+1}(x,t) = \delta v_i(x,t) + \delta\lambda(s)v_i(x,s)\big|_{s=t} - \delta\int_0^t \left[\lambda'(s)v_i(x,s) + \lambda(s)\frac{\partial^2 \tilde{v}_i(x,s)}{\partial x^2} + \lambda g(x,s)\right]ds$$

$$= \left[1+\lambda(s)\big|_{s=t}\right]\delta v_i(x,s) - \delta\int_0^t \left[\lambda'(s)v_i(x,s) + \lambda(s)\frac{\partial^2 \tilde{v}_i(x,s)}{\partial x^2} + \lambda g(x,s)\right]ds$$

The stationary condition would be as follows:

$$1+\lambda(s)\big|_{s=t} = 0, \qquad 0 \le s \le t$$

and

$$\lambda'(s) = 0.$$

Thus

$$\lambda(s) = -1.$$

Therefore the iterative formula for computing $v_i(x,t)$ taking the form:

$$v_{i+1}(x,t) = v_i(x,t) - \int_0^t \lambda(s)\left[\frac{\partial v_i(x,s)}{\partial s} - \frac{\partial^2 v_i(x,s)}{\partial x^2} - g(x,s)\right]ds \qquad i = 0,1,\cdots \qquad (3.41)$$

For simplicity, let $v_0(x,t) = m(x)$, then

$$\int_0^1 v_0(x,t)dx = \int_0^1 m(x)dx = \int_0^1 [u_0(x) - z(x,0)]dx = \int_0^1 u_0(x)dx - \int_0^1 z(x,0)dx$$

$$= \theta_1(0) - \int_0^1 [12\theta_2(0) - 6\theta_1(0)]x dx + \int_0^1 [6\theta_2(0) - 4\theta_1(0)]dx$$

$$= \theta_1(0) - 6\theta_2(0) + 3\theta_1(0) + 6\theta_2(0) - 4\theta_1(0)$$

$$= 0$$

and,

$$\int_0^1 xv_0(x,t)dx = \int_0^1 xm(x)dx = \int_0^1 x[u_0(x) - z(x,0)]dx = \int_0^1 xu_0(x)dx - \int_0^1 xz(x,0)dx$$

$$= \theta_1(0) - \int_0^1 [12\theta_2(0) - 6\theta_1(0)]x^2 dx + \int_0^1 [6\theta_2(0) - 4\theta_1(0)]x dx$$

$$= \theta_2(0) - 4\theta_2(0) + 2\theta_1(0) + 3\theta_2(0) - 2\theta_1(0)$$

$$= 0$$

Then, any initial condition $v_0(x,t)$ given by equation (3.37) must satisfy the homogeneous nonlocal conditions (3.38)-(3.39) help to starting with. Then by substituting $i = 0$ into equation (3.41) one can get:

$$v_1(x,t) = v_0(x,t) - \int_0^t \left[\frac{\partial v_0(x,s)}{\partial s} - \frac{\partial^2 v_0(x,s)}{\partial x^2} - g(x,s) \right] ds$$

$$= m(x) - \int_0^t \left[- \frac{\partial^2 m(x)}{\partial x^2} - g(x,s) \right] ds$$

$$= m(x) + tm''(x) + \int_0^t \left[f(x,s) - \frac{\partial z(x,s)}{\partial s} \right] ds$$

$$= m(x) + tm''(x) + \int_0^t f(x,s) ds - z(x,t) + z(x,0)$$

$$= u_0(x) + tm''(x) + \int_0^t f(x,s) ds - z(x,t)$$

To illustrate this method, consider the following example:

Example (3.2)

