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A Study of Efficient Estimation Methods for the parameters of Extreme Value Distribution by Utilizing Monte Carlo Sampling

A Thesis

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By

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بسم الأب والابن والروح القدس

الإله الواحد أمين

٢٠ المُتَعَقِّلُ في أمرِهِ يَجِدُ خَيرًا , ومنْ
يَتَّكِلْ على الرّبِّ يَهنَأُ . ٢١ الحكيمُ القلبِ
يَشتَهِرُ بِفَهمِهِ , وكلامُهُ العَذبُ يَزيدُهُ
عِلْماً . ٢٢ العقلُ يُنبوعُ حياةٍ لِصاحِبِهِ ,
ومَشورَةُ الأحمق حماقةٌ .٢٣ قلبُ الحكيمِ
يُرشِدُ فَمَهُ , ويَزيدُ كَلامَهُ عِلْماً.

الأمثال ١٦ : ٢٠-٢٣

الإهداء

إلى الله خالق الكون وسيدنا يسوع
المسيح وأمه القديسة مريم العذراء
وكنيستي وعائلتي خاصة أمي (رحمها
الله) هذه امنيتها وأبي العطوف معي
في كل حين وأخوتي وأخواتي
وأقاربي الذين ليس لهم مثيل وأحبائي
وأصدقائي وكل من تعب معي ومن
كان يصلي من أجلي ومن كان
يساعدني في هذه الحياة ....
إني لن أنساكم أبدا

فادي عادل

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Fadi A. Shabo

May 2010

Supervisors Certification

We certify that this thesis entitled " Estimation methods for the parameters of Extreme Value distribution by utilizing Monte Carlo sampling" was prepared by " Fadi Adel Shabo" under our supervision at the College of Science / Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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COMMITTEE CERTIFICATION

We certify that we have read this thesis entitled " *A Study of Efficient Estimation Methods for the parameters of Extreme Value Distribution by Utilizing Monte Carlo Sampling* " and as an examining committee examined the student (*Fadi Adel Shabo*) in its contents and in what it connected with, and that in our opinion it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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Notes and Abbreviations

r.v.	***random variable***
r.s.	***random sample***
s.s.	***sample space***
eq.	***equation***
distn.	***distribution***
p.d.f.	***probability density function***
c.d.f.	***cumulative distribution function***
m.g.f.	***moment generating function***
e.w.	***else were***
IT	***Inverse Transformation***
M.M.	***Moment Method***
m.l.e.	***maximum likelihood estimate***
M.L.M.	***Maximum Likelihood Method***
L.S.M.	***Least Squares Method***
O.S.M.	***Order Statistic Method***
Ext(α,β)	***Extreme Value Distribution with parameters α and β***
W(a,b)	***Weibull Distribution with parameters a and b***
R(x)	***Reliability function of x***
h(x)	***hazard function of x***

Abstract

In this thesis, we consider the extreme value distn. of two parameters for the reason of its appearance in many statistical fields of applications. Mathematical and statistical properties of the distn. such as moments and higher moments are collected and unified and the properties of reliability and hazard functions of the distn. are illustrated.

The chi-square goodness - of - fit is used to test whether the generated samples from the standardized extreme value distn. by Monte Carlo simulation are acceptable for use.

These samples are used to estimate the distn. parameters by four methods of estimation, namely moments method, maximum likelihood method, order statistic method and least squares method.

These methods are discussed theoretically and assessed practically in estimating the reliability and hazard functions. The properties of the estimator, reliability and hazard functions, such as bias, variance, skewness, kurtosis, and mean square error are tabled.

The computer programs are listed in three appendices and the run is made by using "MathCAD 14".

Contents

List of Contents

Contents

Contents

Introduction

Extreme value distributions are the limiting distributions for the minimum or the maximum of a very large collection of random observations from the same arbitrary distn. [26]. Probabilistic extreme value theory is a curios and fascinating blend of an enormous variety of applications involving natural phenomena such as rainfall, floods, wind gusts, air pollution, corrosion, delicate advanced mathematical results on point processes, regularly varying functions, extreme temperatures, large fluctuations in exchange rates, market crashes, and breaking strength of materials. This area of research thus attracted initially the interests of theoretical probabilistic as well as engineers and hydrologists, and only relatively recently of the mainstream statisticians. [24]

A systematic development of the general theory may be regard as having started with the paper by von Bortkiewicz (1922) that deal with the distn. of range in random samples from a normal distn..[24]

In (1943), Gnedenko presented a rigorous foundation for the extreme value theory and provided necessary and sufficient conditions for the weak convergence of the extreme order statistics. [24]

Jenkinson (1955) combined these three extreme-value distributions into one, i.e., the general (or generalized) extreme value (GEV) distribution, by using a transformation of the three-parameter Weibull distribution. [21]

Gumbel made several significant contributions to the extreme value analysis; most of them are detailed in his book length account of statistics of extremes Gumbel (1958). [10][24]

Castillo (1988) has successfully updated Gumbel (1958) and presented many statistical applications of extreme value theory with emphasis on engineering [5].

Reiss (1989) discussed various convergence concepts and rates of convergence associated with extremes (and also with order statistic). [24]

Commonly used distributions for rainfall frequency analysis are the Gumbel distn., and the GEV distn., as applied by many authors (Hosking et al., 1985; Coles et al., 2003). [19][7]

The Gumbel distribution is associate parameters have been determined with the Probability-Weighted Moments (PWM) method, and the GEV distribution in which the three parameters have been calculated with both the PWM method (Hosking et al., 1985) [19] and the L-Moments method (Gellens, 2002) [14].

Based on progressively censored samples, Viveros and Balakrishnan (1994) developed a conditional method of inference to derive exact confidence intervals for the parameters of the extreme value distn. [41]. Beirlant, Teugels and Vynekier (1996) [3] and Reiss and Thomas (1997) [37] provide a lucid practical analysis of extreme values with emphasis on actuarial applications.

Bayesian estimation, prediction and characterization for the extreme value model based on lower record values have been discussed in Mousa (2002) [32]. Smith (2003) provides some applications for financial data [39].

Balakrishnan et al. (2004) discussed in classical framework, the point and interval estimation for parameters of the extreme value distribution based on progressively Type-II censored data, Bayes estimates of the two (unknown) parameters, the reliability and failure rate functions are obtained by using approximation form due to Lindley (1980) [2].

The density of the Gumbel distn. is approximated by a finite mixture counting data from normal distn. using the $\log c_1^2$ density (Frühwirth – Schnatter and Frühwirth, 2007; Frühwirth -Schnatter and Wagner, 2006; Frühwirth -Schnatter et al., 2009). [11][13][12]

The Gumbel distn. sometimes is called doubly exponential [24] and also natural logarithms of Weibull random variables. [30]

Introduction

This thesis involve three chapters. In chapter one, some important mathematical and statistical properties of extreme value distn. and moment properties of the distn. are illustrated and modified. Four methods of estimation for the distn. parameters are discussed theoretically. some concepts and properties the reliability and hazard functions are introduced. Four methods of estimation for the reliability and hazard functions are introduced. Finally, two theorems related to extreme value distn.

In chapter two, some concepts of stochastic simulation are illustrated. One procedure for generating random numbers and random variates from extreme value distn. is discussed theoretically and supported by various examples and one algorithm is illustrated. Finally, chi-square test as best test to goodness-of-fit test to known the distn. is usable or not that observations come from Ext(0,1) are illustrated.

In chapter three, the Monte Carlo simulation results to estimate the parameters, the reliability and the hazard functions given in chapter one practically by one procedure namely (EV-1) that observations come from Ext(1,2) are introduced.

CHAPTER ONE

SOME INTRODUCTORY CONCEPTS OF EXTREME VALUE DISTRIBUTION WITH ESTIMATION METHODS

1.1 Introduction

In this chapter, some mathematical and statistical properties of extreme value distn. are introduced.

This chapter involves six sections. In section (1.2), some fundamental concepts of extreme value distn. are illustrated, while section (1.3), deal with moments and higher moments properties of the distn. In section (1.4), four methods of parameters estimation namely moments method , maximum likelihood method , order statistic method and least squares method are considered. In section (1.5), some concepts and estimation methods to reliability and hazard functions are introduced. In section (1.6), some related theorems concerning the distn. are given.

1.2 Some Fundamentals of Extreme Value Distribution

In this section, some properties of the extreme value distn., are introduced .

Definition (1.1), [33]

A continuous r.v. X is said to have a minimum extreme value distn., denoted by *X~ Ext(α, β),* if X has p.d.f.

$$f(x;\alpha,\beta)=\frac{1}{\beta}e^{(\frac{x-\alpha}{\beta})-e^{(\frac{x-\alpha}{\beta})}} \quad ,-\infty<x<\infty \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

$$-\infty<\alpha<\infty,\ \beta>0.$$

where α and β are respectively known as the scale and shape parameters.

The extreme value distn. depends on two parameters α and β and a wide variety of distribution shapes can be generated by suitable choices of α and β. Figures (1.1) and (1.2) show respectively a graphical representation of some p.d.fs. for fixed α and β varying and for fixed β and α varying.

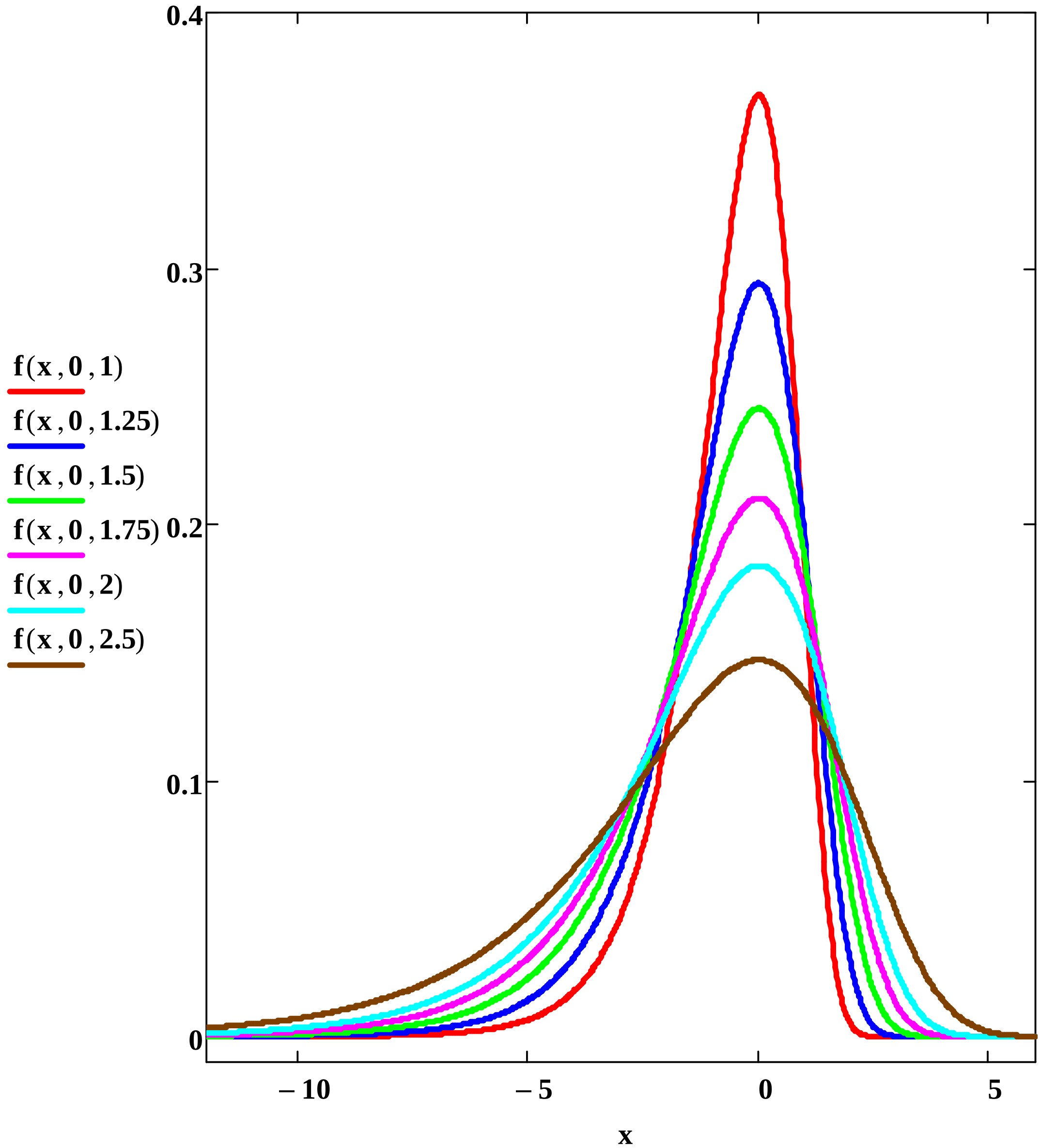


Fig (1.1): Extreme Value p.d.fs. with α = 0 and β = 1, 1.25, 1.5, 1.75, 2, 2.5.

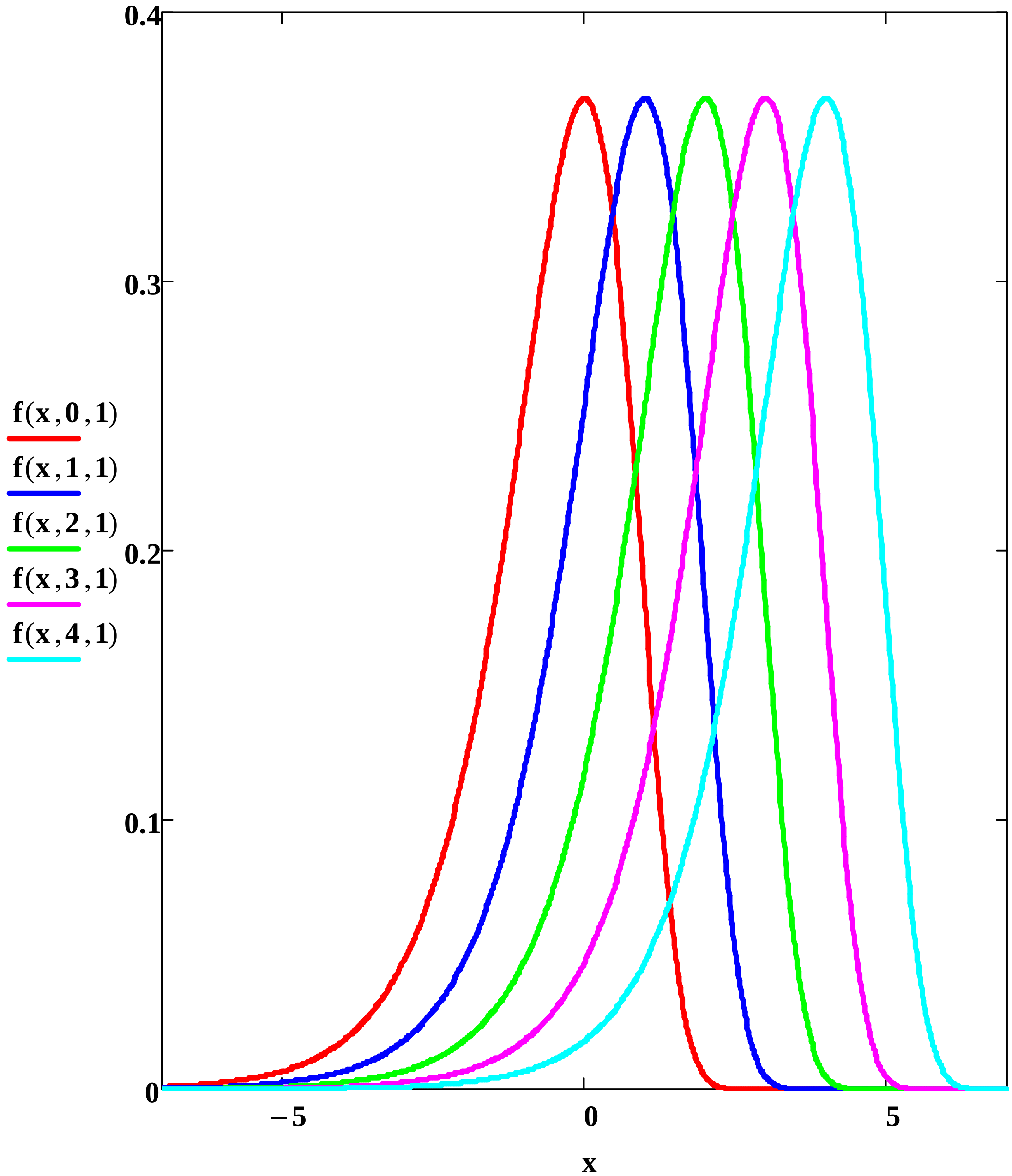


Fig (1.2): Extreme Value p.d.fs. with α = 0, 1, 2, 3, 4 and β =1.

The graph of f(x;α,β) as shown in figure (1.1) and figure (1.2):

1- Have the x-axis as a horizontal asymptote.
2- Increasing for $-\infty < x < \alpha$ and decreasing for $\alpha < x < \infty$.
3- Has maximum point at $(\alpha, \frac{e^{-1}}{b})$.
4- Have two points of inflection at $x = \alpha + \beta[\ln(3 \pm \sqrt{5}) - \ln(2)]$.
5- Concave upward for $-\infty < x < \alpha + \beta[\ln(3 - \sqrt{5}) - \ln(2)]$,

   $\alpha + \beta[\ln(3 + \sqrt{5}) - \ln(2)] < x < \infty$ and Concave downward for

   $\alpha + \beta[\ln(3 - \sqrt{5}) - \ln(2)] < x < \alpha + \beta[\ln(3 + \sqrt{5}) - \ln(2)]$.

The details of the above properties are shown in appendix d.

1.2.1 The Cumulative Distribution Function, [33]

The c.d.f. of minimum extreme value distn. is defined by the following integral:

$$F(x;\alpha,\beta) = Pr(X \le x) = \int_{-\infty}^{x} f(t;\alpha,\beta)\,dt = \int_{-\infty}^{x} \frac{1}{\beta} e^{(\frac{t-\alpha}{\beta}) - e^{(\frac{t-\alpha}{\beta})}}\,dt$$

Set y = $\frac{t-\alpha}{\beta}$ or equivalent $t = \alpha + \beta y$ implies $dt = \beta\,dy$

$$F(x;\alpha,\beta) = \int_{-\infty}^{\frac{x-\alpha}{\beta}} e^{(y - e^{y})}\,dy = \int_{-\infty}^{\frac{x-\alpha}{\beta}} e^{-e^{y}}\, e^{y}\,dy$$

Set $u = e^{y}$ implies $du = e^{y}\,dy$

$$F(x;\alpha,\beta) = \int_{0}^{e^{\frac{x-\alpha}{\beta}}} e^{-u}\,du = 1 - e^{-e^{\frac{(x-a)}{b}}}, \quad -\infty < x < \infty \ldots\ldots\ldots\ldots\ldots (1.2)$$

1.3 Moments and Higher Moments Properties of Extreme Value Distribution [22]

Moments are set of constants used for measuring distn. properties and under certain circumstances they specify the distn. The moments of r.v. X (or distn.) are defined in terms of the mathematical expectation of certain power of X when they exist. For instance,

$\mu'_r = E(X^r)$ is called the r[th] moment of X about the origin and $\mu_r = E[(X-\mu)^r]$ is called the r[th] central moment of X. That is

$$m'_r = E(X^r) = \begin{cases} \sum_x x^r f(x), & X \text{ is discrete r.v.} \\ \int_x x^r f(x)dx, & X \text{ is continuous r.v.} \end{cases}$$

and

$$m_r = E\left[(X-m)^r\right] = \begin{cases} \sum_x (x-m)^r f(x) & ,X \text{ is discrete r.v.} \\ \int_x (x-m)^r f(x)dx & ,X \text{ is continuous r.v.} \end{cases}$$

Provided the sum or integral converges absolutely.

The generating functions reflect certain properties of the distn., they could be used to generate moments. Sometimes they are defining some specific distns., and also have a particular usefulness in connection with sums of independent, r.vs..

First, we shall consider a function of a real t called the moment generating function, denoted by M(t), which can be used to generate moments of r.v X.

For continuous r.v X, the m.g.f. is defined by

$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$, provided the integral converge absolutely.

To find the m.g.f. of extreme value distn., whose p.d.f. is given by eq. (1.1):

$$M(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\beta} e^{(\frac{x-\alpha}{\beta}) - e^{(\frac{x-\alpha}{\beta})}} dx,$$

Set $y = \frac{x-\alpha}{\beta}$ or equivalent $x = \alpha + \beta y$ implies $dx = \beta\, dy$,then

$$M(t) = \int_{-\infty}^{\infty} e^{t(\alpha+\beta y)} e^{y - e^{y}} dy = e^{at} \int_{-\infty}^{\infty} (e^{y})^{\beta t} e^{-e^{y}} e^{y} dy$$

Set $u = e^{y}$ implies $du = e^{y}\, dy$

$$M(t) = e^{at} \int_{0}^{\infty} u^{\beta t} e^{-u}\, du = e^{at}\Gamma(1+\beta t) \int_{0}^{\infty} \frac{1}{\Gamma(1+\beta t)} u^{(1+\beta t)-1} e^{-u}\, du$$

$$= e^{at}\, \Gamma(1+\beta t),\quad t > \frac{-1}{\beta} \quad \text{.....................................(1.3)}$$

where $\Gamma(w) = \int_{0}^{\infty} y^{w-1} e^{-y}\, dy$, $w > 0$ is called gamma function.

To find the moments and other properties of the minimum extreme value distn., we take the logarithm of the m.g.f. of eq. (1.3), then we have

$$\Phi(t) = \ln M(t) = \alpha t + \ln\Gamma(1+\beta t) \quad \text{.......................................(1.4)}$$

The four derivatives of $\Phi(t)$, are given by

$$\left.\begin{array}{l} \Phi'(t) = \alpha + \beta\Psi(1+\beta t) \\ \Phi''(t) = \beta^2\Psi'(1+\beta t) \\ \Phi'''(t) = \beta^3\Psi''(1+\beta t) \\ \Phi''''(t) = \beta^4\Psi'''(1+\beta t) \end{array}\right\} \quad \text{...(1.5)}$$

where $\Psi(z)=\dfrac{d}{dz}\ln\Gamma(z)$ is known as digamma function, when t=0, we have

$$\Phi'(0)=\alpha+\beta\Psi(1)=E(x)\quad\text{.......}(1.6)$$

$$\Phi''(0)=\beta^2\Psi'(1)=Var(x)\quad\text{.......}(1.7)$$

$$\Phi'''(0)=\beta^3\Psi''(1)\quad\text{.......}(1.8)$$

$$\Phi''''(0)=\beta^4\Psi'''(1)\quad\text{.......}(1.9)$$

The details of finding eqs. (1.6) to (1.9) see in appendix d.

