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Stabilization of Nonlinear Stochastic Control System via Output – Feedback Control

A Thesis
Submitted to the Department of Mathematics, College of Science, Al-Nahrain University as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
By

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Shawal **October**

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بسم الله الرحمن الرحيم

{رَبِّ اشْرَحْ لِي صَدْرِي*وَيَسِّرْ لِي أَمْرِي*وَاحْلُلْ عُقْدَةً مِنْ لِسَانِي*يَفْقَهُوا قَوْلِي}

صدق الله العلي العظيم

(سورة طه/الايه ٢٥-٢٨)

الأهداء

الى الشفاه التي أكثرت الدعاء لنا كلما نطقت

الى العين التي رأت فينا أملاً كلما نظرت

الى القلوب التي أزدادت بنا فخراً كلما نبضت

الى أمي وأبي

الى أخوتي

الى من أحبهم ويحبوني

إيناس

Supervisor Certification

I certify that this thesis was prepared under my supervision at the department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:
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Date: / /2008

In view of the available recommendations; I forward this thesis for debate by examination committee.

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Head of the Department Mathematics and Computer Applications
Date: / /2008

Examining Committee's Certification

We certify that we read this thesis entitled " ***Stabilization of Nonlinear Stochastic Control System via Output - Feedback Control***" and as examining committee examined the student, ***Enas Ajil Al-Rekabi*** in its contents and in what it connected with, and that is in our opinion it meet the standard of thesis for the degree of Master of Science, in Mathematics.

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Date: 2009/ /

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Enas Ajil Al-Rekabi

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Contents

Abstract

Stochastic differential equations are one of the most useful areas of the theory of stochastic processes and its applications in mathematics.

Some nonlinear (Itô) [1] dynamic stochastic control system driven by Brownian motion [2] based on dynamic observer have been considered.

Output feedback (observer – based) robust and optimal control law which guarantees global (local) asymptotic stable in probability for the nonlinear stochastic dynamic system are discuss and developed. The necessary theorems regarding the globalty asymptotic stable in the probability of the equilibrium point at the origin of the closed loop stochastic system have been developed and proved. The Lyapunov function approach of stochastic dynamic system has been adapted to justify our proofs.

The inverse optimal stabilization in probability with suitable performance index has also discussed and developed. The necessary mathematical requirements have also been provided. Concluding remarks, future work, computational algorithm based on the theoretical results and illustrations have been presented.

[1] This field of stochastic dynamic is firstly derived and discussed by Kiyosi Itô. In literatures such a stochastic dynamic.

[2] Robert Brown described the motion of a pollen particle suspended in fluid in 1828. It was observed that a particle moved in an irregular, random fashion.

Introduction

Despite major advances in robust stabilization of deterministic nonlinear systems achieved over the last few years and reported in [Krstic, 95],[Freeman, 96] and references therein, the stabilization problem for stochastic systems is yet to be addressed. While not as refined as their deterministic counterparts in [Khalil, 96], Lyapunov techniques for stability analysis of stochastic systems do exist, see, for example, the classical book of Khas'minskii [Kas'minskii, 80] (see also [Kushner,67]). Efforts toward (global) stabilization of stochastic nonlinear systems have been initiated in the work of Florchinger [Florchinger, 93],[Florchinger, 95, a], [Florchinger, 95, b] who, among other things, extended the concept of control Lyapunov functions and Sontag's stabilization formula [Sontag, 89] to the stochastic setting. A breakthrough towards arriving at constructive methods for stabilization of broader classes of stochastic nonlinear systems came with the result of Pan and Basar [Pan&Basar, 96], who derived a robust design for strict-feedback systems motivated by a risk-sensitive cost criterion, [Pan & Bernhard, 95], [James, 94], [Nagai, 96], [Runolfsson, 94], (for other types of optimal control problems, see, e.g., [Haussmann & Suo, 95,a] and [Hussmann & suo, 95, b]).

Stochastic differential equations (SDE's) constitute an ideal mathematical model for a multitude of phenomena and processes encountered in areas such as differential equation, stochastic control, signal processes and mathematical finance, most notably in option pricing (see for example [Øksendal, 98] and [Kloeden & Platen, 92]). Unlike their deterministic counterparts, SDEs do not have explicit solutions, apart from in a few exceptional cases; hence the necessity for a sound theory of their numerical approximation is important.

It is well-known that stochastic integrals and Itô formula play a central role in modern probability theory and its applications to stochastic differential equation concerned by Brownian motion.