Consider the following one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2tx - 2, \quad 0 \le x \le 1, \quad 0 \le t \le 1 \qquad (3.42)$$

together with the initial condition:

$$u(x,0) = x^2 \qquad (3.43)$$

and the nonhomogeneous nonlocal conditions

$$\int_0^1 u(x,t) dx = \frac{1}{3} + \frac{t^2}{2} \qquad (3.44)$$

and

$$\int_0^1 x u(x,t) dx = \frac{1}{4} + \frac{t^2}{3} \qquad (3.45)$$

It is clear that

$$\theta_1(0) = \frac{1}{3} = \int_0^1 u_0(x) ds = \int_0^1 x^2 dx$$

and

$$\theta_2(0) = \frac{1}{4} = \int_0^1 x u_0(x) dx = \int_0^1 x^3 dx$$

That is the compatibility conditions are satisfied. To solve such problem by using the variational iteration method, we must transform it into an equivalent problem given by equations (3.36)-(3.39) with homogeneous nonlocal conditions.

In this case:

$$z(x,t) = (1+t^2)x - \frac{1}{6}$$

$$g(x,t) = -2$$

and

$$m(x) = x^2 - x + \frac{1}{6}.$$

Therefore the nonlocal problem given by equation (3.42)-(3.45) becomes:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - 2, \qquad 0 \le x \le 1, \quad 0 \le t \le 1 \tag{3.46}$$

together with the initial condition:

$$v(x,0) = x^2 - x + \frac{1}{6}, \qquad 0 \le x \le 1 \tag{3.47}$$

and the homogeneous nonlocal conditions

$$\int_0^1 v(x,t)dx = \int_0^1 xv(x,t)dx = 0, \qquad 0 \le t \le 1 \tag{3.48}$$

Let

$$v_0(x,t) = m(x) = x^2 - x + \frac{1}{6}$$

then

$$\frac{\partial v_0}{\partial s} = 0$$

and

$$\frac{\partial^2 v_0}{\partial x^2} = 2.$$

Hence

$$v_1(x,t) = v_0(x,t) - \int_0^t \left[\frac{\partial v_0(x,s)}{\partial s} - \frac{\partial^2 v_0(x,s)}{\partial x^2} - g(x,s) \right] ds$$

$$= x^2 - x + \frac{1}{6} - \int [0 - 2 + 2] ds$$

$$= x^2 - x + \frac{1}{6}$$

Therefore

$$v_i(x,t) = v_0(x,t) = x^2 - x + \frac{1}{6} \qquad i = 1,2,\ldots$$

and this implies that

$$v(x,t) = \lim_{i \to \infty} v_i(x,t) = x^2 - x + \frac{1}{6}$$

Therefore the solution of the original problem is

$$u(x,t) = v(x,t) + z(x,t)$$

$$= x^2 - x + \frac{1}{6} + (1+t^2)x - \frac{1}{6}$$

$$= x^2 - x + \frac{1}{6} + x + t^2 x - \frac{1}{6}$$

$$= x^2 + t^2 x$$

which is the exact solution for the original nonlocal problem.

Example (3.3)

Consider the following one-dimensional nonhomogeneous heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + (1-t)e^{-x}, \qquad 0 \le x \le 1, \qquad 0 \le t \le 1$$

together with the initial condition:

$$u(x,0) = 0, \qquad 0 \le x \le 1$$

and the nonhomogeneous nonlocal conditions:

$$\int_0^1 u(x,t)dx = (1-e^{-1})t, \qquad 0 \le t \le 1$$

and

$$\int_0^1 xu(x,t)dx = (1-2e^{-1})t, \qquad 0 \le t \le 1$$

That is the compatibility conditions are satisfied. To solve such problem by using the variational iteration method, we must transform it into an equivalent problem given by equations (3.36)-(3.39) with homogeneous nonlocal conditions.