Table (1.1) below gives the values of the digamma function and it's first, second and third derivatives for specified values of N. [35]

N	G(N)	ln G(N)	Y(N)	Y¢(N)	Y¢¢(N)	Y¢2(N)
0.1	**9.514**	**2.253**	**-10.424**	**101.433**	**-2001.861**	**60004.513**
0.3	**2.992**	**1.096**	**-3.503**	**12.245**	**-75.273**	**743.142**
0.5	**1.772**	**0.572**	**-1.964**	**4.935**	**-16.829**	**97.409**
0.7	**1.298**	**0.261**	**-1.220**	**2.834**	**-6.435**	**25.879**
0.9	**1.069**	**0.066**	**-0.755**	**1.923**	**-3.202**	**9.739**
1.0	**1.000**	**0.000**	**-0.577**	**1.645**	**-2.404**	**6.494**

The following are the important moments of the distn..

(i) Mean:

E(X) = μ = μ'_1 is called the mean of r.v X. It is a measure of central tendency. By using eq. (1.6), gives

$$\Phi'(0)=m=E(X)=\alpha-\gamma\beta\quad\text{.......}(1.10)$$

where γ = -Y(1) = 0.577 is Euler's constant.

(ii) Variance:

Var (X) = $\sigma^2 = E[(X-\mu)^2]$ is called the variance of r.v. X. It is a measure of dispersion. The use of eq. (1.7), gives

$$\Phi''(0) = s^2 = \beta^2\Psi'(1) = \frac{p^2\beta^2}{6} = (1.645)\beta^2 \ldots\ldots\ldots\ldots\ldots\ldots (1.11)$$

(iii) Coefficient of Variation:

$c.v. = \frac{s}{m}$ is called the variational coefficient of r.v. X. It is a measure of dispersion. By using eqs. (1.10) and (1.11), gives

$$c.v. = \frac{s}{m} = \frac{p\,\beta}{(\alpha-\gamma\beta)\sqrt{6}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.12)$$

(iv) Coefficient of Skewness:

$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{E[(X-\mu)^3]}{[Var(x)]^{3/2}}$ is called the coefficient of Skewness. It is a measure of the departure of the frequency curve from symmetry. If $\gamma_1 = 0$, the curve is not skewed, $\gamma_1 > 0$, the curve is positively skewed, and $\gamma_1 < 0$, the curve is negatively skewed. By using eqs. (1.8) and (1.11), gives

$$\Phi'''(0) = \mu_3 = E[(X-\mu)^3] = \beta^3\Psi''(1) = (-2.404)\beta^3$$

Thus,

$$g_1 = \frac{(-2.404)\beta^3}{[(1.645)\beta^2]^{3/2}} = \frac{(-2.404)\beta^3}{(2.1098)\beta^3} = -1.1395 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.13)$$

(v) Coefficient of Kurtosis:

$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{E[(X-\mu)^4]}{[Var(x)]^2} - 3$ is called the coefficient of kurtosis.

It is a measure of the degree of flattening of the frequency curve. If $\gamma_2 = 0$, the curve is called mesokurtic, if $\gamma_2 > 0$, the curve is called leptokurtic, and if $\gamma_2 < 0$, the curve is called platykurtic. By using eqs. (1.9) and (1.11), gives

$$E[(X-\mu)^4] = \Phi''''(0) + 3(\sigma^2)^2 = \beta^4\Psi'''(1) + 3(1.645\beta^2)^2$$

$$= (6.494)\beta^4 + (8.118)\beta^4 = (14.612)\beta^4$$

Thus,

$$g_2 = \frac{(14.612)\beta^4}{(1.645\beta^2)^2} - 3 = \frac{(14.612)\beta^4}{(2.706)\beta^4} - 3 = 5.4 - 3 = 2.4 \qquad (1.14)$$

1.3.1 Other Central Moments, [22]

(i) Mode:

A mode of a disn. is the value x of r.v. X that maximize the p.d.f. $f(x)$. For continuous distns., the mode x is a solution of

$$\frac{df(x)}{dx} = 0 \quad and \quad \frac{d^2 f(x)}{dx^2} < 0$$

A mode is a measure of location. Also we note that the mode may not exist or we may have more than one mode.

For extreme value case with p.d.f. of eq. (1.1), the logarithm of $f(x)$ is

$$\ln f(x) = -\ln(\beta) + \frac{x-a}{\beta} - e^{(\frac{x-a}{\beta})}$$

$$\frac{d \ln f(x)}{dx} = \frac{1}{\beta} - \frac{1}{\beta} e^{(\frac{x-a}{\beta})}$$

For maximum, set $\frac{d \ln f(x)}{dx} = 0$ implies $\frac{1}{\beta} - \frac{1}{\beta} e^{(\frac{x-a}{\beta})} = 0$

implies $e^{\frac{x-a}{\beta}} = 1$ implies $\frac{x-a}{\beta} = \ln 1 = 0$ implies

the mode is x = α …………………………………………..………… (1.15)

(ii) Median

A median of a disn. is defined to be the value x of r.v X such that $F(x) = \Pr(X \le x) = \frac{1}{2}$. The median is measure of location.

For extreme value case, the c.d.f. given by eq. (1.2), we have

$\frac{1}{2} = 1 - e^{-e^{(\frac{x-\alpha}{\beta})}}$ implies $e^{-e^{(\frac{x-\alpha}{\beta})}} = \frac{1}{2}$ implies $e^{\frac{x-\alpha}{\beta}} = \ln 2$ implies

$\frac{x-\alpha}{\beta} = \ln[\ln(2)]$ implies

The median is x = α + β ln[ln(2)] ……………………………….. (1.16)

1.4 Point Estimation

The point estimation is concerned with inference about the unknown parameters of a distn. from a sample. It provides a single value for each unknown parameter.

The following definitions are needed for the interest of this work.

Definition (1.2) (Statistic), [27]

A statistic is a function of one or more r.vs. which does not depends on any unknown parameters.

Definition (1.3) (Point Estimator), [27]

Any statistic whose value is used to estimate the unknown parameter θ for some function of θ say τ(θ) is called point estimator. Point estimation admits two problems:

First, developing methods of obtaining a statistic, to represent or estimate the unknown parameter in the p.d.f. such statistic.

Second, selecting criteria and technique to define and find best estimator among many possible estimators.

Definition (1.4) (Unbiased Estimator), [34]

An estimator $\hat{\theta} = u(X_1, X_2, \ldots, X_n)$ is defined to be an unbiased estimator of θ if and only if $E(\hat{\theta}) = \theta$ for all $\theta \in \Omega$, where Ω is a parameter space. The term $[E(\hat{\theta}) - \theta]$ is called the bias of the estimator $\hat{\theta}$.

Definition (1.5) (Asymptotically Unbiased Estimator), [34]

An estimator $\hat{\theta} = u(X_1, X_2, \ldots, X_n)$ is defined to be asymptotically unbiased estimator for θ if $\lim_{n\to\infty} E(\hat{\theta}) = \theta$.

Definition (1.6) (Consistent Estimator), [34]

An estimator $\hat{\theta}$ is called consistent estimator for θ if $\hat{\theta}$ converge stochastically to θ, i.e $\lim_{n\to\infty} pr(|\hat{q} - q| < \in) = 1$

1.4.1 Methods of Finding Estimators, [27]

Assume that $X_1, X_2, \ldots, X_n$ be a r.s. of size n from a distn. whose p.d.f. $f(x, q)\, where\; q = (q_1, q_2, \ldots, q_k)$ is a vector of unknown parameters. On the basis of the observed values $x_1, x_2, \ldots, x_n$ of r.vs. $X_1, X_2, \ldots, X_n$ the object is to find statistics, say $U_i = u_i(X_1, X_2, \ldots, X_n),\; i = 1, 2, \ldots, k$, whose values to be used as estimators for $q_i,\; i = 1, 2, \ldots k$.

Several methods could be founds in the literature such as:

Moments method, Maximum likelihood method, Bayesian method, Least squares method, Minimum chi-square method, Minimum distance method and order statistic method.

For extreme value case we shall discuss theoretically four methods of estimation namely the method of moments, the maximum likelihood method, order statistic method and least square method.

1.4.1.1 Estimation of parameters by Moments Method, [34]

Let $X_1, X_2, ..., X_n$ be a r.s of size n from a distn. whose p.d.f. $f(x, \underset{\sim}{\theta})$, where $\underset{\sim}{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ is a vector of k unknown parameters, let $\mu'_r = E\left(X^r\right)$ be the r^th distn. moment about origin and $M_r = \frac{1}{n}\sum_{i=1}^{n} x_i^r$ be the r^th sample moment about origin. The M.M can be described as follows:

Since, we have k unknown parameters, equate μ'_r to M_r at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$. That is $\mu'_r = M_r$ at $\underset{\sim}{\theta} = \underset{\sim}{\hat{\theta}}$, r = 1, 2, …, k. for these k eqs., we find a unique solution for $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k$ and we say that $\hat{\theta}_r, (r = 1, 2, ..., k)$ is an estimate of θ_r obtained by M.M and the corresponding statistic $\hat{\Theta}_r$ is the M.M estimator of θ_r.

For extreme value distn. case, we have two unknown parameters α and β and if a r.s of size n is taken, then we set $m'_r = M_r \ at \ \alpha = \hat{\alpha},\ \beta = \hat{\beta},\ r = 1, 2$. Where $m'_r = E(X^r)$, $M_r = \frac{1}{n}\sum_{i=1}^{n} x_i^r$

For r = 1, we have $m_1' = E(X) = M_1$ and $E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$

from eq. (1.10), gives

$$\hat{m} = \hat{\alpha} - \gamma\hat{\beta} = \bar{X} \quad \text{.......} \quad (1.17)$$

For r = 2, we have $m_2' = E(X^2) = M_2$ and $E(X^2) = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

from eq. (1.11), gives

$Var(X) + [E(X)]^2 = \frac{(n-1)}{n} S^2 + \bar{X}^2$ implies

$Var(X) + \bar{X}^2 = \frac{n-1}{n} S^2 + \bar{X}^2$ implies $\frac{p^2 \hat{\beta}^2}{6} = \frac{n-1}{n} S^2$ implies

$$\hat{\beta} = \frac{S}{p}\sqrt{\frac{6(n-1)}{n}} \quad \text{...(1.18)}$$

where $\hat{\beta}$ is the M.M. estimator for β, and $S = \sqrt{\frac{1}{n-1}[\sum_{i=1}^{n} x_i^2 - n\bar{X}^2]}$ is called standard deviation.

From eqs. (1.18) and (1.19), gives

$$\hat{a} = \bar{X} + \frac{\gamma S}{p}\sqrt{\frac{6(n-1)}{n}} \quad \text{... (1.19)}$$

where $\hat{\alpha}$ is the M.M. estimator for α.

The estimators $\hat{\alpha}$ and $\hat{\beta}$ given by eqs.(1.18) and (1.19) have the following properties : [18]

(i) $\hat{\beta} = \frac{S}{p}\sqrt{\frac{6(n-1)}{n}}$ is approximately an asymptotic unbiased estimator for β, and its variance approach to zero.

Proof:

Since $E(\hat{\beta}) = \frac{1}{p}\sqrt{\frac{6(n-1)}{n}} E(S)$ and $S \to \sigma$, in probability, then

$$E(\hat{\beta}) \to \frac{1}{p}\sqrt{\frac{6(n-1)}{n}} E(s) = \frac{1}{p}\sqrt{\frac{6(n-1)}{n}} \frac{p\beta}{\sqrt{6}} = \beta\sqrt{1-\frac{1}{n}}$$

So $\lim_{n\to\infty} E(\hat{\beta}) ; \beta \lim_{n\to\infty} \sqrt{1-\frac{1}{n}} = \beta\sqrt{(1-0)} = \beta$ .... (1.20)

$\hat{\beta}^2 = \left[\frac{S}{\pi}\sqrt{\frac{6(n-1)}{n}}\right]^2 = \frac{S^2}{\pi^2}\left(\frac{6(n-1)}{n}\right)$, Since $S^2 \to s^2$, then

$$\hat{\beta}^2 \to \frac{6}{\pi^2}(1-\frac{1}{n})\sigma^2$$

$$E(\hat{\beta}^2) \to \frac{6}{\pi^2}(1-\frac{1}{n})E(\sigma^2) = \frac{6}{\pi^2}(1-\frac{1}{n})\sigma^2$$

$$\lim_{n\to\infty} E(\hat{\beta}^2) \cong \frac{6}{\pi^2}(1-0)\sigma^2 = \frac{6}{\pi^2}\frac{\pi^2\beta^2}{6} = \beta^2 \quad \text{.........................} (1.21)$$

From eqs. (1.20) and (1.21), gives

$$Var(\hat{\beta}) = E(\hat{\beta}^2) - [E(\hat{\beta})]^2 \cong \beta^2 - \beta^2 = 0 \quad \text{..............................} (1.22)$$

(ii) $\hat{a} = \bar{X} + \frac{\gamma S}{p}\sqrt{\frac{6(n-1)}{n}}$ is also approximately an asymptotic unbiased estimator for α.

Proof:

Since $E(\hat{\alpha}) = E(\bar{X}) + \frac{\gamma}{p}\sqrt{\frac{6(n-1)}{n}}E(S)$

$$E(\hat{\alpha}) \cong \mu + \frac{\gamma}{p}\sqrt{\frac{6(n-1)}{n}}\sigma = \alpha - \gamma\beta + \frac{\gamma}{p}\frac{p\,\beta}{\sqrt{6}}\sqrt{\frac{6(n-1)}{n}} = \alpha - \gamma\beta + \gamma\beta\sqrt{1-\frac{1}{n}}$$

$$\lim_{n\to\infty} E(\hat{\alpha}) \approx \alpha - \gamma\beta + \gamma\beta\sqrt{1-0} = \alpha \quad \text{..................................} (1.23)$$

$$E(\hat{\alpha}^2) = E\left[\left[\bar{X} + \frac{\gamma S}{p}\sqrt{\frac{6(n-1)}{n}}\right]^2\right]$$

$$= E\left[\bar{X}^2 + \frac{2\gamma\bar{X}S}{p}\sqrt{\frac{6(n-1)}{n}} + \frac{6(n-1)\gamma^2 S^2}{np^2}\right]$$

$$= E(\bar{X}^2) + \frac{2\gamma}{p}\sqrt{\frac{6(n-1)}{n}}E(\bar{X}S) + \frac{6(n-1)\gamma^2}{np^2}E(S^2)$$

Since $\bar{X} \to m$ and $S \to s$ then $\bar{X}S \to ms$

$$E(\hat{\alpha}^2) \to Var(\bar{X}) + [E(\bar{X})]^2 + \frac{2\gamma}{p}\sqrt{\frac{6(n-1)}{n}}ms + \frac{6(n-1)\gamma^2}{np^2}s^2$$

$$= \frac{s^2}{n} + (\alpha-\gamma\beta)^2 + 2\frac{\gamma}{p}\sqrt{6\frac{(n-1)}{n}}(\alpha-\gamma\beta)\frac{p\,\beta}{\sqrt{6}} + \frac{6\gamma^2(n-1)}{np^2}\frac{p^2\beta^2}{6}$$

$$\lim_{n\to\infty} E(\hat{\alpha}^2) = \alpha^2 - 2abg + \gamma^2\beta^2 + 2\sqrt{6}\frac{\gamma}{p}(\alpha-\gamma\beta)\frac{p\beta}{\sqrt{6}} + \frac{6\gamma^2}{p^2}\frac{p^2\beta^2}{6}$$

$$= \alpha^2 - 2\alpha\gamma\beta + \gamma^2\beta^2 + 2\alpha\gamma\beta - 2\gamma^2\beta^2 + \gamma^2\beta^2 = \alpha^2 \quad \ldots\ldots(1.24)$$

From eqs. (1.23) and (1.24), gives

$$Var(\hat{\alpha}) = E(\hat{\alpha}^2) - [E(\hat{\alpha})]^2 \;;\; \alpha^2 - \alpha^2 = 0 \quad \ldots\ldots (1.25)$$

Definition (1.7) (Likelihood function), [34]

The likelihood function of a r.s $X_1, X_2, \ldots, X_n$ of size n from a distn. having p.d.f. $f(x, \underset{\sim}{q})$, where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters, is defined to be the joint p.d.f. of the n r.vs. $X_1, X_2, \ldots, X_n$ which is considered as a function of $\underset{\sim}{\theta}$ and denoted by $L(\underset{\sim}{q}, \underset{\sim}{x})$, that is

$$L(\underset{\sim}{q}, \underset{\sim}{x}) = f(\underset{\sim}{x}, \underset{\sim}{q}) = \prod_{i=1}^{n} f(x_i, \underset{\sim}{q})$$

1.4.1.2 Estimation of Parameters by Maximum Likelihood Method, [34]

Let $L(\underset{\sim}{q}, \underset{\sim}{x})$ be the likelihood function of a r.s $X_1, X_2, \ldots, X_n$ of size n from a distn. whose p.d.f. $f(x, \underset{\sim}{q})$, $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ is a vector of unknown parameters.

Let $\hat{\underset{\sim}{q}} = \underset{\sim}{u}(\underset{\sim}{x}) = \left(u_1(\underset{\sim}{x}), u_2(\underset{\sim}{x}), \ldots, u_k(\underset{\sim}{x})\right)$ be a vector function of the observations $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$. If $\hat{\underset{\sim}{\theta}}$ have the value of $\underset{\sim}{\theta}$ which maximizes $L(\hat{\underset{\sim}{q}}, \underset{\sim}{x})$ then $\hat{\underset{\sim}{\theta}}$ is the m.**l**.e of $\underset{\sim}{\theta}$ and the corresponding statistic $\hat{\underset{\sim}{\Theta}}$ is the M.L.E. of $\underset{\sim}{\theta}$. We note that

(i) Many likelihood functions satisfy the condition that the m.**l**.e is a solution of the likelihood eqs.

$$\frac{\partial L(\underset{\sim}{\theta},\underset{\sim}{x})}{\partial \theta_r}=0 \quad , \text{at } \underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}, \; r = 1,2,\ldots,k$$

(ii) Since $L(\underset{\sim}{\theta},\underset{\sim}{x})$ and $\ln L(\underset{\sim}{\theta},\underset{\sim}{x})$ have their maximum at the same value of $\underset{\sim}{\theta}$ so sometimes it is easier to find the maximum of the logarithm of the likelihood.

In such case, the m.l.e. $\hat{\underset{\sim}{\theta}}$ of $\underset{\sim}{\theta}$ which maximizes $L(\underset{\sim}{\theta},\underset{\sim}{x})$ may be given the solution of the likelihood eqs.

$$\frac{\partial \ln L(\underset{\sim}{\theta},\underset{\sim}{x})}{\partial \theta_r}=0 \quad \text{at } \underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}, \; r = 1,2,\ldots,k$$

For extreme value distn. case

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from Ext(α,β) where the distn. p.d.f. is given by (1.1). The likelihood function is

$$L(\alpha,\beta,\underset{\sim}{x})=f(\underset{\sim}{x},\alpha,\beta)$$

$$=\prod_{i=1}^{n} f(x_i,\alpha,\beta)$$

$$=\prod_{i=1}^{n}\frac{1}{\beta}\mathrm{e}^{(\frac{x_i-\alpha}{\beta})-e^{(\frac{x_i-\alpha}{\beta})}}$$

$$=\beta^{-n}\,\mathrm{e}^{\sum_{i=1}^{n}(\frac{x_i-\alpha}{\beta})}\,\mathrm{e}^{-\sum_{i=1}^{n}e^{(\frac{x_i-\alpha}{\beta})}}$$

$$\ln L(\alpha,\beta;\underset{\sim}{x})=-n\ln(\beta)+\sum_{i=1}^{n}\frac{(x_i-\alpha)}{\beta}-\sum_{i=1}^{n}e^{\frac{(x_i-\alpha)}{\beta}}$$

$$=-n\ln(\beta)-\frac{n\alpha}{\beta}+\sum_{i=1}^{n}\frac{x_i}{\beta}-\sum_{i=1}^{n}e^{\frac{(x_i-\alpha)}{\beta}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.26)$$

$$\frac{\partial \ln L(\alpha,\beta;\underset{\sim}{x})}{\partial \alpha} = \frac{1}{\beta}\sum_{i=1}^{n} e^{\frac{(x_i-\alpha)}{\beta}} - \frac{n}{\beta} \quad \text{.......................................} \quad (1.27)$$

$$\frac{\partial \ln L(\alpha,\beta;\underset{\sim}{x})}{\partial \beta} = \frac{-n}{\beta} + \frac{n\alpha}{\beta^2} - \frac{1}{\beta^2}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{(x_i-\alpha)}{\beta^2} e^{\frac{(x_i-\alpha)}{\beta}} \quad \text{.........} \quad (1.28)$$

Set $\frac{\partial \ln L(\alpha,\beta;\underset{\sim}{x})}{\partial \alpha} = \frac{\partial \ln L(\alpha,\beta;\underset{\sim}{x})}{\partial \beta} = 0 \quad at \ \alpha=\hat{\alpha}, \beta=\hat{\beta}$ then

From eq. (1.27), we have

$$\frac{1}{\hat{\beta}}\sum_{i=1}^{n} e^{\frac{(x_i-\hat{\alpha})}{\hat{\beta}}} - \frac{n}{\hat{\beta}} = 0 \quad \text{..} \quad (1.29)$$

From eq. (1.28), we have

$$\frac{-n}{\hat{\beta}} + \frac{n\hat{\alpha}}{\hat{\beta}^2} - \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{(x_i-\hat{\alpha})}{\hat{\beta}^2} e^{\frac{(x_i-\hat{\alpha})}{\hat{\beta}}} = 0 \quad \text{.........................} \quad (1.30)$$

solution for $\hat{\alpha}$ and $\hat{\beta}$ cannot be found analytically because of the nonlinearity of eq's. (1.29) and (1.30).

An approximate solution for $\hat{\alpha}$ and $\hat{\beta}$ from eq's. (1.29) and (1.30) can be made iteratively by using Newton-Raphson method for solving a non-linear eq's. as follows:

Suppose that:

$$f_1 = f_1(\hat{\alpha}, \hat{\beta}) = \frac{1}{\hat{\beta}}\sum_{i=1}^{n} e^{\frac{x_i-\hat{\alpha}}{\hat{\beta}}} - \frac{n}{\hat{\beta}} \quad \text{and}$$

$$f_2 = f_2(\hat{\alpha}, \hat{\beta}) = \frac{-n}{\hat{\beta}} + \frac{n\hat{\alpha}}{\hat{\beta}^2} - \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{(x_i-\hat{\alpha})}{\hat{\beta}^2} e^{\frac{(x_i-\hat{\alpha})}{\hat{\beta}}}$$

and let $(\hat{\alpha}_{(0)}, \hat{\beta}_{(0)})$ be given initial approximation. If $(\hat{\alpha}_{(s)}, \hat{\beta}_{(s)})$ is the approximate solution of $(\hat{\alpha},\hat{\beta})$ at stage (s), $s = 0,1,2,.......$, then the approximate solution at stage s+1 is given by:

$$\hat{\alpha}_{(s+1)} = \hat{\alpha}_{(s)} + \delta_1 \quad \dots\dots (1.31)$$

$$\hat{\beta}_{(s+1)} = \hat{\beta}_{(s)} + \delta_2 \quad \dots\dots (1.32)$$

In matrix form, we may write:

$$\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} = -\begin{bmatrix}\frac{\partial f_1}{\partial\hat{\alpha}_{(s)}} & \frac{\partial f_1}{\partial\hat{\beta}_{(s)}}\\ \frac{\partial f_2}{\partial\hat{\alpha}_{(s)}} & \frac{\partial f_2}{\partial\hat{\beta}_{(s)}}\end{bmatrix}^{-1}\begin{bmatrix}f_1\\ f_2\end{bmatrix} \quad \dots\dots (1.33)$$

Provided that:

$$\begin{vmatrix}\frac{\partial f_1}{\partial\hat{\alpha}_{(s)}} & \frac{\partial f_1}{\partial\hat{\beta}_{(s)}}\\ \frac{\partial f_2}{\partial\hat{\alpha}_{(s)}} & \frac{\partial f_2}{\partial\hat{\beta}_{(s)}}\end{vmatrix} \neq 0$$

Set:

$$a = \frac{\partial f_1}{\partial\hat{\alpha}_{(s)}} = \frac{-1}{\hat{\beta}^2}\sum_{i=1}^{n} e^{\frac{(x_i-\hat{\alpha})}{\hat{\beta}}}$$

$$b = \frac{\partial f_1}{\partial\hat{\beta}_{(s)}} = \frac{\partial f_2}{\partial\hat{\alpha}_{(s)}} = \frac{n}{\hat{\beta}^2} - \frac{1}{\hat{\beta}^2}\sum_{i=1}^{n} e^{\frac{(x_i-\hat{\alpha})}{\hat{\beta}}} - \sum_{i=1}^{n}\frac{(x_i-\hat{\alpha})}{\hat{\beta}^3} e^{\frac{(x_i-\hat{\alpha})}{\hat{\beta}}}$$

$$c = \frac{\partial f_2}{\partial\hat{\beta}_{(s)}} = \frac{n}{\hat{\beta}^2} - 2\frac{n\hat{\alpha}}{\hat{\beta}^3} + \frac{2}{\hat{\beta}^3}\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}\frac{(x_i-\hat{\alpha})}{\hat{\beta}^3} e^{\frac{(x_i-\alpha)}{\beta}}$$

$$-\sum_{i=1}^{n}\left[\frac{x_i-\hat{\alpha}}{\hat{\beta}^2}\right]^2 e^{\frac{(x_i-\hat{\alpha})}{\hat{\beta}}}$$

We have:

$$\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} = -\begin{bmatrix}a & b\\ b & c\end{bmatrix}^{-1}\begin{bmatrix}f_1\\ f_2\end{bmatrix} = -\frac{1}{ac-b^2}\begin{bmatrix}c & -b\\ -b & a\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix}, \text{ for } ac-b^2 \neq 0$$

Then:

$$\delta_1 = -\frac{1}{ac-b^2}(cf_1 - bf_2) \quad \text{and} \quad \delta_2 = -\frac{1}{ac-b^2}(-bf_1 + af_2)$$

and according eqs. (1.31) and (1.32), we have:

$$\hat{\alpha}_{(s+1)} = \hat{\alpha}_{(s)} - \frac{1}{ac-b^2}(cf_1 - bf_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.34)$$

$$\hat{\beta}_{(s+1)} = \hat{\beta}_{(s)} - \frac{1}{ac-b^2}(-bf_1 + af_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.35)$$

where the initial values $\alpha_0 = \alpha$ and $\beta_0 = \beta$.

1.4.1.3 Estimation of Parameters by Order Statistic Method, [29]:

This method can be described as follows:

Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from distn. p.d.f. $f(x, \underset{\sim}{\theta})$ where $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, is a vector of k-unknown parameters.

Let $Y_1 < Y_2 < \ldots < Y_n$ represent the arrangement of the sample set $\{X_i\}$ in ascending order of magnitude. Let $\mu'_r = E(X^r)$ be the r^{th} distn. moment about the origin and $M_r = \frac{1}{n}\sum_{i=0}^{n} x_i^r$ is the r^{th} sample moment about the origin, $r = 1, 2, 3, \ldots$

In this method, we equate $\mu'_1 = M_1$ at $\theta_i = \hat{\theta}_i$, i =1,2,…k and ranking $E(Y_i) = Y_i$ beginning with i = 1 until i = k-1 this process will gives k eqs.

to provide a unique solution for θ_i at $\hat{\theta}_i$, i = 1, 2, … , k.

For extreme value distn. case:

We have two unknown parameters α and β and if we take a r.s. of size n from Ext(α, β), we let Y_1 represent the first order statistic of the sample.

From the statistic theory the p.d.f. of Y_1 is:

$g_1(y_1) = n[1 - F(y_1)]^{n-1} f(y_1)$

where $f(y_1)$ and $F(y_1)$ represent the p.d.f. and c.d.f. of y_1 as given in eqs. (1.1) and (1.2) respectively. Then

$$g_1(y_1) = n\left[1-[1-e^{-e^{(\frac{y_1-\alpha}{\beta})}}]\right]^{n-1} \frac{1}{\beta} e^{\frac{y_1-\alpha}{\beta} - e^{(\frac{y_1-\alpha}{\beta})}}$$

$$= \frac{n}{\beta} e^{[\frac{y_1-\alpha}{\beta} - ne^{\frac{(y_1-\alpha)}{\beta}}]} , -\infty < y_1 < \infty \ , -\infty < \alpha < \infty \ , \beta > 0$$

To find $E(Y_1)$,we consider the m.g.f. of Y_1

$$M_{Y_1}(t) = E(e^{ty_1}) = \int_{-\infty}^{\infty} e^{ty_1} \frac{n}{\beta} e^{[\frac{y_1-\alpha}{\beta} - ne^{\frac{(y_1-\alpha)}{\beta}}]} dy_1$$

Let $z = \dfrac{y_1 - \alpha}{\beta}$, then $\beta\, dz = dy_1$

$$M_{Y_1}(t) = n\int_{-\infty}^{\infty} e^{t(\alpha+\beta z)} e^{(z-ne^z)} dz = ne^{\alpha t} \int_{-\infty}^{\infty} (e^z)^{\beta t} e^{-n(e^z)} e^z dz$$

Let $u = e^z$ implies $du = e^z\, dz$

$$M_{Y_1}(t) = ne^{\alpha t}\int_0^{\infty}(u)^{\beta t}e^{-un}\,du = ne^{\alpha t}\int_0^{\infty}u^{(1+\beta t)-1}e^{\frac{-u}{1/n}}\,du = \frac{ne^{\alpha t}\Gamma(1+\beta t)}{n^{1+\beta t}}$$

$$= \frac{e^{\alpha t}\Gamma(1+\beta t)}{n^{\beta t}} \quad \ldots\ldots (1.36)$$

where $\Gamma(w) = \int_0^{\infty} y^{w-1}e^{-y}\,dy$, $w > 0$ is called gamma distn..

Set $\Phi_{Y_1}(t) = \text{Ln } M_{Y_1}(t) = \alpha t + \text{Ln } \Gamma(1 + \beta t) - \beta t \text{ Ln } (n)$

$\Phi'_{Y_1}(t) = \alpha + \beta\psi(1 + \beta t) - \beta \ln(n) = \alpha + \beta[\psi(1 + \beta t) - \ln(n)]$

$\Phi'_{Y_1}(0) = \alpha + \beta[\psi(1) - \ln(n)] = E(Y_1)$ …………………………….. (1.37)

where $\Psi(z) = \frac{d}{dz}\ln\Gamma(z)$ is known as digamma function.

Now, we apply the order statistic method by setting:

$\mu'_1 = \hat{\alpha} - \gamma\hat{\beta} = \bar{X}$ and $E(Y_1) = Y_1$ at $\alpha = \hat{\alpha}$, $\beta = \hat{\beta}$, which leads to:

$\hat{\alpha} + \hat{\beta}[\psi(1) - \ln(n)] = Y_1$ ………………………………..… (1.38)

$\hat{\alpha} = \bar{X} + \gamma\hat{\beta}$ ……………………………………………….. (1.39)

From eqs. (1.38) and (1.39) the estimators of β and α are respectively:

$$\hat{\beta} = \frac{\bar{X} - Y_1}{\ln(n)} \quad \ldots\ldots (1.40)$$

$\hat{\alpha} = \bar{X} + \gamma\hat{\beta}$ ………………………………………..……….(1.41)

where $\psi(1) = -\gamma = -0.577$.

The estimators $\hat{\alpha}$ and $\hat{\beta}$ given by eqs. (1.40) and (1.41), have the following properties:

(i) $\hat{\beta} = \dfrac{\bar{X} - Y_1}{\ln(n)}$ is an unbiased estimator for β, and its variance approach to zero.

Proof:

Since $E(\hat{\beta}) = E\left[\dfrac{\bar{X} - Y_1}{\ln(n)}\right] = \dfrac{1}{\ln(n)}\left[E(\bar{X}) - E(Y_1)\right]$

$$= \frac{1}{\ln(n)}[\mu - \alpha + \beta(\ln(n) + \gamma)]$$

$$= \frac{1}{\ln(n)}[\alpha - \gamma\beta - \alpha + \beta\ln(n) + \gamma\beta)] = \beta \quad \text{......................(1.42)}$$

$$E(\hat{\beta}^2) = \frac{1}{(\ln(n))^2}E[(\bar{X} - Y_1)^2] = \frac{1}{(\ln(n))^2}[E(\bar{X}^2) - 2E(\bar{X}Y_1) + E(Y_1^{\,2})]$$

$$= \frac{1}{(\ln(n))^2}[\frac{\sigma^2}{n} + \mu^2 - 2E(\bar{X}Y_1) + \sigma^2 + (\mu - \beta\ln(n))^2]$$

$$E(\bar{X}Y_1) = E(\frac{1}{n}Y_1\sum_{i=1}^{n}X_i) = \frac{1}{n}E[\min(X_i)\sum_{i=1}^{n}X_i]$$

$$= \frac{1}{n}E[\min(X_i)X_1 + \min(X_i)X_2 + \ldots + (\min(X_i))^2 + \ldots + \min(X_i)X_n]$$

$$= \frac{1}{n}E\left[X_i^2 + \sum_{\substack{j=1 \\ i\neq j}}^{n-1}X_iX_j\right] = \frac{1}{n}\left[E(X_i^2) + \sum_{\substack{j=1 \\ i\neq j}}^{n-1}E(X_iX_j)\right]$$

$$= \frac{1}{n}\left[E(X_i^2) + \sum_{\substack{j=1 \\ i\neq j}}^{n-1}E(X_i)E(X_j)\right]$$

Where X_i and X_j are independent

$$E(\bar{X}Y_1) = \frac{1}{n}\left[Var(X_i) + \left[E(X_i)\right]^2 + \sum_{\substack{j=1 \\ i\neq j}}^{n-1}E(X_i)E(X_j)\right]$$

$$E(\bar{X}Y_1)=\frac{1}{n}\Big[\sigma^2+\mu^2+\sum_{\substack{j=1\\ i\neq j}}^{n-1}\mu^2\Big]=\frac{1}{n}\Big[\sigma^2+\mu^2+(n-1)\mu^2\Big]=\frac{1}{n}\Big[\sigma^2+n\mu^2\Big]$$

Hence:

$$E(\hat{\beta}^2)=\frac{1}{(\ln(n))^2}\left[\frac{\sigma^2}{n}+\mu^2-2(\frac{\sigma^2}{n}+\mu^2)+\sigma^2+(\mu-\beta\ln(n))^2\right]$$

$$=\frac{1}{(\ln(n))^2}\left[\frac{\sigma^2}{n}+\mu^2-2\frac{\sigma^2}{n}-2\mu^2+\sigma^2+\mu^2-2\mu\beta\ln(n)+(\beta\ln(n))^2\right]$$

$$=\frac{1}{(\ln(n))^2}\left[\sigma^2-\frac{\sigma^2}{n}-2\mu\beta\ln(n)+(\beta\ln(n))^2\right]$$

$$=\frac{\sigma^2}{(\ln(n))^2}-\frac{\sigma^2}{n(\ln(n))^2}-\frac{2\mu\beta}{\ln(n)}+\beta^2$$

$$\lim_{n\to\infty}E(\hat{\beta}^2)\approx 0-0-0+\beta^2=\beta^2 \quad \text{...} \quad (1.43)$$

From eqs. (1,42) and (1.43), gives

$$\text{Var}(\hat{\beta})=E(\hat{\beta}^2)-E[(\hat{\beta})]^2\;;\;\beta^2-\beta^2=0 \quad \text{.................................} \quad (1.44)$$

(ii) $\hat{\alpha}=\bar{X}+\gamma\hat{\beta}$ is an unbiased estimator for α, and its variance approach to zero.

Proof:

Since $E(\hat{\alpha})=E(\bar{X}+\gamma\hat{\beta})=E(\bar{X})+\gamma E(\hat{\beta})=\mu+\gamma\beta=\alpha-\gamma\beta+\gamma\beta$

$$=\alpha \quad \text{...} \quad (1.45)$$

$$E(\hat{\alpha}^2)=E[(\bar{X}+\gamma\hat{\beta})^2]=E[(\bar{X}^2+2\gamma\hat{\beta}\bar{X}+\gamma^2\hat{\beta}^2]$$

$$=E(\bar{X}^2)+2\gamma E(\bar{X}\hat{\beta})+\gamma^2E(\hat{\beta}^2)$$

$$=\text{Var}(\bar{X})+[E(\bar{X})]^2+2\gamma E(\hat{\beta})E(\bar{X})+\gamma^2E(\hat{\beta}^2)$$

$$=\frac{\sigma^2}{n}+\mu^2+2\gamma\beta\mu+\gamma^2\beta^2$$

$$E(\hat{\alpha}^2) = \frac{\sigma^2}{n} + (\alpha - \gamma\beta)^2 + 2\gamma\beta(\alpha - \gamma\beta) + \gamma^2\beta^2$$

$$= \frac{\pi^2\beta^2}{6n} + \alpha^2 - 2\alpha\beta\gamma + \gamma^2\beta^2 + 2\gamma\alpha\beta - 2\gamma^2\beta^2 + \gamma^2\beta^2 = \frac{\pi^2\beta^2}{6n} + \alpha^2$$

$$\lim_{n\to\infty} E(\hat{\alpha}^2) = \alpha^2 \qquad (1.46)$$

From eqs. (1.45) and (1.46), gives

$$Var(\hat{\alpha}) = E(\hat{\alpha}^2) - [E(\hat{\alpha})]^2 \; ; \; \alpha^2 - \alpha^2 = 0 \qquad (1.47)$$

1.4.1.4 Estimation of Parameters by Least Squares Method, [29]

The Least squares method is a general technique for estimating parameters in fitting a set of points to generate a curve whose trend might be linear, quadratic, or of higher order. In order to utilize this method, the error terms due to experiment must satisfy the following conditions:

(i) They have zero mean.

(ii) They have the same variance.

(iii) They must be uncorrelated.

For good results of fitting curve to the data set, the error must be minimized as small as possible.

Let us assume that we have a set of n data points (x_i, t_i) through which we desire to pass a straight line. This line is representing the best fit in the least square sense.

Suppose that the best fitting straight line to the data (x_i, t_i) is $x = \lambda_0 + \lambda_1 t$, where λ_0 and λ_1 are two unknown parameters representing respectively the vertical intercept and the slope, as shown in fig (1.3).

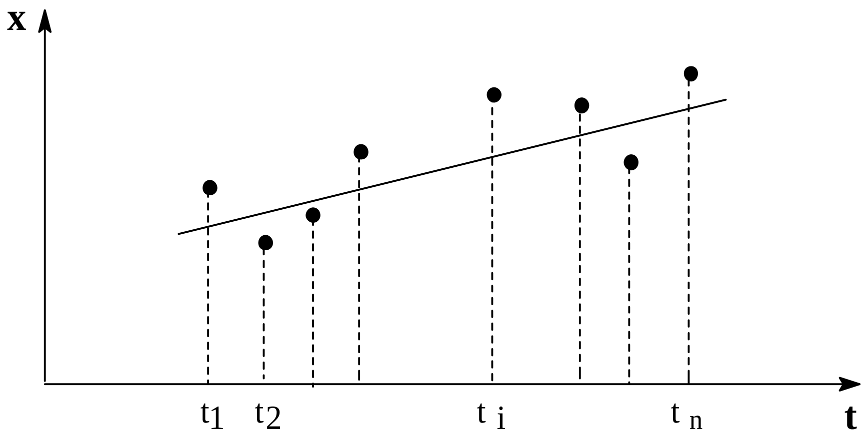


Figure (1.3) The best fitted line to the data (x_i, t_i).

The ordinate x_i as given by the general line is $\lambda_0 + \lambda_1 t_i$. The difference between these two values is the error of fit at the i^{th} point $e_i = x_i - (\lambda_0 + \lambda_1 t_i)$.

Let the sum of squares of all errors at the data points be:

$$\Omega = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (x_i - \lambda_0 - \lambda_1 t_i)^2$$

For minimum, we set:

$$\frac{\partial \Omega}{\partial \lambda_0} = 0 \text{ and } \frac{\partial \Omega}{\partial \lambda_1} = 0, \text{ at } \lambda_0 = \hat{\lambda}_0, \lambda_1 = \hat{\lambda}_1$$

$$\left.\frac{\partial \Omega}{\partial \lambda_0}\right|_{\substack{\lambda_0=\hat{\lambda}_0 \\ \lambda_1=\hat{\lambda}_1}} = -2\sum_{i=1}^{n} (x_i - \hat{\lambda}_0 - \hat{\lambda}_1 t_i) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.48)$$

$$\left.\frac{\partial \Omega}{\partial \lambda_1}\right|_{\substack{\lambda_0=\hat{\lambda}_0 \\ \lambda_1=\hat{\lambda}_1}} = -2\sum_{i=1}^{n} (x_i - \hat{\lambda}_0 - \hat{\lambda}_1 t_i) t_i = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.49)$$

From (1.48) and (1.49), we can get two eqs. as:

$$n\hat{\lambda}_0 + \hat{\lambda}_1 \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} x_i \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.50)$$

$$\hat{\lambda}_0 \sum_{i=1}^{n} t_i + \hat{\lambda}_1 \sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} t_i x_i \quad \ldots\ldots (1.51)$$

Equations (1.50) and (1.51) are simultaneous algebraic eqs. for the two parameters λ_0 and λ_1.

In matrix notation (1.50) and (1.51) may be written as:

$$\underset{\sim}{A}\, \underset{\sim}{\hat{\lambda}} = \underset{\sim}{b} \quad \ldots\ldots (1.52)$$

where:

$$\underset{\sim}{A} = \begin{bmatrix} n & \sum_{i=1}^{n} t_i \\ \sum_{i=1}^{n} t_i & \sum_{i=1}^{n} t_i^2 \end{bmatrix}, \quad \underset{\sim}{\hat{\lambda}} = \begin{bmatrix} \hat{\lambda}_0 \\ \hat{\lambda}_1 \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} t_i x_i \end{bmatrix}$$

The solution of eq.(1.52) is:

$$\underset{\sim}{\hat{\lambda}} = \underset{\sim}{A}^{-1} \underset{\sim}{b} \text{ if and only if } |\underset{\sim}{A}| \neq 0.$$

Thus, whenever the data points t_i, $\forall$ i are given, then the matrix $\underset{\sim}{A}$ and the vector $\underset{\sim}{b}$ may be computed and hence $\underset{\sim}{\hat{\lambda}}$ is determined as follows:

$$\hat{\lambda}_0 = \frac{\bar{X} \sum_{i=1}^{n} t_i^2 - \bar{t} \sum_{i=1}^{n} t_i x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t} \sum_{i=1}^{n} t_i} \quad \ldots\ldots (1.53)$$

$$\hat{\lambda}_1 = \frac{\sum_{i=1}^{n} t_i x_i - \bar{t} \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t} \sum_{i=1}^{n} t_i} \quad \ldots\ldots (1.54)$$

provided that $\left(\sum_{i=1}^{n} t_i^2 - \bar{t} \sum_{i=1}^{n} t_i \right) \neq 0$, where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$, $\bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i$.

For extreme value distn. case:

Suppose that $X_1, X_2, \ldots, X_n$ be a random sample of size n from extreme value distn. having cumulative function:

$$F(x) = \Pr(X \le x) = 1 - e^{-e^{\frac{(x-\alpha)}{\beta}}}, \quad -\infty < x < \infty$$

We set $u_i = F(x_i)$, then $u_i = 1 - e^{-e^{\frac{(x-\alpha)}{\beta}}}$, which implies:

$$x_i = \alpha + \beta \, \mathrm{Ln}(-\ln(u_i)), \; i = 1, 2, \ldots, n \quad \ldots\ldots (1.55)$$

where $0 < u < 1$ implies $0 < 1\text{-}u < 1$.

Set $y_i = x_i$, $t_i = \mathrm{Ln}(-\ln(u_i))$, $i = 1, 2, \ldots, n$ and $\hat{\lambda}_0 = \alpha$, $\hat{\lambda}_1 = \beta$.