It is clear that

$$\theta_1(0) = \int_0^1 u_0(x)ds = 0$$

and

$$\theta_2(0) = \int_0^1 xu_0(x)dx = 0$$

In this case:

$$z(x,t) = 6(1-3e^{-1})tx - 2(1-4e^{-1})t$$

$$z(x,0) = 0$$

$$\frac{\partial z}{\partial t} = 6(1-3e^{-1})x - 2(1-4e^{-1})$$

$$g(x,t) = e^{-x}(1-t) - 6(1-3e^{-1})x + 2(1-4e^{-1})$$

$$m(x) = 0 - z(x,0) = 0$$

$$v_1(x,t) = \int_0^t \left[e^{-x}(1-s) - 6(1-3e^{-1})x + 2(1-4e^{-1})\right]ds$$

$$= e^{-x}(t - \frac{t^2}{2}) - 6(1-3e^{-1})xt + 2(1-4e^{-1})t$$

$$v_2(x,t) = e^{-x}(t - \frac{t^2}{2}) - 6(1-3e^{-1})xt + 2(1-4e^{-1})t -$$

$$\int_0^t \left[e^{-x}(1-s) - 6(1-3e^{-1})x + 2(1-4e^{-1}) - e^{-x}(s - \frac{s^2}{2}) + 6(1-3e^{-1})x - 2(1-4e^{-1}) \right] ds$$

$$v_2(x,t) = e^{-x}(t - \frac{t^3}{6}) - 6(1-3e^{-1})xt + 2(1-4e^{-1})t$$

By continuing in this manner one can get

$$v_i(x,t) = e^{-x}\left[t - \frac{t^{i+1}}{(i+1)!} \right] - 6(1-3e^{-1})xt + 2(1-4e^{-1})t, \qquad i = 1,2,\ldots$$

Hence,

$$v(x,t) = \lim_{i\to\infty} v_i(x,t) = te^{-x} - 6(1-3e^{-1})xt + 2(1-4e^{-1})t$$

In this case

$$u(x,t) = v(x,t) + z(x,t)$$

$$= te^{-x} - 6(1-3e^{-1})xt + 2(1-4e^{-1})t + 6(1-3e^{-1})xt - 2(1-4e^{-1})t$$

$$= te^{-x}$$

Which is the exact solution of the original problem.

Conclusions

Form this present thesis, we can conclude the following:

1. The separation of variables and the eigenfunction expansion method gave the solvability for special types of nonlocal problems.
2. In the application of the varaitional iterational method, it is noted that every initial approximation to the solution of the nonlocal problems must satisfy the local and nonlocal conditions that associated with these problems.
3. In a similar manner, one can easily use the varaitional iterational method to solve the one-dimensional nonhomogeneous Laplace equation with nonhomogeneous nonlocal conditions.

Recommendations for Future Work

The variational iterational method is a provided methods, from the present study, we can recommend the following problems as future works:

1. Describe other types of nonlocal conditions uses series type.
2. Use the varaitional iterational method for solving the nonlinear nonlocal problems.
3. Study the multi-dimensional nonlocal problems and use the separation of variables, the eigenfunction expansion method and the varaitional iterational method to solve them
4. Devote the varaitional iterational method as a technique for solving partial integro-differential equations with nonlocal conditions.

5. The main advantage of the varaitional iterational method is that it can be applied directly for all types of nonlinear partial differential and integral equations, homogeneous or nonhomogeneous, with constant or variable coefficients

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المستخلص

هذا العمل يختص بالمسائل اللا محلية حيث يمكن تقسيم هدفه الأساسي الى ثلاثة اقسام، والتي يمكن ادراجها كالتالي:

اولاً: تقديم بعض الطرق التحليلة، والمسماة بطريقة فصل المتغيرات وطريقة توسيع الدالة الذاتية، المستخدمة لحل بعض أنواع المعادلات التفاضلية الجزئية الخطية مع الشروط اللامحلية.

ثانياً : استخدام طريقة التكرار التغايري لحل بعض انواع خاصة من المعادلات التفاضلية الجزئية ذات البعد الواحد مع الشروط اللامحلية.

ثالثاً : عرض بعض التطبيقات اللامحلية والتي تظهر في مشكلة المرونة الحرارية باستخدام طرق فصل المتغيرات وتوسيع الدالة الذاتية والتكرار التغايري مع حلولها.

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بعض الطرق التحليلية لحل بعض انواع المسائل اللا محلية

رسالة

مقدمة إلى كلية العلوم – جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير في علوم الرياضيات

مِن قبل

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