Then $y_i = \lambda_0 + \lambda_1 t_i$, $i = 1, 2, \ldots, n$; where $\hat{\alpha} = \hat{\lambda}_0$, $\hat{\beta} = \hat{\lambda}_1$

Utilizing eq.(1.52) for obtaining the estimator $\hat{\lambda}_0$ and $\hat{\lambda}_1$, therefore;

The least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ can be obtained from the (1.55):

$$\hat{\alpha} = \hat{\lambda}_0 \quad \ldots\ldots (1.56)$$

$$\hat{\beta} = \hat{\lambda}_1 \quad \ldots\ldots (1.57)$$

The estimators $\hat{\alpha}$ and $\hat{\beta}$ given by (1.56) and (1.57) have the following properties:

(i) $\hat{\beta} = \dfrac{\sum_{i=1}^{n} t_i x_i - \bar{t}\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i}$ is an unbiased estimator.

Set $S_{tt} = \sum_{i=1}^{n}(t_i - \bar{t})^2 = \sum_{i=1}^{n}(t_i - \bar{t})t_i = \sum_{i=1}^{n} t_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} t_i\right)^2$

$$S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i - \bar{x})x_i = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$$

$$S_{tx} = \sum_{i=1}^{n}(t_i - \bar{t})(x_i - \bar{x}) = \sum_{i=1}^{n}(t_i - \bar{t})x_i = \sum_{i=1}^{n} t_i x_i - \frac{1}{n}\left(\sum_{i=1}^{n} t_i\right)\left(\sum_{i=1}^{n} x_i\right)$$

So $\hat{\beta}$ may be written as : $\hat{\beta} = \dfrac{S_{tx}}{S_{tt}}$

Since $$E(\hat{\beta}) = E\left(\frac{S_{tx}}{S_{tt}}\right) = \frac{1}{S_{tt}} E(S_{tx}) = \frac{1}{S_{tt}} E\left[\sum_{i=1}^{n}(t_i - \bar{t})x_i\right]$$

$$= \frac{1}{S_{tt}} \sum_{i=1}^{n}(t_i - \bar{t})E(x_i) = \frac{1}{S_{tt}} \sum_{i=1}^{n}(t_i - \bar{t})(\alpha + \beta t_i)$$

$$= \frac{1}{S_{tt}}\left[\alpha \sum_{i=1}^{n}(t_i - \bar{t}) + \beta \sum_{i=1}^{n}(t_i - \bar{t})t_i\right]$$

Hence:

$$E(\hat{\beta}) = \frac{1}{S_{tt}}(\beta S_{tt}) = \beta \quad \text{.......... (1.58)}$$

since $$Var(\hat{\beta}) = Var\left(\frac{S_{tx}}{S_{tt}}\right) = \frac{1}{S_{tt}^2} Var(S_{tx}) = \frac{1}{S_{tt}^2} Var\left[\sum_{i=1}^{n}(t_i - \bar{t})x_i\right]$$

$$= \frac{1}{S_{tt}^2} \sum_{i=1}^{n} Var\left[(t_i - \bar{t})x_i\right] = \frac{1}{S_{tt}^2} \sum_{i=1}^{n}(t_i - \bar{t})^2 Var(x_i) = \frac{1}{S_{tt}^2} \sum_{i=1}^{n}(t_i - \bar{t})^2 \sigma^2$$

Because $x_1, \mathbf{K}, x_n$ are independent hence:

$$Var(\hat{\beta}) = \frac{1}{S_{tt}^2}\sigma^2 S_{tt} = \frac{\sigma^2}{S_{tt}} = \frac{\pi^2\beta^2}{6\sum_{i=1}^{n}(t_i - \bar{t})^2} \quad \text{.......... (1.59)}$$

(ii) $\hat{\alpha}$ is an unbiased estimator. From eq.(1.50), we have:

Since $\hat{\alpha} = \bar{X} - \hat{\beta}\,\bar{t}$ implies $E(\hat{\alpha}) = E(\bar{X} - \hat{\beta}\,\bar{t}) = E(\bar{X}) - \bar{t}\,E(\hat{\beta})$

Since $x_i = \alpha + \beta t_i + e_i$, then:

$$\sum_{i=1}^{n} x_i = n\alpha + \beta \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} e_i$$

Which implies that :

$\bar{X} = \alpha + \beta\,\bar{t} + \bar{e}$ implies $E(\bar{X}) = \alpha + \beta\,\bar{t} + 0$

where $E(\bar{e}) = 0$

Hence:

$$E(\hat{\alpha}) = \alpha + \beta\bar{t} - \beta\bar{t} = \alpha \quad \text{.......................................} (1.60)$$

Since $\operatorname{Var}(\hat{\alpha}) = \operatorname{Var}(\bar{X} - \hat{\beta}\,\bar{t}) = \operatorname{Var}(\bar{X}) + \bar{t}^{\,2}\operatorname{Var}(\hat{\beta}) - 2\bar{t}\operatorname{Cov}(\bar{X}, \hat{\beta})$

$$= \frac{\pi^2\beta^2}{6n} + \bar{t}^{\,2}\frac{\pi^2\beta^2}{6\sum_{i=1}^{n}(t_i - \bar{t})^2} - 2\bar{t}\operatorname{Cov}(\bar{X}, \hat{\beta})$$

Since $\operatorname{Cov}(\bar{X}, \hat{\beta}) = \operatorname{Cov}\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{S_{tx}}{S_{tt}}\right)$

$$= \frac{1}{nS_{tt}}\operatorname{Cov}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n}(t_i - \bar{t})x_i\right)$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n}\operatorname{Cov}(x_i, (t_i - \bar{t})x_i)$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n}\left[E\left[(t_i - \bar{t})x_i^2\right] - E(x_i)E\left[(t_i - \bar{t})x_i\right]\right]$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n}(t_i - \bar{t})\left[E(x_i^2) - (E(x_i))^2\right]$$

$$= \frac{1}{nS_{tt}}\sum_{i=1}^{n}(t_i - \bar{t})\operatorname{Var}(x_i) = \frac{1}{nS_{tt}}\sigma^2\sum_{i=1}^{n}(t_i - \bar{t})$$

$$\lim_{n\to\infty}\operatorname{Cov}(\bar{X}, \hat{b}) = 0$$

$$\operatorname{Var}(\hat{\alpha}) \to \frac{\pi^2\beta^2}{6}\left[\frac{1}{n} + \frac{\bar{t}^{\,2}}{\sum_{i=1}^{n}(t_i - \bar{t})^2}\right] \quad \text{...} (1.61)$$

1.5 Reliability and Hazard Functions of Extreme Value Distn.

In this section, we illustrate some concepts, relations, properties, estimation for the reliability and hazard functions.

1.5.1 Some Concepts of Reliability and Hazard Functions

Initially, we shall represent the definition of the reliability and hazard functions. Billington and Allen (1983). [4] define reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered. Chicken & Posner (1998) define hazard as situation can cause harm. Harm is taken to imply injury, damage, loss of performance and finances [6].

Reliability theory is a general theory about systems failure. Reliability theory was originally developed for estimating the reliabilities of physical devices. The source of the reliability failures of physical devices is typically the physical deterioration of the materials used in their construction. This physical deterioration provides the basis of stochastic reliability modeling, since the deterioration is assumed to vary randomly with time. It predicts the late-life mortality deceleration with subsequent leveling-off, as well as the late-life mortality plateaus. The theory explains why mortality rates (hazard rates) increase exponentially with age (Gompertz law) in many species, by taking into account the initial flaws (defects) in newly formed systems. It also explains why organisms prefer to die according to the Gompertz law, while technical devices usually fail according to the Weibull (power) law. Theoretical conditions are specified when organisms die according to the Weibull law: organisms should be relatively free of initial flaws and defects. The theory makes it possible to find a general failure law applicable to all adult and extreme old ages, where the Gompertz and the Weibull laws are just special cases of this more general failure law. Therefore, reliability theory seems to be a promising approach for developing a comprehensive theory of aging and longevity integrating mathematical methods with specific biological knowledge. [28]

In the reliability modeling, minimum extreme value distributions are frequently encountered, e.g., if a system consists of *n* identical components in series, and the system fails when the first of these components fails, then system failure times are the minimum of *n* random component failure times. Extreme value theory says that, independent of the choice of component model, the system model will approach a Weibull as *n* becomes large. The same reasoning can also be applied at a component level, if the component failure occurs when the first of many similar competing failure processes reaches a critical level. The reliability function is denoted by $R(x;\alpha,\beta)$, is given by

$$\mathrm{R}(x;a,b) = \Pr(X > x) = 1-\Pr(X \le x) = 1-F(x;a,b) \quad \text{………. (1.62)}$$

For engineering systems, failure rates or hazard rates are terms applied to the first failure times for a population of non-repairable components or to non-repairable systems. The failure rate is defined for non-repairable populations as the (instantaneous) rate of failure for the survivors to time *x* during the next instant of time. The failure rate (or hazard rate) is denoted by $h(x;\alpha,\beta)$ and calculated from

$$h(x;a,b) = \frac{f(x;a,b)}{1-F(x;a,b)} = \frac{f(x;a,b)}{R(x;a,b)} \quad \text{…………………………….... (1.63)}$$

The failure rate is sometimes called a conditional failure rate. The cumulative hazard function, denoted by $H(x;\alpha,\beta)$, [26] is defined to be

$$H(x;a,b) = \int_0^x h(w;a,b)\,dw \quad \text{……………………………………… (1.64)}$$

The failure rate function is often used to indicate the health condition of a working device. A high failure rate indicates a bad health condition because the probability for the device to fail in the next instant of time is high. [30]

1.5.2 Some Important Relations, [30][26]

It is obvious that one of the functions $f(x;a,b), F(x;a,b)$, $R(x;a,b)$, $h(x;a,b)$, $H(x;a,b)$ is adequate to specify completely the lifetime distribution of a device. These functions are satisfy the well-known relations

1- $h(x;a,b)=\dfrac{\frac{-d}{dx}R(x;a,b)}{R(x;a,b)}=\dfrac{-d}{dx}\ln[R(x;a,b)]$

2- $R(x;a,b)=e^{-\int_0^x h(w\, a,b)dw}$

3- $f(x;a,b)=h(x;a,b).R(x;a,b)=h(x;a,b).e^{-\int_0^x h(w;a,b)dw}$

4- $F(x;a,b) = 1 - e^{-H(x;a,b)}$

5- $H(x;a,b) = -\ln R(x;a,b)$

1.5.3 Properties of Reliability and Hazard Functions of the Extreme Value Distribution, [26]

In this section, we shall give some mathematical properties of the reliability and the hazard functions of the extreme value distn.

The reliability function of the extreme value distn. can be obtained in terms of the c.d.f. of eq.(1.2) as follows

$$R(x;\alpha,\beta) = 1 - F(x;\alpha,\beta) = 1-\left[1-e^{-e^{\frac{x-\alpha}{\beta}}}\right]=e^{-e^{\frac{x-\alpha}{\beta}}} \quad \text{……………..} \ (1.65)$$

The reliability function given by (1.65) satisfy the following:

1- $0 < R(x;a,b) \le 1$

2- R(0) = 1 and R(∞) = 0.

3- The function R(x) is a non-increasing function of x.

4- The function R(x) is continuous from the left at each x.

The hazard function of the extreme value distn. can be obtained in terms of the p.d.f of eq.(1.1) and the reliability function of eq. (1.65) as follows:

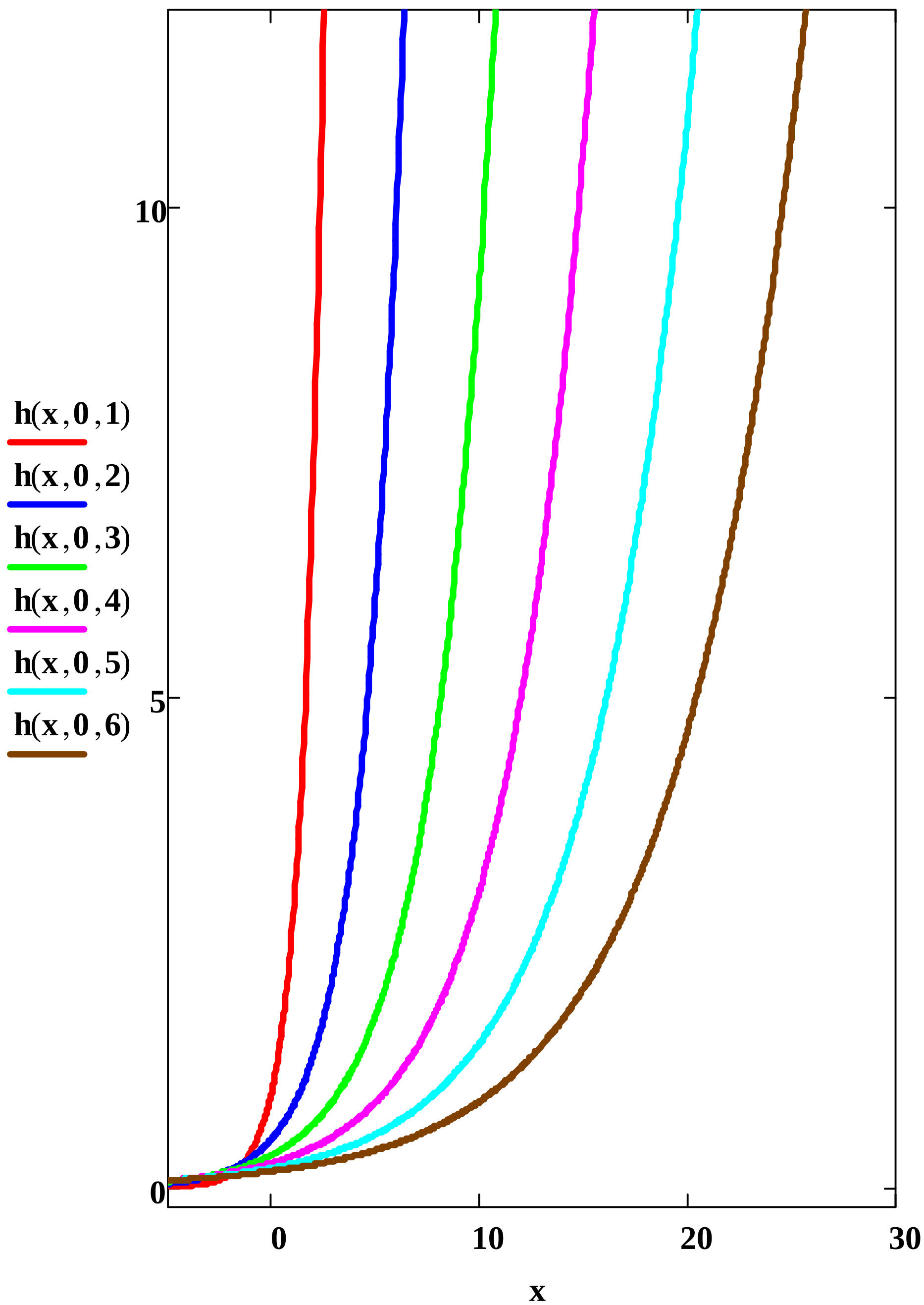
$$h(x;\alpha,\beta) = \frac{f(x;a,b)}{R(x;a,b)} = \frac{\frac{1}{\beta}e^{\frac{x-\alpha}{\beta}-e^{\frac{x-\alpha}{\beta}}}}{e^{-e^{\frac{x-\alpha}{\beta}}}} = \frac{1}{\beta}e^{\frac{x-\alpha}{\beta}} \quad \text{……..…………….} (1.66)$$

The hazard function given by (1.66) satisfy the following:

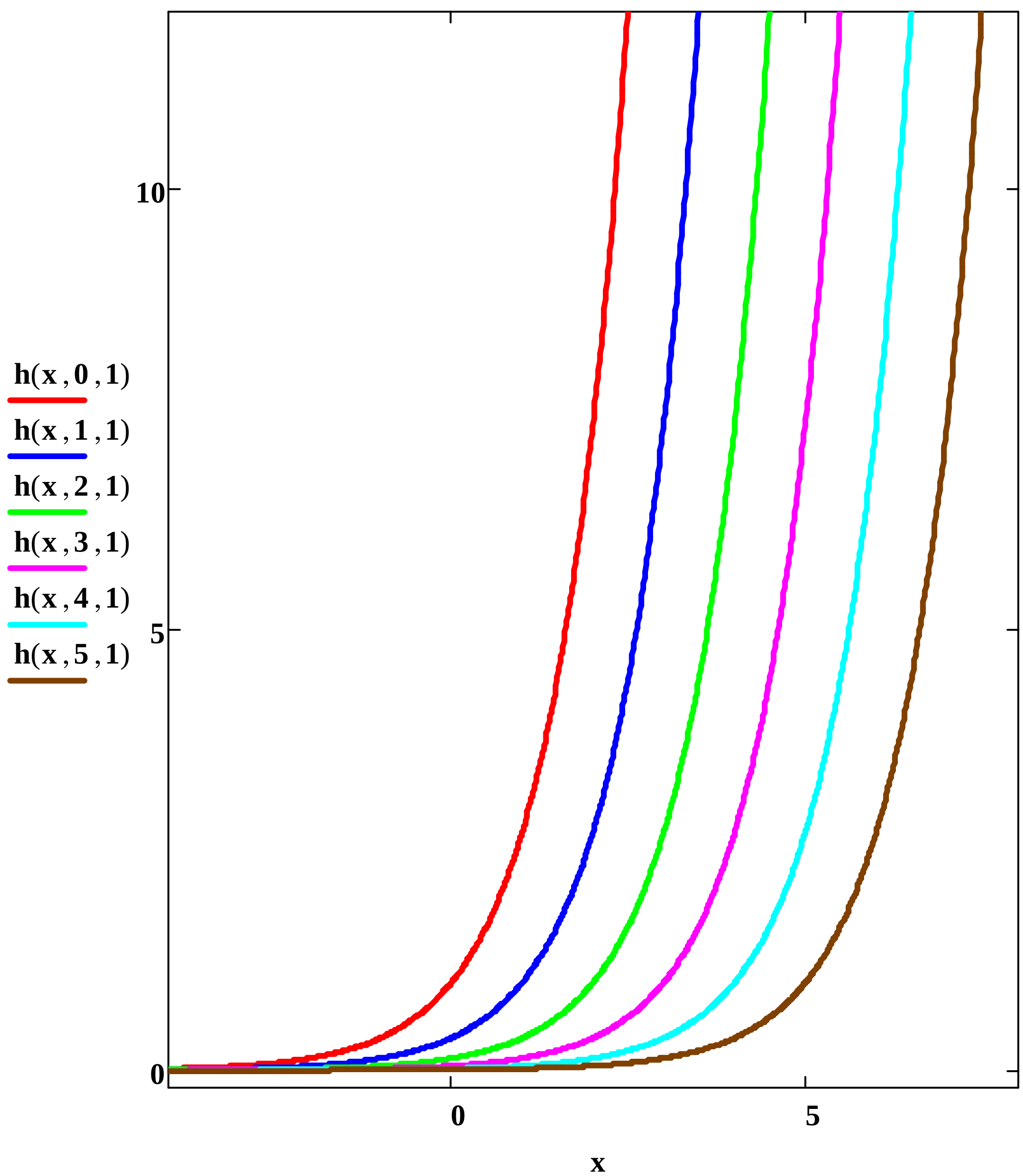
1. *h(x;α,β)* is an increasing function for all x and it is Concave upward for

$$\frac{1}{b}e^{-\frac{a}{b}} < x < \infty .$$

Several typical failure rate curves are given in Figures (1.4.a) and (1.4.b). Inspection of these curves makes it obvious that the failure rate is monotonic for all α and b.



Fig(1.4.a): failure rate of Extreme Value distribution with α = 0 and b = 1, 2, 3, 4, 5, 6



Fig(1.4.b): failure rate of Extreme Value distribution with

α = 0, 1, 2, 3, 4, 5 and b =1

1.5.4 Estimation of the Reliability and Hazard Functions of Extreme Value Distribution

The estimators of the parameters by four methods of estimation that given in section (1.4) can be used to estimate the reliability and the hazard functions as follows.

1.5.4.1 Estimation by M.M

The M.M estimators of β and α as given by (1.18) and (1.19) are

$$\hat{\beta} = \frac{S}{p}\sqrt{\frac{6(n-1)}{n}} \qquad \text{and} \qquad \hat{\alpha} = \bar{X} + g\hat{b}$$

accordingly the estimators of $R(x;\alpha,\beta)$ and $h(x;\alpha,\beta)$ is now obtained by replacing α and b in (1.65) and (1.66) by their estimates $\hat{a}$ *and* $\hat{b}$ given in (1.18) and (1.19). Accordingly the estimator of *R(x)* is:

$$\hat{R}_{M.M}(x;\hat{\alpha}_{M.M},\hat{\beta}_{M.M}) = e^{-e^{\frac{x-\hat{\alpha}}{\hat{\beta}}}} \qquad \text{.....................................(1.67)}$$

and the estimator of *h(x)* is:

$$\hat{h}_{M.M}(x;\hat{\alpha}_{M.M},\hat{\beta}_{M.M}) = \frac{1}{\hat{\beta}}e^{(\frac{x-\hat{\alpha}}{\hat{\beta}})} \qquad \text{................................. (1.68)}$$

1.5.4.2 Estimation by MLM

The M.L.M. of α and β as given by (1.34) and (1.35) are

$$\hat{\alpha}_{(s+1)} = \hat{\alpha}_{(s)} - \frac{1}{ac-b^2}(cf_1 - bf_2) \quad \text{and} \quad \hat{\beta}_{(s+1)} = \hat{\beta}_{(s)} - \frac{1}{ac-b^2}(-bf_1 + af_2)$$

accordingly the estimators of $R(x;\alpha,\beta)$ and $h(x;\alpha,\beta)$ is now obtained by replacing a and b in (1.65) and (1.66) by their estimates $\hat{a}$ *and* $\hat{b}$ given in (1.34) and (1.35). Accordingly the estimator of *R(x)* is:

$$\hat{R}_{M.L.M}(x;\hat{\alpha}_{M.L.M.},\hat{\beta}_{M.L.M.}) = e^{-e^{\frac{x-\hat{\alpha}}{\hat{\beta}}}} \quad \text{.............................(1.69)}$$

and the estimator of *h(x)* is:

$$\hat{h}_{M.L.M}(x;\hat{\alpha}_{M.L.M.},\hat{\beta}_{M.L.M.}) = \frac{1}{\hat{\beta}}e^{(\frac{x-\hat{\alpha}}{\hat{\beta}})} \quad \text{........................... (1.70)}$$

1.5.4.3 Estimation by O.S.M

The O.S.M estimators of β and α as given by (1.40) and (1.41) are

$$\hat{\beta} = \frac{\bar{X} - Y_1}{\ln(n)} \quad \text{and} \quad \hat{\alpha} = \bar{X} + g\hat{b}$$

accordingly the estimators of R(x;α,β) and h(x;α,β) is now obtained by replacing α and b in (1.65) and (1.66) by their estimates $\hat{a}$ and $\hat{b}$ given in (1.40) and (1.41). Accordingly the estimator of *R(x)* is:

$$\hat{R}_{O.S.M}(x;\hat{\alpha}_{O.S.M.},\hat{\beta}_{O.S.M.}) = e^{-e^{\frac{x-\hat{\alpha}}{\hat{\beta}}}} \quad \text{................................. (1.71)}$$

and the estimator of *h(x)* is:

$$\hat{h}_{O.S.M}(x;\hat{a}_{O.S.M.},\hat{b}_{O.S.M.}) = \frac{1}{\hat{\beta}}e^{(\frac{x-\hat{\alpha}}{\hat{\beta}})} \quad \text{............................... (1.72)}$$

1.5.4.4 Estimation by L.S.M

The L.S.M estimators of β and α as given by (1.56) and (1.57) are

$$\hat{\beta} = \frac{\sum_{i=1}^{n} t_i x_i - \bar{t}\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i} \quad \text{and} \quad \hat{\alpha} = \frac{\sum_{i=1}^{n} t_i x_i - \bar{t}\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} t_i^2 - \bar{t}\sum_{i=1}^{n} t_i}$$

accordingly the estimators of R(x;α,β) and h(x;α,β) is now obtained by replacing α and b in (1.65) and (1.66) by their estimates $\hat{a}$ *and* $\hat{b}$ given in (1.56) and (1.57). Accordingly the estimator of *R(x)* is:

$$\hat{R}_{L.S.M}(x;\hat{\alpha}_{L.S.M.},\hat{\beta}_{L.S.M.}) = e^{-e^{\frac{x-\hat{\alpha}}{\hat{\beta}}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.73)$$

and the estimator of *h(x)* is:

$$\hat{h}_{L.S.M}(x;\hat{\alpha}_{L.S.M.},\hat{\beta}_{L.S.M.}) = \frac{1}{\hat{\beta}} e^{(\frac{x-\hat{\alpha}}{\hat{\beta}})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.74)$$

1.6 Some Related Theorems, [33]

Theorem (1.1)

If the r.v $X \sim W(a,b)$, then the r.v. $Y = \ln(X) \sim Ext(a,b)$ where $a = e^{-\frac{a}{b}}$,$b = \frac{1}{b}$

Theorem (1.2)

If the r.v $X \sim Ext(a,b)$, then the r.v $Y = -X \sim Ext(-a,b)$

CHAPTER TWO

MONTE CARLO SAMPLING

2.1 Introduction

In this chapter, we shall give some definitions, concepts and historical review about Monte Carlo simulation.

2.1.1 Definition, [16]

Simulation in a " wide sense " is defined as a numerical technique for conducting experiments on a digital computer which involve certain types of mathematical and logical models that describe the behavior of system over extended periods of real time.

For example, designing games, training pilots on flight conditions, film to simulate objects, a telephone communication system, a large scale military battle (to evaluate defensive or offensive weapon system) and network traffic simulation.

Where as simulation in a " narrow sense " (also called stochastic simulation) is defined as experimenting with the model over time, it includes sampling stochastic variates from probability distn. Often simulation is viewed as a “Method of Last Resort” to be used when every things else has failed. Software building and technical development have made simulation one of the most widely used and accepted tools for designers in the system analysis and operation research.

2.2 Monte Carlo Simulation

Stochastic simulation is sometimes called Monte Carlo simulation, because sampling from a particular distribution involve the use of random numbers. [38]

Historically, The name “Monte Carlo” was coined by Metropolis (1946) (inspired by Ulam’s interest in poker) during the Manhattan

Project of World War II, because of the similarity of statistical simulation to games of chance, and because Monte Carlo, the capital of Monaco was a center for gambling. [1]

Courant [8] showed the equivalence of the behavior of certain random walks to solutions of certain partial differential equations. Early use of Monte Carlo was the sampling experiment that led student W. S. Gosset (1908) to the discovery of the distribution of the t-statistic and the correlation coefficient of the distn.. [15] In the 1930s, Enrico Fermi made some numerical experiments that called Monte Carlo calculations, which is the first of used a random number method to calculated the properties of the newly-discovered neutron. [36]

During the Second World War, von Neumann, Fermi, Ulam, and Metropolis and the beginnings of modern digital computers gave a strong impetus to the advancement of Monte Carlo. In the late 1940s and early 1950s, there was a surge of interest. Papers appeared that described the new method and how it could be used to solve problems in statistical mechanics, radiation transport, economic modeling, and other fields. [40][9]

The two most influential developments of that kind were the improvements in methods for the transport equation, especially reliable methods of ‘‘importance sampling’’ [23] and the invention of the algorithm of Metropolis *et. al.*. The resulting successes have borne out the optimistic expectations of the pioneers of the 1940s. In (1948) Fermi, Metropolis and Ulam obtained Monte Carlo estimates for the eigenvalues of Schrödinger equation. [31]

The main requirement to use Monte Carlo method for simulation of a physical system is that it must be possible to describe the system in terms of p.d.f., also called partition function (Z). Once the p.d.f. or Z for a system is known, then the simulation begins by random “sampling” from the p.d.f., and subsequently determining the desired properties of the sample by conducting some kind of a "trial" and subjecting the outcome to a reasonable test such as chi-square-goodness of fit test. Many trials are outcomes of all of these trials are recorded. The final step in the Monte Carlo method is that the behavior of the overall system is obtained by computing the average of outcomes of the trails conducted. [1]

Monte Carlo methods provide approximate solutions to a variety of mathematical problems by performing statistical sampling experiments. Monte Carlo methods are a collection of different methods that all basically perform the same process. This process involves performing many simulations using random numbers and probability to get an approximation of the answer to the problem. [20]

Monte Carlo simulation is widely used in many fields in Mathematics and Statistical Physics to numerical solution of complex multi-dimensional partial differentiation and integration problems, also it is used for simulating quantum systems to solve optimization problems in operations researches. [36]

Also in Engineering, Monte Carlo simulation is used to estimate reliability of mechanical components in mechanical engineering. Effective life of pressure vessels in chemical engineering. While in electronics engineering and circuit design, circuits in computer chips are simulated using Monte Carlo methods for estimating the probability of fetching instructions in memory buffers. [16][36]

Also, Monte Carlo Simulation is used in financial [16], and phenomena modeling such as the calculation of risk in business. Monte Carlo are useful in studying systems with a large number of coupled degrees of freedom, such as fluids, disordered materials, strongly coupled solids, and cellular structures. [36]

Markov Chain Monte Carlo simulation methods that have been widely used in recent years in econometrics and statistics. [17]

2.3 Random Number Generation, [25]

The best means of obtaining unpredictable random numbers is by measuring physical phenomena such as radioactive decay, thermal noise in semiconductors, sound samples taken in a noisy environment, and even digitized images of a lava lamp.

However, few computers (or users) have access to the kind of specialized hardware required for these sources, and must rely on other means of obtaining random data. The term "practically strong randomness" is used here to represent randomness which isn't cryptographically strong by the usual definitions but which is as close to it as is practically possible.

We say that, the random numbers generated by any method is a "good" one if the random numbers are uniformly distributed, statistically independent and reproducible, more over the method is necessarily fast and requires minimum capacity in the computer memory.

The Congruential methods for generating pseudorandom numbers are designed specifically to satisfy as many of these requirements as possible.

These methods produce a nonrandom sequence of numbers according to some recursive formula based on calculating the residues module of some integer m of a linear transformation. The Congruential methods are based on a fundamental congruence relationship, which may be formulated as:

$$X_{i+1} = (aX_i + c)(\bmod m),\ i = 1,2,...,m\ . \qquad (2.1)$$

where a is the multiplier, c is the increment , and m is the modulus (a, c, m are non-negative integers), (mod m) mean that eq.(2.1) can be written as:

$$X_{i+1} = aX_i + c - m\left[\frac{aX_i + c}{m}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.2)}$$

Where z = $\frac{aX_i + c}{m}$ is the greater integer in z

Given an initial starting value X_1 with fixed values of a, c and m, then eq. (2.2) yields congruence relationship (modulo m) for any values i of the sequence $\{X_i\}$. The seq. $\{X_i\}$ will repeat itself in at most m steps and will be therefore periodic. For instant:

Let $a = c = X_1 = 4$, and $m = 3$,then the sequence obtained from the recursive formula

$X_{i+1} = (4X_i + 4)(\text{mod } 3)$ is $X_i = 4, 2, 0, 1, 2, \ldots$.

The random number on the unit interval [0,1] can be obtained by:

$$U_i = \frac{X_i}{m},\ i = 1, 2, \ldots, m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.3)}$$

It follows from eq.(2.3) that $X_i \le m$, $\forall i$, this inequality mean that the period of the generator cannot exceed m, that is, the sequence $\{X_i\}$ contains at most m distinct numbers. So we should choose m as large as possible to ensure, a sufficiently large sequence of distinct numbers in the cycle.

It is noted in the literature, that good statistical result can be achieved from computers by choosing $a = 2^{7+1}$, $c = 1$, and $m = 2^{35}$.

2.4 Random Variates Generation from Continuous Distribution

Many methods and procedures are proposed in the literatures for generating random numbers from different distributions. We shall utilize the inverse transform method, (IT).

2.4.1 Inverse Transform Method

One of the more useful ways of generating random variates is through the inverse transformation techniques which is based on the following theorem:

Theorem (2.1), [38]

The random variable $U = F(X) \sim U(0, 1)$ if and only if the random variable $X = F^{-1}(U)$ has c.d.f pr $(X \le x) = F(x)$.

The algorithm of generating random variates by inverse transform method can be described by the steps of IT-algorithm:

IT-Algorithm:

1. Generate U from U(0, 1).
2. Set U = F(X).
3. Return $X = F^{-1}(U)$ if the inverse exist.
4. Deliver X as a random variable generated from the p.d.f f(x).
5. Stop.

As an application of IT-Algorithm, we shall consider the following examples:

Example (2.1):

Consider, we wish to generate a r.v. X, where the distn. p.d.f:

$$f(x) = \begin{cases} \dfrac{e^{-\left(\frac{x}{1-x}\right)}}{(1-x)^2}, & 0 < x < 1 \\ 0, & e.w. \end{cases}$$

then, the c.d.f. of this p.d.f.

$$F(x) = pr(X \le x) = \int_{-\infty}^{x} f(t)\, dt = \int_{0}^{x} \frac{e^{-(\frac{t}{1-t})}}{(1-t)^2}\, dt$$

Set $u = \frac{t}{1-t}$ implies $du = \frac{1}{(1-t)^2}\, dt$

$$F(x) = \int_{0}^{\frac{x}{1-x}} e^{-u} du = 1\text{-}e^{-(\frac{x}{1-x})}$$

Set u = F(x), implies:

$$x = \frac{\ln(u)}{\ln(u)-1}, \quad 0 < u < 1$$

Apply IT-Algorithm:

1. Generate U from U(0, 1).
2. Set $X = \frac{\ln(U)}{\ln(U)-1}$.
3. Deliver X as a random variable generated from $f(x) = \frac{e^{-(\frac{x}{1-x})}}{(1-x)^2}$.
4. Stop.

Example (2.2):

If a r.v. X required from the distn. whose distn. p.d.f:

$$f(x) = \begin{cases} \frac{x-2a}{(b-a)^2}, & 2a < x < a+b \\ \frac{2b-x}{(b-a)^2}, & a+b < x < 2b \\ 0, & \text{e.w.} \end{cases}$$

then, the c.d.f of this p.d.f is:

$$F(x)= pr(X \le x) = \begin{cases} 0, & x \le 2a \\ \int_{2a}^{x} \frac{(t-2a)}{(b-a)^2} dt, & 2a < x < a+b \\ \int_{2a}^{a+b} \frac{(t-2a)}{(b-a)^2} dt + \int_{a+b}^{x} \frac{(2b-t)}{(b-a)^2} dt, & a+b \le x < 2b \\ 1, & x \ge 2b \end{cases}$$

So: $$F(x) = \begin{cases} 0, & x \le 2a \\ \frac{1}{2}[\frac{x-2a}{b-a}]^2, & 2a < x < a+b \\ 1-\frac{1}{2}[\frac{2b-x}{b-a}]^2, & a+b \le x < 2b \\ 1, & x \ge 2b \end{cases}$$

For $2a < x < a+b$, set $u = F(x) \Rightarrow u = \frac{1}{2}[\frac{x-2a}{b-a}]^2$, implies:

$x = 2a+(b-a)\sqrt{2u}$, for $0 < u \le \frac{1}{2}$

For $a+b \le x < 2b$, set $u = F(x) \Rightarrow u = 1-\frac{1}{2}[\frac{2b-x}{b-a}]^2$, implies:

$x = 2b-(b-a)\sqrt{2u}$, for $\frac{1}{2} < u < 1$

Apply IT-Algorithm:

1. Read a and b.
2. Generate U from U(0, 1).
3. If $0 < U \le \frac{1}{2}$ set $X = 2a+(b-a)\sqrt{2U}$: go to step (5).
4. Else, set $X = 2b-(b-a)\sqrt{2U}$.
5. Deliver X as a random variable generated from

$$f(x) = \begin{cases} \dfrac{x-2a}{(b-a)^2}, & 2a<x<a+b \\ \dfrac{2b-x}{(b-a)^2}, & a+b<x<2b \end{cases}.$$

6. Stop.

We note that:

To apply the inverse transform method, the c.d.f F(x) must exist in a form for which the corresponding inverse transform can be applied analytically.

Some probability distn., it's either impossible or possible to find the inverse transform , that is, to solve, $u = F(x) = \int_{-\infty}^{x} f(t)\,dt$.

For example:

1. $X \sim Exp(\lambda)$, where $f(x) = \frac{1}{\lambda} e^{-x/\lambda}$, $0 < x < \infty$ (possible).
2. $X \sim G(2, 1)$, where $f(x) = xe^{-x}$, $0 < x < \infty$ (difficult).
3. $X \sim N(0, 1)$, where $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $-\infty < x < \infty$ (impossible).

2.5 Procedure for generating Random Variates of Extreme Value Distribution

In this section, we shall consider the procedure for generating random variates from extreme value distn. by utilizing theorem (2.1).

2.5.1 Procedure (EV-1):

This procedure is based on Inverse Transform method given by theorem (2.1):

From eq. (1.2), the c.d.f. of extreme value distn. is:

$$F(x;\alpha,\beta) = 1 - e^{-e^{\frac{x-\alpha}{\beta}}}$$

Setting u = F(x; α,β) implies $u = 1 - e^{-e^{\frac{x-\alpha}{\beta}}}$, implies that:

$$x = \alpha + \beta \, \text{Ln}(-\text{Ln}(u))$$

The (EV-1) algorithm describe the necessary steps for generating random variates by the inverse transform method.

Algorithm (EV-1):

1. Read α, β.
2. Generate U form U(0, 1).
3. Set $X = \alpha + \beta \text{Ln}(-\text{Ln}(u))$.
4. Deliver X as a r.v. generated from Ext(α, β).
5. Stop.

2.6 Goodness _ of _Fit Test for Extreme Value Observations, [34]

We shall subject the observations of extreme value distn. that obtained from a computer by simulation to a test to see whether or not it will be acceptable for use.

Many goodness of fit tests are available could be found throughout the literature such as Chi-Square, Kolmogorov-Smirnov, Sign-Rank, Median, Mann-Whitney tests etc. We shall utilize chi-square goodness of fit test which is considered as the best known test of all statistical tests. Such test can described as follows:

Suppose $X_1, X_2, \ldots, X_n$ be a r.s. of size n from distn. whose c.d.f $F(x) = pr(X \le x)$ is unknown and we wish to test the null hypothesis that the observations

$$H_0 : F(x) = F_0(x) \text{ versus } H_1 : F(x) \ne F_0(x)$$

Where $F_0(x)$ is completely specified c.d.f.

we assume that the n observations have been grouped into k mutually exclusive cells. Let P_i be the probability that the outcome of X_i of the sample fallen in the cells i and let O_i be the number of the observed of the cell i and let e_i be the expected number of cell i, i=1, 2, ….., k.

Then we have the following table :_

Cell i	1	2	3	……………………………	K	Total
O_i	O_1	O_2	O_3	……………………………	O_k	N
e_i	e_1	e_2	e_3	……………………………	e_k	

Where $n = \sum_{i=0}^{k} O_i = \sum_{i=0}^{k} e_i$

Since, $O_i \sim b(n, p_i)$, with the expected ($e_i = n\,p_i$). The test statistic suggested by person is $y = \sum_{i=0}^{k} \frac{[O_i - e_i]^2}{e_i}$, which tends to be small, when H_0 is true and large when H_0 is false the exact distn. of r.v. y is quite complicated for large n.

The distn. of r.v. y is approximately chi_ square with k-1 degrees of freedom. i.e. as n→∞, y ~ $\chi^2(k-1)$, under H_0 (when H_0 is true) we expect pr(Y ≤ $\chi^2_{1-\alpha}$) = 1- α, where α is the significance level of the test.

In particular, usually we take α = 0.01, 0.05, or 0.1 and quintile $\chi^2_{1-\alpha}$ that correspond to the probability (1-α) given in chi_ square table

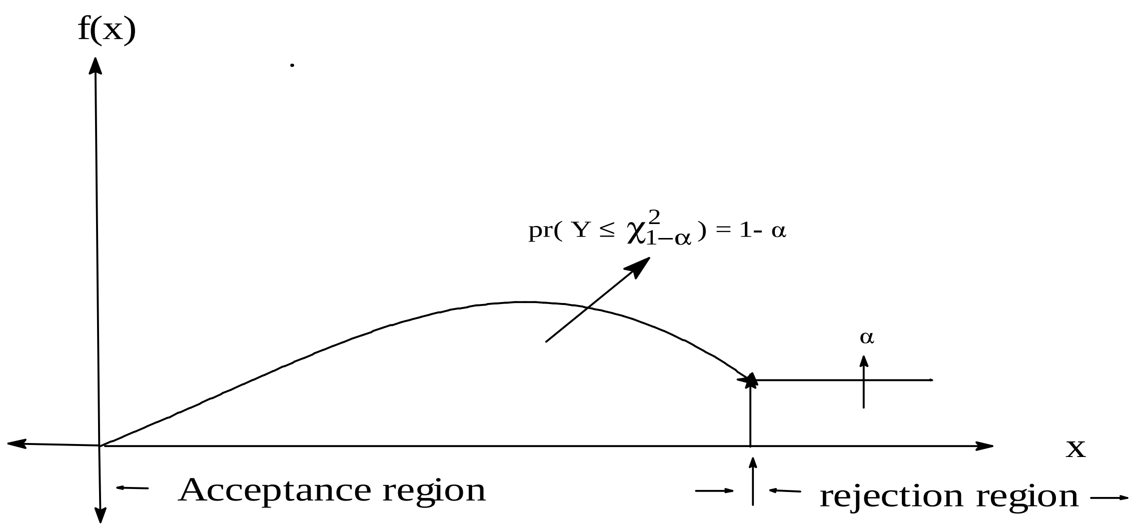


Figure (2.1) The Chi-Square Goodness-of-Fit Test.

We applied this test on a sample of size n=100 to test whether the observations come from Ext (0,1) where $F_0(x;\alpha,\beta) = 1-e^{-e^x}$. We take the number of cells k=10 with pooling method will reduce the number of cell takes and we observe the value of test statistic $Y = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$ with α=0.05 level of significant and the number of repetition 500 is used.

It has been found that 94% of 500 trials are accepted. So we accept that the observations obtained by the computer as a real observations come from Ext(0,1).

CHAPTER THREE

MONTE CARLO RESULTS

3.1 Introduction

In this chapter, we shall utilize Monte Carlo method to estimate the parameters of extreme value distribution by moment method, maximum likelihood method, order statistic method and least squares method as given in eq.s (1.18), (1.19), (1.34), (1.35), (1.40), (1.42), (1.56) and (1.57) of chapter one, The simulated samples of extreme value are observed by monte carlo method according to the procedure given in sections (2.5.1). These estimators are used to estimate the reliability and the hazard functions by four methods given in section (1.5) of chapter one.

3.2 The Estimates of the Parameters Using Procedure (EV-1)

To access the results obtained by the four methods of estimation, we generate samples of size n = 5(1) 10(2) 20(5) 30(10) 100 from extreme value distn. and repetition 500 is used.

A computer program (1) is made in Appendix (A) uses procedure (EV-1) of section (2.5.1) which utilizing the Inverse Transform Method.

The estimators by the four methods of estimations are displayed in table (3.1).

Table (3.1)

Parameters estimation.

Sample size n	Estimation of $(\hat{a}, \hat{b})$			
	M.M	M.L.M	O.S.M	L.S.M
5	(0.734,1.677)	(5.956,6.735)	(0.929, 2.083)	(1,2)
6	(0.828,1.690)	(3.639,5.642)	(0.968,2.080)	(1,2)
7	(0.842,1.727)	(2.448,4.793)	(1.023,2.072)	(1,2)
8	(0.874,1.760)	(2.169,3.041)	(1.022,1.958)	(1,2)
9	(0.881,1.836)	(0.207,1.286)	(1.018,2.04)	(1,2)
10	(0.892,1.844)	(1.819,2.803)	(1.015,2.039)	(1,2)
12	(0.896,1.868)	(1.683,2.629)	(1.012, 2.037)	(1,2)
14	(0.901,1.870)	(1.482,1.573)	(0.989, 1.965)	(1,2)
16	(0,920,1.882)	(0.584,1.601)	(0.990, 1.970)	(1,2)
18	(0.937,1.909)	(0.636,1.756)	(1.009,1.978)	(1,2)
20	(0.938,1.910)	(0.725,2.253)	(0.992, 2.022)	(1,2)
25	(0.961,1.911)	(0.783,1.826)	(1.008,2.016)	(1,2)
30	(0.965,1.914)	(0.832,2.111)	(0.993,2.014)	(1,2)
40	(0.970,1.951)	(0.880,1.921)	(1.006,1.990)	(1,2)
50	(0.971,1.958)	(0.889,1.944)	(0.995, 1.992)	(1,2)
60	(0.979,1.968)	(0.902,2.064)	(1.005,1.993)	(1,2)
70	(0.985,1.970)	(0.924,1.988)	(1.004,1.994)	(1,2)
80	(0.989,1.971)	(0.932,2.011)	(0.996,1.995)	(1,2)
90	(0.990,1.979)	(0.944,2.004)	(1.003,1.998)	(1,2)
100	(0.991,1.988)	(0.956,1.997)	(0.999,2.001)	(1,2)

Table (3.1) shows that the L.S.M. give exact estimate values for a and b because the estimators $\hat{a}$ and $\hat{\beta}$ are unbiased as shown in eqs. (1.56) and (1.57).

For all samples sizes the O.S.M. is best than M.M. and M.L.M. while all methods are adequate for moderate and large samples. In small samples, we note that the M.L.M. give estimate higher than the expectation and this might be due to the given bound of ending the estimation.

3.3 The Bias of Estimators Using Procedure (EV-1)

The biases of estimators $\hat{\alpha}$ and $\hat{\beta}$ which can be obtained by:

$$\text{Bias}(\hat{\alpha}) = \hat{\alpha} - \alpha$$

$$\text{Bias}(\hat{\beta}) = \hat{\beta} - \beta$$

Tables (3.2) and (3.3) show the biases of estimators $(\hat{\alpha})$ and $(\hat{\beta})$ obtained by the four methods of estimation:

Table (3.2)

Bias of Estimator ($\hat{a}$).

Sample size n	*Bias of Estimation ($\hat{a}$)*			
	M.M	*M.L.M*	*O.S.M*	*L.S.M*
5	-0.266	4.956	-0.071	0
6	-0.172	2.639	-0.032	0
7	-0.158	1.448	0.023	0
8	-0.126	1.169	0.022	0
9	-0.119	-0.893	0.018	0
10	-0.108	0.819	0.015	0
12	-0.104	0.683	0.012	0
14	-0.099	0.482	-0.011	0
16	-0.080	-0.416	-0.010	0
18	-0.063	-0.364	9.354×10^{-3}	0
20	-0.062	-0.275	-7.721×10^{-3}	0
25	-0.039	-0.217	8.464×10^{-3}	0
30	-0.035	-0.168	-6.844×10^{-3}	0
40	-0.030	-0.120	6.153×10^{-3}	0
50	-0.029	-0.111	-4.82×10^{-3}	0
60	-0.021	-0.098	5.306×10^{-3}	0
70	-0.015	-0.976	3.798×10^{-3}	0
80	-0.011	-0.068	-3.783×10^{-3}	0
90	-9.892×10^{-3}	-0.056	3.118×10^{-3}	0
100	-9.266×10^{-3}	-0.044	-5.478×10^{-4}	0

Table (3.3)
Bias of Estimator ($\hat{b}$).

Sample size n	Bias of Estimation ($\hat{b}$)			
	M.M	M.L.M	O.S.M	L.S.M
5	-0.323	4.735	0.083	0
6	-0.310	3.642	0.080	0
7	-0.273	2.793	0.072	0
8	-0.240	1.041	-0.042	0
9	-0.164	-0.714	0.040	0
10	-0.156	0.803	0.039	0
12	-0.132	0.629	0.037	0
14	-0.130	-0.427	-0.035	0
16	-0.118	-0.399	-0.030	0
18	-0.091	-0.244	-0.022	0
20	-0.090	0.253	0.022	0
25	-0.089	-0.174	0.016	0
30	-0.086	0.111	0.014	0
40	-0.049	-0.079	-9.509×10^{-3}	0
50	-0.042	-0.056	-7.544×10^{-3}	0
60	-0.032	0.064	-7.043×10^{-3}	0
70	-0.03	-0.012	-6.327×10^{-3}	0
80	-0.029	0.011	-5.134×10^{-3}	0
90	-0.021	0.004	-1.564×10^{-3}	0
100	-0.012	-0.003	9.192×10^{-4}	0

Tables (3.2) and (3.3) show that the simulated biases of the estimators $\hat{a}$ and $\hat{b}$ given by L.S.M. coincide with the theoretical biases given by eqs. (1.56) and (1.57).

For small and moderate samples the biases of M.M. and O.S.M. are better than those given by M.L.M.

3.4 The Variance of Estimators Using Procedure (EV-1)

The variances of estimator ($\hat{\alpha}$) are shown in table (3.4), where the true values of variances are given:

1- Equation (1.24) by moments method.

2- Equation (1.46) by order statistic method.

3- Equation (1.60) by least squares method.

4- While the variance concern the M.L.M. is excluded because the non linearity appearance to eqs. (1.34) and (1.35)

Table (3.4) show the variance of estimator ($\hat{\alpha}$) where the true value of variance ($\hat{\alpha}$) are shown in parenthesis.

Table (3.4)

Variance of Estimator ($\hat{a}$).

Sample size n	*Variance of Estimation ($\hat{a}$)*			
	M.M	***M.L.M***	***O.S.M***	***L.S.M***
5	1.021 (0)	331.120	1.136 (0)	0 (2.459)
6	0.930 (0)	262.316	1.045 (0)	0 (1.782)
7	0.630 (0)	171.976	0.740 (0)	0 (1.368)
8	0.608 (0)	111.897	0.721 (0)	0 (1.138)
9	0.506 (0)	82.016	0.614 (0)	0 (0.989)
10	0.462 (0)	61.010	0.563 (0)	0 (0.883)
12	0.408 (0)	42.643	0.540 (0)	0 (0.727)
14	0.319 (0)	33.031	0.432 (0)	0 (0.604)
16	0.308 (0)	24.806	0.416 (0)	0 (0.526)
18	0.283 (0)	14.151	0.386 (0)	0 (0.466)
20	0.207 (0)	9.613	0.293 (0)	0 (0.420)
25	0.162 (0)	5.722	0.256 (0)	0 (0.327)
30	0.153 (0)	1.031	0.244 (0)	0 (0.272)
40	0.126 (0)	0.642	0.224 (0)	0 (0.201)
50	0.090 (0)	0.183	0.171 (0)	0 (0.161)
60	0.079 (0)	0.097	0.170 (0)	0 (0.133)
70	0.066 (0)	0.061	0.166 (0)	0 (0.114)
80	0.054 (0)	0.052	0.156 (0)	0 (0.100)
90	0.049 (0)	0.041	0.130 (0)	0 (0.089)
100	0.048 (0)	0.035	0.121 (0)	0 (0.080)

The variances of estimator ($\hat{\beta}$) are shown in table (3.5), where the true values of variances are given:

1- Equation (1.22) by moments method.

2- Equation (1.43) by order statistic method.

3- Equation (1.58) by least squares method.

4- While the variance concern the M.L.M. is excluded because the non linearity appearance to eqs. (1.34) and (1.35).

Table (3.5) show the variance of estimator ($\hat{\beta}$) where the true value of variance ($\hat{\beta}$) are shown in parenthesis.

Table (3.5)

Variance of Estimator ($\hat{b}$).

Sample size n	Variance of Estimation ($\hat{b}$)			
	M.M	M.L.M	O.S.M	L.S.M
5	0.693 (0)	301.351	1.658 (0)	0 (2.248)
6	0.500 (0)	241.012	1.276 (0)	0 (1.595)
7	0.465 (0)	160.516	1.155 (0)	0 (1.225)
8	0.413 (0)	100.443	1.064 (0)	0 (0.988)
9	0.404 (0)	87.511	0.970 (0)	0 (0.761)
10	0.341 (0)	61.167	0.933 (0)	0 (0.634)
12	0.326 (0)	49.797	0.863 (0)	0 (0.498)
14	0.271 (0)	33.224	0.767 (0)	0 (0.412)
16	0.245 (0)	26.012	0.698 (0)	0 (0.346)
18	0.180 (0)	19.778	0.565 (0)	0 (0.281)
20	0.165 (0)	11.654	0.486 (0)	0 (0.252)
25	0.141 (0)	6.031	0.483 (0)	0 (0.197)
30	0.125 (0)	1.245	0.453 (0)	0 (0.162)
40	0.105 (0)	0.731	0.426 (0)	0 (0.114)
50	0.076 (0)	0.231	0.390 (0)	0 (0.089)
60	0.069 (0)	0.091	0.380 (0)	0 (0.072)
70	0.063 (0)	0.059	0.345 (0)	0 (0.062)
80	0.050 (0)	0.044	0.339 (0)	0 (0.053)
90	0.049 (0)	0.042	0.327 (0)	0 (0.047)
100	0.044 (0)	0.039	0.308 (0)	0 (0.042)

Tables (3.4) and (3.5) show that the variances of the estimators $\hat{\alpha}$ and $\hat{\beta}$ by obtained M.M.,O.S.M. and L.S.M. the true and approximated variances respectively are zero values by eqs. (1,25), (1.22), (1.47), (1.44), (1.61), (1.59).

The variances of estimators $\hat{\alpha}$ and $\hat{\beta}$ practically given by M.M., M.L.M., O.S.M. and theoretically L.S.M. respectively converge to zero as sample sizes increase. Also, note that the variance of estimator $\hat{\alpha}$ by M.M and O.S.M. are adequate in all sample sizes while the variance of estimator $\hat{\beta}$by obtained M.M. is better than O.S.M. and M.L.M. in small and moderate samples. In large sample, the variances of estimators $\hat{\alpha}$ and $\hat{\beta}$ obtained by M.L.M. is better than M.M. and O.S.M.

3.5 The Skewness of Estimators Using Procedure (EV-1)

The skewness of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{Skewness}\,(\hat{\alpha}) = \frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\alpha_i)^3 - 3\hat{\alpha}\sum_{i=1}^{n}(\alpha_i)^2 + 3\hat{\alpha}^2\sum_{i=1}^{n}(\alpha_i) - \hat{\alpha}^3\right]}{(\sigma^2)^{3/2}}$$

$$\text{Skewness}\,(\hat{\beta}) = \frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\beta_i)^3 - 3\hat{\beta}\sum_{i=1}^{n}(\beta_i)^2 + 3\hat{\beta}^2\sum_{i=1}^{n}(\beta_i) - \hat{\beta}^3\right]}{(\sigma^2)^{3/2}}$$

Tables (3.6) and (3.7) show the skewness of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the three methods of estimation.

Table (3.6)

Skewness of Estimator ($\hat{a}$).

Sample size n	*Skewness of Estimation ($\hat{a}$)*		
	M.M	***M.L.M***	***O.S.M***
5	0.022	-18.052	0.368
6	0.216	-15.819	0.575
7	0.806	-7.124	1.252
8	1.293	-6.456	1.577
9	1.829	-5.809	2.152
10	1.857	-1.600	2.245
12	2.296	1.993	2.500
14	4.361	5.247	3.676
16	4.629	6.233	3.753
18	4.956	8.858	4.083
20	7.518	11.463	5.123
25	13.323	20.731	7.469
30	14.615	24.853	8.298
40	21.468	37.739	9.923
50	33.752	43.958	14.629
60	43.122	52.013	14.789
70	55.583	61.862	15.841
80	72.442	79.120	17.242
90	85.756	89.346	21.629
100	93.316	97.334	24.931

Table (3.7)

Skewness of Estimator ($\hat{b}$).

Sample size n	Skewness of Estimation ($\hat{b}$)		
	M.M	M.L.M	O.S.M
5	9.276	-21.035	5.710
6	14.726	-17.229	7.274
7	17.469	-13.712	8.214
8	22.129	-7.033	9.383
9	24.389	-3.967	9.804
10	32.073	-0.059	10.736
12	36.006	8.588	12.022
14	47.088	20.673	13.233
16	55.470	34.221	15.255
18	91.980	51.463	20.475
20	104.042	82.679	23.284
25	131.325	105.877	23.663
30	157.888	164.057	26.529
40	218.168	246.318	29.252
50	355.896	543.756	35.861
60	418.579	909.222	37.598
70	485.000	1.23×10^3	39.919
80	704.112	1.698×10^3	41.076
90	710.785	1.967×10^3	44.963
100	837.481	2.033×10^3	47.768

We note that the evaluation of skewness of $\hat{\alpha}$ and $\hat{\beta}$ given by L.S.M. converge to the infinity because the zero value of the variance of $\hat{\alpha}$ and $\hat{\beta}$.

Table (3.6) show that there is a small skewness of $\hat{\alpha}$ to the right of the estimators $\hat{\alpha}$ by M.M. and O.S.M. and rapidly increase away from normality for large samples. While there is sever skewness of $\hat{\alpha}$ given by M.L.M. in both direction.

Table (3.7) shows that for all sample sizes there is a heavy skewness of $\hat{\beta}$ to the right of the estimators $\hat{\beta}$ given by M.M. and O.S.M., while there is a sever skewness of $\hat{\beta}$ given by M.L.M. in both direction. This indicate that the distn. of $\hat{\beta}$ is away from normality.

3.6 The Kurtosis of Estimators Using Procedure (EV-1)

The kurtosis of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) which can be obtained by:

$$\text{Kurtosis}(\hat{\alpha})=\frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\alpha_i)^4-4\hat{\alpha}\sum_{i=1}^{n}(\alpha_i)^3+6\hat{\alpha}^2\sum_{i=1}^{n}\alpha_i-4\hat{\alpha}^3\sum_{i=1}^{n}\alpha_i\right]}{(\sigma^2)^2}-3$$

$$\text{Kurtosis }(\hat{\beta})=\frac{\frac{1}{n}\left[\sum_{i=1}^{n}(\beta_i)^4-4\hat{\beta}\sum_{i=1}^{n}(\beta_i)^3+6\hat{\beta}^2\sum_{i=1}^{n}\beta_i-4\hat{\beta}^3\sum_{i=1}^{n}\beta_i\right]}{(\sigma^2)^2}-3$$

Tables (3.8) and (3.9) show the kurtosis of estimators ($\hat{\alpha}$) and ($\hat{\beta}$) by the three methods of estimation.

Table (3.8)

Kurtosis of Estimator ($\hat{a}$).

Sample size n	*Kurtosis of Estimation ($\hat{a}$)*		
	M.M	***M.L.M***	***O.S.M***
5	-0.131	319.035	-0.285
6	-0.620	268.492	-1.059
7	-1.149	198.275	-1.601
8	-2.022	153.623	-2.518
9	-2.605	128.338	-3.002
10	-2.957	99.365	-3.103
12	-3.287	83.438	-3.198
14	-7.347	67.100	-4.712
16	-7.892	42.906	-5.656
18	-8.821	21.836	-6.101
20	-15.141	7.496	-8.422
25	-32.459	0.406	-14.264
30	-36.777	-24.510	-16.643
40	-60.593	-70.345	-20.366
50	-109.872	-108.313	-34.478
60	-151.956	-133.772	-34.954
70	-212.216	-245.939	-35.959
80	-301.959	-340.550	-40.649
90	-379.396	-400.250	-57.213
100	-423.287	-448.422	-69.700

Table (3.9)

Kurtosis of Estimator ($\hat{b}$).

Sample size n	Kurtosis of Estimation ($\hat{b}$)		
	M.M	M.L.M	O.S.M
5	-15.120	747.593	-4.263
6	-30.376	594.523	-5.815
7	-37.661	424.041	-6.041
8	-56.839	311.857	-14.760
9	-66.747	269.538	-14.887
10	-96.003	186.850	-16.668
12	-112.716	136.369	-19.829
14	-164.946	101.539	-23.425
16	-206.870	89.163	-28.309
18	-411.859	68.721	-50.180
20	-485.491	43.048	-59.796
25	-664.637	-87.155	-61.783
30	-848.811	-242.537	-72.256
40	-1.311×10^{3}	-565.830	-83.850
50	-2.522×10^{3}	-2.772×10^{3}	-110.122
60	-3.129×10^{3}	-8.813×10^{3}	-115.703
70	-3.808×10^{3}	-1.319×10^{4}	-130.699
80	-6.261×10^{3}	-2.027×10^{4}	-135.952
90	-6.338×10^{3}	-2.266×10^{4}	-150.758
100	-7.895×10^{3}	-2.577×10^{4}	-165.758

We note that the evaluation of kurtosis of $\hat{a}$ and $\hat{b}$ given by L.S.M. converge to infinity from the left because the zero values of the variance of $\hat{a}$ and $\hat{b}$.

Table (3.8) shows that the evaluation of kurtosis of estimator $\hat{a}$ there is a platykurtic by M.M. and O.S.M. the values rapidly increase away from normality for large samples. While there is sever kurtosis of $\hat{a}$ given by M.L.M. in both direction, where in small and moderate samples give leptokurtic and in large sample give platykurtic.

Table (3.9) shows that the evaluation of kurtosis of estimator $\hat{b}$ there is heavy kurtosis to the platykurtic given by M.M. and O.S.M., while there is a sever kurtosis of $\hat{b}$ given by M.L.M. in both direction, where in small and moderate samples give leptokurtic and in large sample give platykurtic. This indicate that the distn. of $\hat{b}$ is away from normality.

3.7 Mean Square Error of Estimators Using Procedure (EV-1), [33]

The mean square error of estimators $(\hat{\alpha})$ and $(\hat{\beta})$ which can be obtained by:

$$\text{m.s.e}\ (\hat{\alpha}) = \text{Variance}(\hat{\alpha}) + [\text{bias}(\hat{\alpha})]^2$$

$$\text{m.s.e}\ (\hat{\beta}) = \text{Variance}(\hat{\beta}) + [\text{bias}(\hat{\beta})]^2$$

Tables (3.10) and (3.11) show the mean square error of estimators $(\hat{\alpha})$ and $(\hat{\beta})$ by the four methods of estimation.

Table (3.10)

Mean square error of Estimator ($\hat{a}$).

Sample size n	***Mean square error of Estimation ($\hat{a}$)***			
	M.M	***M.L.M***	***O.S.M***	***L.S.M***
5	1.092	355.682	1.141	0
6	0.960	269.280	1.046	0
7	0.655	174.073	0.741	0
8	0.624	113.264	0.722	0
9	0.520	82.813	0.614	0
10	0.474	61.681	0.563	0
12	0.419	43.110	0.540	0
14	0.329	33.263	0.432	0
16	0.314	24.979	0.416	0
18	0.287	14.283	0.386	0
20	0.211	9.689	0.293	0
25	0.164	5.660	0.256	0
30	0.154	1.059	0.244	0
40	0.127	0.656	0.224	0
50	0.091	0.195	0.171	0
60	0.080	0.107	0.170	0
70	0.066	0.067	0.166	0
80	0.054	0.057	0.156	0
90	0.049	0.044	0.130	0
100	0.044	0.037	0.121	0

Table (3.11)

Mean square error of Estimator ($\hat{b}$).

Sample size n	Mean square error of Estimation ($\hat{b}$)			
	M.M	M.L.M	O.S.M	L.S.M
5	0.797	323.771	1.665	0
6	0.596	254.276	1.282	0
7	0.540	168.317	1.160	0
8	0.471	101.527	1.066	0
9	0.431	88.020	0.972	0
10	0.365	61.812	0.935	0
12	0.343	50.193	0.864	0
14	0.288	33.406	0.768	0
16	0.259	26.171	0.699	0
18	0.188	20.022	0.565	0
20	0.173	11.718	0.487	0
25	0.149	6.061	0.483	0
30	0.132	1.257	0.453	0
40	0.107	0.737	0.428	0
50	0.078	0.234	0.390	0
60	0.070	0.095	0.380	0
70	0.064	0.059	0.345	0
80	0.051	0.044	0.339	0
90	0.049	0.042	0.327	0
100	0.044	0.039	0.308	0

We note that the evaluation of m.s.e. of $\hat{a}$ and $\hat{b}$ given by L.S.M. are zero values because the variances and biases are zero values of $\hat{a}$ and $\hat{b}$.

Table (3.10) and Table (3.11) show that the evaluation of m.s.e. $\hat{a}$ and $\hat{b}$ by obtained M.M., M.L.M. and O.S.M. are decreasing as sample size increase because the variance and the bias values of these methods are decreasing as sample size increase. In small and moderate samples the M.M. and O.S.M. better than M.L.M. while in large sample these methods are edequate as the variance of these methods because the bias is very small value.

3.8 Reliability and Hazard functions of Estimators Using Procedure (EV-1)

In this section, we shall use the estimators in sub-section (1.6.3) to estimate the reliability and hazard functions. The estimators in table (3.1) are used to find the estimates of the reliability and the hazard functions by four methods given in section (1.6.3), the result is display in tables (3.12) and (3.13), the biased of the estimators shown in tables (3.14) and (3.15).

Table (3.12)

Estimation of R(x)

Sample size n	*Estimation of* $\hat{R}(x)$				
	True Value	*M.M*	*M.L.M*	*O.S.M*	*L.S.M*
5	0.520	0.519	0.482	0.521	0.520
6	0.528	0.526	0.489	0.530	0.528
7	0.535	0.530	0.492	0.531	0.535
8	0.539	0.539	0.519	0.539	0.539
9	0.556	0.555	0.540	0.555	0.556
10	0.562	0.561	0.558	0.561	0.562
12	0.567	0.567	0.565	0.567	0.567
14	0.570	0.569	0.568	0.569	0.570
16	0.574	0.572	0.570	0.572	0.574
18	0.578	0.578	0.577	0.578	0.578
20	0.588	0.589	0.586	0.589	0.588
25	0.600	0.600	0.599	0.600	0.600
30	0.607	0.607	0.607	0.607	0.607
40	0.638	0.637	0.639	0.637	0.638
50	0.645	0.645	0.646	0.645	0.645
60	0.653	0.653	0.654	0.653	0.653
70	0.687	0.687	0.688	0.687	0.687
80	0.718	0.718	0.719	0.718	0.718
90	0.722	0.722	0.723	0.722	0.722
100	0.725	0.725	0.726	0.725	0.725

Table (3.13)

Estimation of h(x)

Sample size n	*Estimation of* $\hat{h}(x)$				
	True Value	*M.M*	*M.L.M*	*O.S.M*	*L.S.M*
5	4.019	13.186	0.617	3.319	4.019
6	5.891	15.658	1.184	6.006	5.891
7	8.794	32.051	2.885	8.368	8.794
8	17.767	42.677	12.885	17.405	17.767
9	20.097	52.177	26.383	19.649	20.097
10	39.102	75.982	47.146	29.036	39.102
12	68.741	144.546	88.761	47.341	68.741
14	72.315	195.298	112.160	70.796	72.315
16	668.604	1.542×10^{3}	731.208	662.653	668.604
18	1.358×10^{3}	2.855×10^{3}	3.328×10^{3}	1.816×10^{3}	1.358×10^{3}
20	2.765×10^{3}	5.491×10^{3}	6.7×10^{3}	2.369×10^{3}	2.765×10^{3}
25	1.794×10^{4}	4.158×10^{4}	4.57×10^{4}	2.314×10^{4}	1.794×10^{4}
30	1.913×10^{4}	4.309×10^{4}	5.09×10^{4}	2.627×10^{4}	1.913×10^{4}
40	4.625×10^{4}	6.655×10^{4}	7.469×10^{4}	4.926×10^{4}	4.625×10^{4}
50	2.803×10^{5}	3.792×10^{5}	4.461×10^{5}	3.013×10^{5}	2.803×10^{5}
60	6.502×10^{5}	1.100×10^{6}	2.47×10^{6}	7.143×10^{5}	6.502×10^{5}
70	3.519×10^{7}	4.623×10^{7}	4.98×10^{7}	2.173×10^{7}	3.519×10^{7}
80	1.064×10^{9}	1.698×10^{9}	7.695×10^{9}	1.083×10^{9}	1.064×10^{9}
90	2.117×10^{9}	3.28×10^{9}	4.277×10^{9}	1.23×10^{9}	2.117×10^{9}
100	1.704×10^{10}	2.092×10^{10}	3.334×10^{10}	1.338×10^{10}	1.704×10^{10}

We note that the $\hat{R}(x)$ and $\hat{h}(x)$ have exact values as R(x) and h(x) by using L.S.M. because the estimations of $\hat{a}$ and $\hat{b}$ have exact values as α and β

Table (3.12) shows that the $\hat{R}(x)$ values in M.M., M.L.M. and O.S.M. are very near to the exact values of R(x) for all sample sizes. Furthermore, the $\hat{R}(x)$ values are increasing values as sample sizes increase.

Table (3.13) shows that $\hat{h}(x)$ values in M.M., M.L.M. and O.S.M. are converge to infinity as samples sizes increase as shown in fig (1.4.a) and fig (1.4.b). Furthermore, we see for large and moderate samples there is higher difference in variation of $\hat{h}(x)$ given by these methods with respect to the true values of h(x).

Table (3.14)

Bias of Estimator $\hat{R}(x)$.

Sample size n	*Bias of Estimation $\hat{R}(x)$*			
	M.M	*M.L.M*	*O.S.M*	*L.S.M*
5	-1.488×10^{-3}	-3.823×10^{-3}	4.783×10^{-4}	0
6	-2.406×10^{-3}	-5.747×10^{-3}	1.87×10^{-3}	0
7	-5.205×10^{-3}	-3.728×10^{-3}	-4.372×10^{-3}	0
8	-8.355×10^{-4}	-9.576×10^{-4}	-1.749×10^{-5}	0
9	-1.751×10^{-3}	-4.13×10^{-3}	-1.309×10^{-3}	0
10	-1.518×10^{-3}	-3.822×10^{-3}	-1.404×10^{-3}	0
12	5.747×10^{-4}	1.103×10^{-4}	2.019×10^{-4}	0
14	-2.203×10^{-4}	-5.957×10^{-4}	-2.227×10^{-4}	0
16	-1.319×10^{-4}	-2.486×10^{-4}	-1.519×10^{-3}	0
18	-4.4×10^{-4}	-1.19×10^{-4}	-4.322×10^{-4}	0
20	4.763×10^{-4}	3.645×10^{-4}	4.4×10^{-4}	0
25	4.941×10^{-4}	3.811×10^{-4}	4.663×10^{-4}	0
30	4.293×10^{-4}	3.73×10^{-4}	3.779×10^{-4}	0
40	-1.282×10^{-3}	-3.832×10^{-3}	-1.502×10^{-3}	0
50	-8.997×10^{-5}	-2.261×10^{-5}	-1.172×10^{-4}	0
60	1.419×10^{-4}	1.639×10^{-4}	1.427×10^{-4}	0
70	5.245×10^{-5}	9.465×10^{-5}	9.266×10^{-6}	0
80	1.997×10^{-4}	8.364×10^{-4}	1.341×10^{-4}	0
90	-2.051×10^{-4}	-2.139×10^{-4}	-1.737×10^{-4}	0
100	-2.724×10^{-4}	-7.226×10^{-4}	-2.689×10^{-4}	0

Table (3.15)
Bias of Estimator $\hat{h}(x)$.

Sample size n	Bias of Estimation $\hat{h}(x)$			
	M.M	**M.L.M**	**O.S.M**	**L.S.M**
5	9.166	-2.636	-0.700	0
6	9.767	-4.556	0.115	0
7	23.257	-3.563	-0.426	0
8	24.910	0.897	-0.362	0
9	32.080	3.404	-0.448	0
10	36.881	0.109	-10.065	0
12	75.805	78.664	-21.400	0
14	122.982	57.96	-1.519	0
16	873.493	160.084	-5.951	0
18	1.496×10^3	933.225	457.995	0
20	2.726×10^3	1.097×10^3	-395.657	0
25	2.364×10^4	0.962×10^4	520.317	0
30	2.395×10^4	3.487×10^4	7.138×10^3	0
40	2.03×10^4	1.572×10^4	3.017×10^3	0
50	9.885×10^4	1.444×10^4	2.094×10^4	0
60	4.498×10^5	3.635×10^5	6.412×10^4	0
70	1.103×10^7	3.094×10^7	-1.346×10^7	0
80	6.34×10^8	4.486×10^8	1.915×10^7	0
90	1.163×10^9	3.438×10^9	-8.87×10^8	0
100	3.875×10^9	-5.394×10^9	-3.665×10^9	0

We note that the biases of the R(x) and h(x) by using L.S.M. are zero values because the $\hat{R}(x)$ and $\hat{h}(x)$ values are the exact values to R(x) and h(x).

Table (3.14) shows that the biases of $\hat{R}(x)$ by obtained M.M., M.L.M. and O.S.M. are converge to zero value in all sample sizes.

Table (3.15) shows that the biases of $\hat{h}(x)$ by obtained M.M., M.L.M. and O.S.M. are converge to infinity as samples sizes increase. In large samples the difference among biases the three methods.

CONCLUSIONS AND FUTURE WORK

Conclusions

1. For all sample sizes the estimators $\hat{a}$ and $\hat{\beta}$ by obtained O.S.M. is superior than M.M. and M.L.M.
2. The L.S.M. give exact estimate values for a and b because the estimators of $\hat{a}$ and $\hat{\beta}$ are unbiased.
3. The simulated biases of the estimators $\hat{a}$ and $\hat{b}$ given by L.S.M. coincide with the theoretical biases.
4. The variances of the estimators $\hat{\alpha}$ and $\hat{\beta}$ obtained by the three methods of estimation which M.M., O.S.M. and L.S.M. are rapidly approach to true zero values as sample sizes increase.
5. The skewness of the estimators $\hat{\alpha}$ and $\hat{\beta}$ and the kurtosis of the estimators $\hat{\alpha}$ and $\hat{\beta}$ given by L.S.M. converge to the infinity because the zero value of the variances of the estimators $\hat{\alpha}$ and $\hat{\beta}$.
6. The estimators $\hat{\alpha}$ and $\hat{\beta}$ obtained by M.M. and O.S.M. are skewed to the right as a simple size increase. While the estimators $\hat{\alpha}$ and $\hat{\beta}$ by M.L.M. skewed to both direction.
7. The estimators $\hat{\alpha}$ and $\hat{\beta}$ obtained by M.M. and O.S.M. are kurtosis to the leptokurtic as a simple size increase. While the estimators $\hat{\alpha}$ and $\hat{\beta}$ by M.L.M. kurtosis to both direction.

Conclusions

8. The evaluation of m.s.e. of $\hat{a}$ and $\hat{b}$ given by L.S.M. are zero values because the variances and biases are zero values of $\hat{a}$ and $\hat{b}$.

9. The $\hat{R}(x)$ and $\hat{h}(x)$ have exact values as R(x) and h(x) by using L.S.M. because the estimations of $\hat{a}$ and $\hat{b}$ have exact values as α and β

10. The $\hat{R}(x)$ values in M.M., M.L.M. and O.S.M. are very near to the exact values of R(x) for all sample sizes.

11. The $\hat{h}(x)$ values in M.M., M.L.M. and O.S.M. are converge to infinity as sample sizes increase as shown in fig (1.4.a) and fig (1.4.b).

12. The biases of the $\hat{R}(x)$ and $\hat{h}(x)$ obtained by M.M., M.L.M. and O.S.M. are respectively converge to zero value and infinity as sample sizes increase, while the L.S.M. gives exact biases.

13. The disadvantage of Monte Carlo methods depends on generating pseudorandom variates and that might carry dirty data, and that might effect the results of M.L.M. of estimation $\hat{\alpha}$ and $\hat{\beta}$ when we use Newton-Raphson iteration.

Future Work

1. This work can be used for Generalized Extreme Value distn. of three parameters, maximum Extreme Value distn. and other life distn.
2. Another methods of estimation could be used to estimate the distn. parameters, R(x) and h(x) such as minimum Chi-square, minimum distance, Bayesian method, …..etc.
3. It can generate r.v$^{'s}$. from Extreme Value distn. by other new procedures which can be compared with other used procedures.
4. The bias of estimation is a r.v. of unknown distribution which can be investigated approximately by using well-known statistical tests such as Kolmogorov-Smirnov Goodness-of-Fit Test, Serial Test, …etc.

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APPENDICES

Appendix A
Computer Programs of Estimation Methods

Program (1) :procedure (EV-1)

Enter your values of a, b, k and n

```
a :=        b :=        n :=        k := 

    i := 0..n − 1      j := 0..k − 1

u := | for  j ∈ 0..k − 1
     |   ww_j ← | for  i ∈ 0..n − 1
     |          |   wk_i ← rnd(1)
     |          | wk
     | ww

x := | for  i ∈ 0..n − 1
     |   kwk ← | for  j ∈ 0..k − 1
     |         |   kk_j ← a + b·ln(−ln(u_j))
     |         | kk
     | kwk
```

Appendix B

Appendix B

Computer Programs of Estimation Methods

Program 2: Estimation by Moments Method

Enter your values of a, b, k and n

$a :=$ ▮ $\quad b :=$ ▮ $\quad n :=$ ▮ $\quad k :=$ ▮

$i := 0..n-1 \quad j := 0..k-1$

$$u := \text{for } j \in 0..k-1 \quad ww_j \leftarrow \text{for } i \in 0..n-1 \quad wk_i \leftarrow \text{rnd}(1)$$

$u_0 =$

$$x := \text{for } i \in 0..n-1 \quad kwk \leftarrow \text{for } j \in 0..k-1 \quad kk_j \leftarrow a + b\cdot\ln\left(-\ln\left(u_j\right)\right)$$

$x_0 =$

$$wq := \text{for } j \in 0..k-1 \quad ew_j \leftarrow \frac{1}{n}\cdot\sum_{i=0}^{n-1}\left(x_j\right)_i$$

$wq_0 =$

$$gf := \text{for } j \in 0..k-1 \quad tr_j \leftarrow \left[\sum_{i=0}^{n-1}\left[\left(x_j\right)_i\right]^2\right]$$

$gf_0 =$

$$ggf := \text{for } j \in 0..k-1 \quad ty_j \leftarrow \sqrt{\frac{\left[gf_j - n\cdot\left(wq_j\right)^2\right]}{n-1}}$$

$gg_0 =$

$$gfgf := \text{for } j \in 0..k-1 \quad tty_j \leftarrow \frac{ggf_j}{\pi}\cdot\sqrt{\frac{6\cdot(n-1)}{n}}$$

$gfgf_0 =$

$$b1 := \frac{1}{k}\cdot\sum_{i=0}^{k-1}\left(gfgf_i\right)$$

$b1 =$ ▮

$$aaa_j := wq_j + 0.577\cdot gfgf_j$$

$aaa_0 =$

$$a1 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} aaa_i$$

$a1 =$ ▮

Appendix B

$$\sigma11 := \frac{1}{k-1}\cdot\left[\sum_{i=0}^{k-1}\left(aaa_i - a1\right)^2\right]$$

$(\sigma11) = ▮$

$$sk11 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(aaa_i\right)^3 - 3\cdot(a1)\cdot\sum_{i=0}^{k-1}\left(aaa_i\right)^2 + 3\cdot(a1)^2\cdot\sum_{i=0}^{k-1} aaa_i - (a1)^3\right]}{\sigma11^{\frac{3}{2}}}$$

$(sk11) = ▮$

$$ku11 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(aaa_i\right)^4 - 4\cdot(a1)\cdot\sum_{i=0}^{k-1}\left(aaa_i\right)^3 + 6\cdot(a1)^2\cdot\sum_{i=0}^{k-1}\left(aaa_i\right)^2 - 4\cdot(a1)^3\cdot\sum_{i=0}^{k-1} aaa_i + (a1)^4\right]}{\sigma11^2} - 3$$

$$mse11 := \sigma11 + (bais11)^2$$

$mse11 = ▮$

$(ku11) = ▮$

$$bais12 := b1 - b$$

$bais12 = ▮$

$$\sigma12 := \frac{1}{k-1}\cdot\left[\sum_{i=0}^{k-1}\left(gfgf_i - b1\right)^2\right]$$

$\sigma12 = ▮$

$$sk12 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(gfgf_i\right)^3 - 3\cdot(b1)\cdot\sum_{i=0}^{k-1}\left(gfgf_i\right)^2 + 3\cdot(b1)^2\cdot\sum_{i=0}^{k-1} gfgf_i - (b1)^3\right]}{\sigma12^{\frac{3}{2}}}$$

$sk12 =$

$$ku12 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(gfgf_i\right)^4 - 4\cdot(b1)\cdot\sum_{i=0}^{k-1}\left(gfgf_i\right)^3 + 6\cdot(b1)^2\cdot\sum_{i=0}^{k-1}\left(gfgf_i\right)^2 - 4\cdot(b1)^3\cdot\sum_{i=0}^{k-1} gfgf_i + (b1)^4\right]}{\sigma12^2} - 3$$

$$mse12 := \sigma12 + (bais12)^2$$

$ku12 = ▮$

$mse12 = ▮$

reaw := for $j \in 0..k-1$

$$wew_j \leftarrow e^{-e^{\frac{-a+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b}}}$$

$$rea := \frac{1}{k}\cdot\sum_{i=0}^{k-1} reaw_i$$

$rea = ▮$

hazw := for $j \in 0..k-1$

$$wew_j \leftarrow \frac{1}{b}\cdot e^{\frac{-a+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b}}$$

$$haz := \frac{1}{k}\cdot\sum_{i=0}^{k-1} hazw_i$$

$haz = ▮$

Appendix B

$$rea11 := \text{for } j \in 0..k-1$$
$$\quad wew_j \leftarrow e^{-e^{\frac{-a1+\sum_{i=0}^{n-1}(x_j)_i}{b1}}}$$

$$rea1 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} rea11_i$$

$$rea1 = ▮$$

$$haz311 := \text{for } j \in 0..k-1$$
$$\quad weew_j \leftarrow \frac{1}{b1}\cdot e^{\frac{-a1+\sum_{i=0}^{n-1}(x_j)_i}{b1}}$$

$$haz61 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} haz311_i$$

$$haz61 = ▮$$

$$ba1 := rea1 - rea$$

$$ba1 = ▮$$

$$ha1 := haz61 - haz$$

$$ha1 = ▮$$

Program 3: Estimation by Maximum Likelihood Method

Enter your values of a, b, k and n

$$a := ▮ \quad b := ▮ \quad n := ▮ \quad k := ▮$$
$$i := 0..n-1 \quad j := 0..k-1$$

$$u := \text{for } j \in 0..k-1$$
$$\quad ww_j \leftarrow \text{for } i \in 0..n-1$$
$$\quad\quad wk_i \leftarrow rnd(1)$$

$$u_0 =$$

$$x := \text{for } i \in 0..n-1$$
$$\quad kwk \leftarrow \text{for } j \in 0..k-1$$
$$\quad\quad kk_j \leftarrow a + b\cdot\ln(-\ln(u_j))$$

$$x_0 =$$

Appendix B

$$aa_0 := a \qquad bb_0 := b$$

$$
\begin{aligned}
f144 := &\ \text{for } w \in 0..k-1 \\
&\quad dq_w \leftarrow \text{for } j \in 0..k-1 \\
&\qquad f1_j \leftarrow \frac{-n}{bb_j} + \frac{1}{bb_j}\cdot\sum_{i=0}^{n-1} e^{\frac{(x_w)_i - aa_j}{bb_j}} \\
&\qquad f2_j \leftarrow \frac{-n}{bb_j} + \frac{n\cdot aa_j}{(bb_j)^2} - \frac{1}{(bb_j)^2}\cdot\sum_{i=0}^{n-1}(x_w)_i + \sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^2}\right]\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right] \\
&\qquad f3_j \leftarrow \frac{-1}{(bb_j)^2}\cdot\sum_{i=0}^{n-1} e^{\frac{(x_w)_i - aa_j}{bb_j}} \\
&\qquad f4_j \leftarrow \frac{n}{(bb_j)^2} - \frac{1}{(bb_j)^2}\cdot\sum_{i=0}^{n-1} e^{\frac{(x_w)_i - aa_j}{bb_j}} - \sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^3}\right]\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right] \\
&\qquad f5_j \leftarrow \frac{n}{(bb_j)^2} - \frac{2\cdot n\cdot aa_j}{(bb_j)^3} + \frac{2}{(bb_j)^3}\cdot\sum_{i=0}^{n-1}(x_w)_i - 2\cdot\sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^3}\right]\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right] - \sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^2}\right]^2\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right] \\
&\qquad aa_{j+1} \leftarrow aa_j - \left[\frac{f5_j\cdot f1_j - f4_j\cdot f2_j}{f3_j\cdot f5_j - (f4_j)^2}\right] \\
&\qquad bb_{j+1} \leftarrow bb_j - \left[\frac{f3_j\cdot f2_j - f4_j\cdot f1_j}{f3_j\cdot f5_j - (f4_j)^2}\right] \\
&\qquad f6_j \leftarrow \sqrt{\left[\frac{f3_j\cdot f2_j - f4_j\cdot f1_j}{f3_j\cdot f5_j - (f4_j)^2}\right]^2 + \left[\frac{f5_j\cdot f1_j - f4_j\cdot f2_j}{f3_j\cdot f5_j - (f4_j)^2}\right]^2} \\
&\qquad \text{break if } f6_j < 10^{-6} \\
&\quad dq_w \leftarrow aa_{j+1}
\end{aligned}
$$

$f144 = ▮$

Appendix B

$$f14 := \text{for } w \in 0..k-1$$

$$dq_w \leftarrow \text{for } j \in 0..k-1$$

$$f1_j \leftarrow \frac{-n}{bb_j} + \frac{1}{bb_j}\cdot\sum_{i=0}^{n-1} e^{\frac{(x_w)_i - aa_j}{bb_j}}$$

$$f2_j \leftarrow \frac{-n}{bb_j} + \frac{n\cdot aa_j}{(bb_j)^2} - \frac{1}{(bb_j)^2}\cdot\sum_{i=0}^{n-1}(x_w)_i + \sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^2}\right]\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right]$$

$$f3_j \leftarrow \frac{-1}{(bb_j)^2}\cdot\sum_{i=0}^{n-1} e^{\frac{(x_w)_i - aa_j}{bb_j}}$$

$$f4_j \leftarrow \frac{n}{(bb_j)^2} - \frac{1}{(bb_j)^2}\cdot\sum_{i=0}^{n-1} e^{\frac{(x_w)_i - aa_j}{bb_j}} - \sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^3}\right]\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right]$$

$$f5_j \leftarrow \frac{n}{(bb_j)^2} - \frac{2\cdot n\cdot aa_j}{(bb_j)^3} + \frac{2}{(bb_j)^3}\cdot\sum_{i=0}^{n-1}(x_w)_i - 2\cdot\sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^3}\right]\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right] - \sum_{i=0}^{n-1}\left[\left[\frac{(x_w)_i - aa_j}{(bb_j)^2}\right]^2\cdot e^{\frac{(x_w)_i - aa_j}{bb_j}}\right]$$

$$aa_{j+1} \leftarrow aa_j - \left[\frac{f5_j\cdot f1_j - f4_j\cdot f2_j}{f3_j\cdot f5_j - (f4_j)^2}\right]$$

$$bb_{j+1} \leftarrow bb_j - \left[\frac{f3_j\cdot f2_j - f4_j\cdot f1_j}{f3_j\cdot f5_j - (f4_j)^2}\right]$$

$$f6_j \leftarrow \sqrt{\left[\frac{f3_j\cdot f2_j - f4_j\cdot f1_j}{f3_j\cdot f5_j - (f4_j)^2}\right]^2 + \left[\frac{f5_j\cdot f1_j - f4_j\cdot f2_j}{f3_j\cdot f5_j - (f4_j)^2}\right]^2}$$

$$\text{break if } f6_j < 10^{-6}$$

$$dq_w \leftarrow bb_{j+1}$$

$f14 =$

$$a2 := \frac{1}{k}\cdot\sum_{i=0}^{k-1}(f144_i) \qquad b2 := \frac{1}{k}\cdot\sum_{i=0}^{k-1}(f14_i)$$

$a2 =$ ▮ $b2 =$ ▮

$$bais21 := a2 - a$$

$bais21 =$ ▮

$$\sigma 21 := \frac{1}{k-1}\cdot\sum_{j=0}^{k-1}(f144_j - a2)^2$$

$\sigma 21 =$ ▮

Appendix B

$$sk21 := \frac{\frac{1}{k}\cdot\left[\sum_{j=0}^{k-1}\left(f144_j\right)^3 - 3\cdot a2\cdot\sum_{j=0}^{k-1}\left(f144_j\right)^2 + 3\cdot a2^2\cdot\sum_{j=0}^{k-1} f144_j - a2^3\right]}{\sigma21^{\frac{3}{2}}}$$

$(sk21) =$ ▮

$$ku21 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(f144_i\right)^4 - 4\cdot a2\cdot\sum_{i=0}^{k-1}\left(f144_i\right)^3 + 6\cdot a2^2\cdot\sum_{i=0}^{k-1}\left(f144_i\right)^2 - 4\cdot a2^3\cdot\sum_{i=0}^{k-1} f144_i + a2^4\right]}{\sigma21^2} - 3$$

$(ku21)$:

$$mse21 := \sigma21 + (bais21)^2$$

$mse21 =$ ▮

$$bais22 := b2 - b$$

$bais22 =$ ▮

$$\sigma22 := \frac{1}{k-1}\cdot\sum_{i=0}^{k-1}\left(f14_i - b2\right)^2$$

$(\sigma22) =$ ▮

$$sk22 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(f14_i\right)^3 - 3\cdot b2\cdot\sum_{i=0}^{k-1}\left(f14_i\right)^2 + 3\cdot b2^2\cdot\sum_{i=0}^{k-1} f14_i - b2^3\right]}{\sigma22^{\frac{3}{2}}}$$

$(sk22) =$

$$ku22 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(f14_i\right)^4 - 4\cdot b2\cdot\sum_{i=0}^{k-1}\left(f14_i\right)^3 + 6\cdot b2^2\cdot\sum_{i=0}^{k-1}\left(f14_i\right)^2 - 4\cdot b2^3\cdot\sum_{i=0}^{k-1} f14_i + b2^4\right]}{\sigma22^2} - 3$$

$$mse22 := \sigma22 + (bais22)^2$$

$(ku22)$:

$mse22 =$ ▮

$$rea221 := \text{for } j \in 0..k-1 \quad wew_j \leftarrow e^{-e^{\frac{-a2+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b2}}}$$

$$rea232 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} rea221_i$$

$rea232 =$ ▮

$$haz221 := \text{for } j \in 0..k-1 \quad weew_j \leftarrow \frac{1}{b2}\cdot e^{\frac{-a2+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b2}}$$

$$haz223 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} haz221_i$$

$haz223 =$ ▮

$$ba2 := rea232 - rea$$

$ba2 =$ ▮

$$ha2 := haz223 - haz$$

$ha2 =$ ▮

Appendix B

Program 4: Estimation by Order Statistic Method

Enter your values of a, b, k and n

$a :=$ ▮ $b :=$ ▮ $n :=$ ▮ $k :=$ ▮

$i := 0..n-1$ $j := 0..k-1$

$$u := \text{for } j \in 0..k-1 \quad ww_j \leftarrow \text{for } i \in 0..n-1 \quad wk_i \leftarrow \text{rnd}(1)$$

$u_0 =$

$$x := \text{for } i \in 0..n-1 \quad kwk \leftarrow \text{for } j \in 0..k-1 \quad kk_j \leftarrow a + b\cdot\ln(-\ln(u_j))$$

$x_0 =$

$$gfh := \text{for } j \in 0..k-1 \quad tty_j \leftarrow \min(x_j)$$

$gfh_0 =$ ▮

$$y := \text{for } j \in 0..k-1 \quad tty_j \leftarrow \frac{wq_j - gfh_j}{\ln(n)}$$

$y_0 =$ ▮

$$b3 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} y_i$$

$b3 =$ ▮

$$ay_j := wq_j + 0.577y_j$$

$ay_0 =$ ▮

$$a3 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} ay_i$$

$a3 =$ ▮

$$bais31 := a3 - a$$

$bais31 =$ ▮

$$\sigma31 := \frac{1}{k-1}\cdot\sum_{i=0}^{k-1}(ay_i - a3)^2$$

$\sigma31 =$ ▮

$$sk31 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}(ay_i)^3 - 3\cdot a3\cdot\sum_{i=0}^{k-1}(ay_i)^2 + 3\cdot a3^2\cdot\sum_{i=0}^{k-1}(ay_i) - a3^3\right]}{\sigma31^{\frac{3}{2}}}$$

$sk31 =$ ▮

$$ku31 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}(ay_i)^4 - 4\cdot a3\cdot\sum_{i=0}^{k-1}(ay_i)^3 + 6\cdot a3^2\cdot\sum_{i=0}^{k-1}(ay_i)^2 - 4\cdot a3^3\cdot\sum_{i=0}^{k-1} ay_i + a3^4\right]}{\sigma31^2} - 3$$

$ku31 =$ ▮

$$mse31 := \sigma31 + (bais31)^2$$

$mse31 =$ ▮

$$bais32 := b3 - b$$

$bais32 =$ ▮

Appendix B

$$\sigma32 := \frac{1}{k-1}\cdot\sum_{i=0}^{k-1}\left(y_i - b3\right)^2$$

$\sigma32 = ▮$

$$sk32 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(y_i\right)^3 - 3\cdot b3\cdot\sum_{i=0}^{k-1}\left(y_i\right)^2 + 3\cdot b3^2\cdot\sum_{i=0}^{k-1}y_i - b3^3\right]}{\sigma32^{\frac{3}{2}}}$$

$sk32 = ▮$

$$ku32 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(y_i\right)^4 - 4\cdot b3\cdot\sum_{i=0}^{k-1}\left(y_i\right)^3 + 6\cdot b3^2\cdot\sum_{i=0}^{k-1}\left(y_i\right)^2 - 4\cdot b3^3\cdot\sum_{i=0}^{k-1}y_i + b3^4\right]}{\sigma32^2} - 3$$

$$mse32 := \sigma32 + (bais32)^2$$

$ku32 = ▮$

$mse32 = ▮$

$$rea31 := \text{for } j \in 0..k-1 \quad wew_j \leftarrow e^{-e^{\frac{-a3+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b3}}}$$

$rea31_0 = ▮$

$$rea3 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} rea31_i$$

$rea3 = ▮$

$$haz31 := \text{for } j \in 0..k-1 \quad weew_j \leftarrow \frac{1}{b3}\cdot e^{\frac{-a3+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b3}}$$

$haz31_0 = ▮$

$$haz3 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} haz31_i$$

$haz3 = ▮$

$$ba3 := rea3 - rea$$

$ba3 = ▮$

$$ha3 := haz3 - haz$$

$ha3 = ▮$

Appendix B

Program 5: Estimation by Least Squares Method

Enter your values of a, b, k and n

$a :=$ ▮ $\quad b :=$ ▮ $\quad n :=$ ▮ $\quad k :=$ ▮

$i := 0..n-1 \qquad j := 0..k-1$

$$u := \text{for } j \in 0..k-1 \quad ww_j \leftarrow \text{for } i \in 0..n-1 \quad wk_i \leftarrow \text{rnd}(1)$$

$u_0 =$

$$x := \text{for } i \in 0..n-1 \quad kwk \leftarrow \text{for } j \in 0..k-1 \quad kk_j \leftarrow a + b\cdot\ln(-\ln(u_j))$$

$x_0 =$

$$tt := \text{for } i \in 0..n-1 \quad kwk \leftarrow \text{for } j \in 0..k-1 \quad kk_j \leftarrow \ln(-\ln(u_j))$$

$tt_0 =$ ▮

$$twq := \text{for } j \in 0..k-1 \quad ew_j \leftarrow \frac{1}{n}\cdot\sum_{i=0}^{n-1}(tt_j)_i$$

$twq_0 =$ ▮

$$tq := \text{for } j \in 0..k-1 \quad ew_j \leftarrow \frac{\left[twq_j\cdot\sum_{i=0}^{n-1}(x_j)_i\right]-\left[\sum_{i=0}^{n-1}\left[(tt_j)_i\cdot(x_j)_i\right]\right]}{\left[twq_j\cdot\sum_{i=0}^{n-1}(tt_j)_i\right]-\left[\sum_{i=0}^{n-1}\left[(tt_j)_i\right]^2\right]}$$

$tq_0 =$ ▮

$$b4 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} tq_i$$

$b4 =$ ▮

$$aa4_j := wq_j - twq_j\cdot tq_j$$

$aa4_0 =$ ▮

$$a4 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} aa4_i$$

$a4 =$ ▮

$bais41 =$ ▮

$$bais41 := a4 - a$$

$$\sigma41 := \frac{1}{k-1}\cdot\sum_{j=0}^{k-1}\left(aa4_j - a4\right)^2$$

$\sigma41 =$ ▮

$$var1 := \text{for } j \in 0..k-1 \quad ew_j \leftarrow \frac{\pi^2\cdot b^2}{6}\cdot\left[\frac{1}{n}+\frac{(twq_j)^2}{\sum_{i=0}^{n-1}\left[(tt_j)_i - twq_j\right]^2}\right]$$

$var1 =$ ▮

$$v1 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} var1_i$$

$v1 =$ ▮

Appendix B

var2 := for $j \in 0..k-1$

$$ew_j \leftarrow \frac{\pi^2 \cdot b^2}{6\sum_{i=0}^{n-1}\left[\left(tt_j\right)_i - twq_j\right]^2}$$

var2 = ▮

$$v2 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} var2_i$$

v2 = ▮

$$sk41 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(aa4_i\right)^3 - 3\cdot a4\cdot\sum_{i=0}^{k-1}\left(aa4_i\right)^2 + 3\cdot a4^2\cdot\sum_{i=0}^{k-1}\left(aa4_i\right) - a4^3\right]}{\sigma 41^{\frac{3}{2}}}$$

sk41 = ▮

$$ku41 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(aa4_i\right)^4 - 4\cdot a4\cdot\sum_{i=0}^{k-1}\left(aa4_i\right)^3 + 6\cdot a4^2\cdot\sum_{i=0}^{k-1}\left(aa4_i\right)^2 - 4\cdot a4^3\cdot\sum_{i=0}^{k-1} aa4_i + a4^4\right]}{\sigma 41^2} - 3$$

ku41 = ▮

$$mse41 := \sigma 41 + (bais41)^2$$

mse41 = ▮

$$bais42 := b4 - b$$

bais42 = ▮

$$\sigma 42 := \frac{1}{k-1}\cdot\sum_{j=0}^{k-1}\left(tq_j - b4\right)^2$$

σ42 = ▮

$$sk42 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(tq_i\right)^3 - 3\cdot b4\cdot\sum_{i=0}^{k-1}\left(tq_i\right)^2 + 3\cdot b4^2\cdot\sum_{i=0}^{k-1} tq_i - b4^3\right]}{\sigma 42^{\frac{3}{2}}}$$

sk42 = ▮

$$ku42 := \frac{\frac{1}{k}\cdot\left[\sum_{i=0}^{k-1}\left(tq_i\right)^4 - 4\cdot b4\cdot\sum_{i=0}^{k-1}\left(tq_i\right)^3 + 6\cdot b4^2\cdot\sum_{i=0}^{k-1}\left(tq_i\right)^2 - 4\cdot b4^3\cdot\sum_{i=0}^{k-1} tq_i + b4^4\right]}{\sigma 42^2} - 3$$

ku42 = ▮

$$mse42 := \sigma 42 + (bais42)^2$$

mse42 = ▮

rea41 := for $j \in 0..k-1$

$$wew_j \leftarrow e^{-e^{\frac{-a4+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b4}}}$$

$rea41_0$ = ▮

$$rea4 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} rea41_j$$

rea4 = ▮

haz41 := for $j \in 0..k-1$

$$weew_j \leftarrow \frac{1}{b4}\cdot e^{\frac{-a4+\sum_{i=0}^{n-1}\left(x_j\right)_i}{b4}}$$

$haz41_0$ = ▮

$$haz4 := \frac{1}{k}\cdot\sum_{i=0}^{k-1} haz41_j$$

haz4 = ▮

$$ba4 := rea4 - rea$$

ba4 = ▮

$$ha4 := haz4 - haz$$

ha4 = ▮

Appendix C

Appendix C

Computer Program of Chi-Square Goodness-of-Fit Test for Ext(0,1)

Program (6) :Chi-Square Goodness–of-Fit Test

Enter your values of k and n

```
n := ■          k := ■

i := 0.. n − 1      j := 0.. k − 1

u := for  j ∈ 0.. k − 1
        ww_j ← for  i ∈ 0.. n − 1                 u_0 = ■
                  wk_i ← rnd(1)

x := for  i ∈ 0.. n − 1
        kwk ← for  j ∈ 0.. k − 1                                      x_0 = ■
                 kk_j ← ln(−ln(u_j))

qq := for  j ∈ 0.. k − 1
          for  t ∈ 0.. 4
             v_t ← 0
          tf_j ← for  i ∈ 0.. n − 1
                     v_0 ← v_0 + 1  if  2 ≥ (x_j)_i > .6
                     v_1 ← v_1 + 1  if  .6 ≥ (x_j)_i > −.5
                     v_2 ← v_2 + 1  if  −.5 ≥ (x_j)_i > −1.5              qq_0 = ■
                     v_3 ← v_3 + 1  if  −1.5 ≥ (x_j)_i > −2.4
                     v_4 ← v_4 + 1  if  −2.4 ≥ (x_j)_i > −7

          yy_0 := 18.39      yy_1 := 36.78      yy_2 := 25.4   yy_3 := 11.8   yy_4 := 7.54

     rrf := for  j ∈ 0.. k − 1
               rt_j ← for  i ∈ 0.. 9                rrf_0 = ■
                         yy
```

Appendix C

$$ddd_j := \sum_{i=0}^{4} \frac{\left[\left(qq_j\right)_i - \left(rrf_j\right)_i\right]^2}{\left(rrf_j\right)_i} \qquad ddd = ▮$$

$$yu := 0$$

$$fr := \text{for } j \in 0..k-1$$

$$\quad yu \leftarrow yu + 1 \text{ if } ddd_j > 9.49 \qquad fr = ▮$$

$$ty := \frac{fr}{k} \qquad ty = ▮$$

Appendix D

The p.d.f. of extreme value distn. is :

$$f(x;a,b)=\frac{1}{\beta}e^{(\frac{x-\alpha}{\beta})-e^{(\frac{x-\alpha}{\beta})}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

The details about the graph of y = f(x;α,β),

The term $\frac{1}{\beta}e^{(\frac{-\alpha}{\beta})-e^{(\frac{-\alpha}{\beta})}} \neq 0$, that implies the x-axis is an horizontal asymptote.

The first derivative of y, gives

$$y'=\frac{1}{\beta^2}e^{(\frac{x-\alpha}{\beta})-e^{(\frac{x-\alpha}{\beta})}}\left[1-e^{(\frac{x-\alpha}{\beta})}\right]=0$$

because $e^{(\frac{x-\alpha}{\beta})-e^{(\frac{x-\alpha}{\beta})}} > 0$ and $\frac{1}{\beta^2} > 0$, $(\beta > 0)$ implies

$1-e^{(\frac{x-\alpha}{\beta})}=0$ implies $e^{(\frac{x-\alpha}{\beta})}=1$ implies $x=a$ implies

y increasing function for $-\infty < x < a$ and decreasing function for $a < x < \infty$, and having the maximum point at $(a, \frac{1}{\beta}e^{-1})$

The second derivative of y, gives

$$y'=\frac{1}{\beta^3}e^{(\frac{x-\alpha}{\beta})-e^{(\frac{x-\alpha}{\beta})}}\left[\left[1-e^{(\frac{x-\alpha}{\beta})}\right]^2-e^{(\frac{x-\alpha}{\beta})}\right]=0$$

because $e^{(\frac{x-\alpha}{\beta})-e^{(\frac{x-\alpha}{\beta})}} > 0$ and $\frac{1}{\beta^3} > 0$, $(\beta > 0)$ implies

Appendix D

$$\left[1-e^{(\frac{x-\alpha}{\beta})}\right]^2 - e^{(\frac{x-\alpha}{\beta})} = 0 \text{ implies } 1-3e^{(\frac{x-\alpha}{\beta})} + \left[e^{(\frac{x-\alpha}{\beta})}\right]^2 = 0$$

implies $x = a + b\ln\left[\frac{3\pm\sqrt{5}}{2}\right]$ implies

y concave upward for $-\infty < x < a + b\ln\left[\frac{3-\sqrt{5}}{2}\right]$ and

$a + b\ln\left[\frac{3+\sqrt{5}}{2}\right] < x < \infty$ while y concave downward for

$a + b\ln\left[\frac{3-\sqrt{5}}{2}\right] < x < a + b\ln\left[\frac{3+\sqrt{5}}{2}\right]$

Set $x = a + b\ln\left[\frac{3\pm\sqrt{5}}{2}\right]$ in eq. (1), gives

$y = \frac{1}{b}\left[\frac{3\pm\sqrt{5}}{2}\right]e^{-\left[\frac{3\pm\sqrt{5}}{2}\right]}$, implies

y have two points of inflection at $(a + b\ln\left[\frac{3\pm\sqrt{5}}{2}\right], \frac{1}{b}\left[\frac{3\pm\sqrt{5}}{2}\right]e^{-\left[\frac{3\pm\sqrt{5}}{2}\right]})$

Appendix D

To finding the moments and other properties of extreme value distn. , then we have :

$$\Phi(t)=\ln M(t)$$

$$\Phi'(t)=\frac{M'(t)}{M(t)}$$

$$\Phi'(0)=\frac{M'(0)}{M(0)}=\frac{m}{1}=E(X)$$

$$\Phi''(t)=\frac{M(t)M''(t)-[M'(t)]^2}{[M(t)]^2}$$

$$\Phi''(0)=\frac{M(0)M''(0)-[M'(0)]^2}{[M(0)]^2}=\frac{M''(0)-[M'(0)]^2}{1}=E(X^2)-m^2=E[(X-m)^2]=Var(X)$$

$$\Phi'''(t)=\frac{[M(t)]^2[M(t)M'''(t)+M'(t)M''(t)-2M'(t)M''(t)]-2[M(t)M''(t)-[M'(t)]^2]M(t)M'(t)}{[M(t)]^4}$$

$$=\frac{[M(t)]^2M'''(t)-M(t)M'(t)M''(t)-2M(t)M'(t)M''(t)+2[M'(t)]^3}{[M(t)]^3}$$

$$=\frac{[M(t)]^2M'''(t)-3M(t)M'(t)M''(t)+2[M'(t)]^3}{[M(t)]^3}=\frac{M'''(t)}{M(t)}-3\frac{M'(t)M''(t)}{[M(t)]^2}+2\left[\frac{M'(t)}{M(t)}\right]^3$$

$$\Phi'''(0)=\frac{M'''(0)}{M(0)}-3\frac{M'(0)M''(0)}{[M(0)]^2}+2\frac{[M'(0)]^3}{[M(0)]^3}=\frac{E(X^3)}{1}-3\frac{mE(X^2)}{1}+2\frac{m^3}{1}=E[(X-m)^3]$$

$$\Phi''''(t)=\frac{M(t)M''''(t)-M'(t)M'''(t)}{[M(t)]^2}-3\left[\frac{[M(t)]^2[M'(t)M'''(t)+[M''(t)]^2]-2M(t)M''(t)[M'(t)]^2}{[M(t)]^4}\right]$$

$$+6\left[\frac{M'(t)}{M(t)}\right]^2\frac{M(t)M''(t)-[M'(t)]^2}{[M(t)]^2}$$

$$\Phi''''(0)=\frac{M(0)M''''(0)-M'(0)M'''(0)}{[M(0)]^2}-3\left[\frac{[M(0)]^2[M'(0)M'''(0)+[M''(0)]^2]-2M(0)M''(0)[M'(0)]^2}{[M(0)]^4}\right]$$

$$+6\left[\frac{M'(0)}{M(0)}\right]^2\frac{M(0)M''(0)-[M'(0)]^2}{[M(0)]^2}$$

$$=\frac{E(X^4)-mE(X^3)}{1}-3\left[\frac{mE(X^3)+[E(X^2)]^2-2m^2E(X^2)}{1}\right]+6\left[\frac{m}{1}\right]^2\frac{E(X^2)-m^2}{1}$$

$$=E(X^4)-mE(X^3)-3mE(X^3)-3[E(X^2)]^2+6m^2E(X^2)+6m^2E(X^2)-6m^4$$

Appendix D

$$
\begin{aligned}
\Phi''''(0) &= E(X^4) - 4mE(X^3) - 3[Var(X) + m^2]^2 + 12m^2[Var(X) + m^2] - 6m^4 \\
&= E(X^4) - 4mE(X^3) - 3[Var(X)]^2 - 6m^2Var(X) - 3m^4 + 12m^2Var(X) + 12m^4 - 6m^4 \\
&= E(X^4) - 4mE(X^3) - 3[Var(X)]^2 - 2m^4 + 6m^2Var(X) + 3m^4 \\
&= E(X^4) - 4mE(X^3) + 6m^2Var(X) + 3m^4 - 3[Var(X)]^2 \\
&= E[(X - m)^4] - 3[Var(X)]^2
\end{aligned}
$$

$$E[(X - m)^4] = \Phi''''(0) + 3[Var(X)]^2$$

الخلاصة

تم التطرق في هذه الرسالة الى توزيع القيمة المتطرفة ذو المعلمتين بسبب ظهوره في العديد من مجالات الاحصاء وتطبيقاته. خواص الرياضية والاحصائية لهذا التوزيع كالعزوم والعزوم العليا قد جمعت ووحدت، بالاضافة الى استعراض خواص كل من دالتي المعولية والمخاطرة.

استخدم أختبار مربع كاي لجودة التوفيق لاختبار العينات المولدة من توزيع القيمة المتطرفة القياسية بطريقة محاكاة مونت كارلو لمعرفة فيما اذا كانت قابلة للاستخدام.

استخدمت هذه العينات في تخمين معالم التوزيع باربع طرق وهي طريقة العزوم، طريقة الترجيح الأعظم، طريقة العزوم المعدلة للعينات المرتبة احصائياً وطريقة المربعات الصغرى

نوقشت هذه الطرق نظرياً وطبقت عملياً في تخمين دالتي المعولية والمخاطرة. خواص المخمنات وضعت في جداول كمقياس التحيز والتباين ومعامل الالتواء ومعامل التفلطح ومقياس معدل مربع الخطأ.

تم وضع برامج الحاسوب في ثلاث ملحقات و نفذت باستخدام برنامج (MathCAD 14).

جمهورية العراق
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قسم الرياضيات و تطبيقات الحاسوب

دراسة كفاءة طرق التقدير لمعلمات توزيع القيمة المتطرفة باستخدام معاينة مونت كارلو

رسالة

مقدمة إلى كلية العلوم - جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير في علوم الرياضيات

مِنْ قِبَل

فادي عادل ابراهيم يونان شعيو

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