The theory of Itô stochastic differential equations is one of the most beautiful and most useful areas of the theory of stochastic processes. However, until recently the range of investigations in this theory have been, in our view, unjustifiably restricted: only equations were studied which can, in analogy with the deterministic case, be called ordinary stochastic equations. The situation has begun to change in the last 10-16 years. The necessity of considering equations combining the features of partial differential equations and Itô equations has appeared both in the theory of stochastic processes and in related areas. [Krylov&Rozovskii, 07]

Despite huge popularity of the linear-quadratic-Gaussian control problem, the stabilization problem for nonlinear stochastic systems has been receiving relatively little attention until recently.

In [Deng &Krstic, 97, a] and [Deng & Kristic, 97, b], they designed simpler inverse optimal control laws for strict-feedback systems which guarantee global asymptotic stability in probability and whose algorithms can be directly coded in symbolic software.

Some stochastic differential system for linear quadratic stochastic system (optimal control system) as well as robust controller has been adapted. Stabilized and optimal controller designer for some modified nonlinear stochastic dynamic system are derived and modified. The work of this thesis is to generated of the previous work of the literatures [Hua Deng &Krstic, 97], [Hua Deng, 97],[Deng & Kristic, 99], [Florchinger, 93], and [Florchinger, 95, a].

Based on the previous work, this thesis we design a robust and optimal control law which guarantees global asymptotic stability in probability. The design is fully systematic and its algorithm can be directly coded in symbolic software (for examples Matlab software).

We deal with nonlinear systems in which the equilibrium at the origin is preserved even in the presence of noise because the noise vector field is vanishing at the origin. This means that we exclude linear systems with additive noise.

Another preparatory comment of potential interest with technical expertise in robust designs is that the Lyapunov function that we construct is not of the form $V = \sum z_i^2$ but of the form $V = \sum z_i^4$ but in our work, the form $V = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P \tilde{x})^2$ has been adapted. The quartic form (fourth-order) is employed in order to handle some special terms in the Lyapunov analysis which arise due to the some class of Itô differentiation rule system and Itô rule.

This thesis consists of three chapters. The first chapter deals with the basic concept of stochastic dynamic system.

In chapter two, the necessary mathematical principles concerning stochastic integration, Itô formula, Itô SDE, existence and uniqueness of Itô SDEs, as well as some solvable examples have been presented.

In chapter three, two results have presented. First, we design an out put feedback (observer – based) back stepping control law which guarantees global asymptotic stability in probability is presented. The stabilizing control laws which are also optimal with respect to meaningful cost functional are described. The algorithms and examples are also given.

Future work, concluding remarks, appendixes, and references are presented.

This chapter presents basic concepts of stochastic dynamic system which is divided into three sections, the first one describe the set of algebra, the second section deals with the random variable while the third section deals with the stochastic processes and some of its kinds.

1.1 ALGEBRA OF SETS :

The collection of all elementary outcomes of a random experiment is called ***sample space*** and is denoted by Ω. In the terminology, the sample space is termed as the ***universal*** set. Thus, the sample space Ω is a set consisting of mutually exclusive, collectively exhaustive listing of all possible outcomes of a random experiment. That is, $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ denotes the set of all finite outcomes, $\Omega = \{\omega_1, \omega_2, \dots\}$ denotes the set of all countably infinite outcomes, and , $\Omega = \{0 \le t \le T\}$ denotes the set of uncountably infinite outcomes.

Let Ω represent the sample space which is a collection of ω-points as defined earlier. The various set operations are ***complementation***, ***union*** and ***intersection***. Let A and B be two subsets of the sample space Ω, denoted by $A \subset \Omega$, $B \subset \Omega$. The complement of A, denoted by A^c, represents the set of all ω-points not contained in A, i.e,

$$A^c = \{\omega: \ \omega \notin A\} \tag{1.1}$$

Evidently the complement of Ω is the empty set ϕ. The union of sets A and B, denoted by $A \cup B$ or $A+B$, represents the occurrence of ω-points in either A or B. Similarly, the intersection of sets A and B, denoted by $A \cap B$ or AB, represents the occurrence of ω-points in A and B. Clearly, if there is no commonality of ω-points in A and B, then $A \cap B$ is the empty set ϕ.

Example (1.1):

Let Ω be the ω-points on the real line R.

$$\Omega = \{\omega:\ -\infty < \omega < \infty\}$$

Define

$$A = \{\omega:\ \omega \in (-\infty, a)\} = \{\omega: \omega < a\}\ \ a \in R$$

$$B = \{\ \omega:\ \omega \in (b, c)\} = \{\omega: b < \omega < c\}\ b \in R$$

Then the set operations yield

$$A^c = \{a \leq \omega < \infty\}$$

$$B^c = \{-\infty < \omega \leq b\} \cup \{c \leq \omega < \infty\}$$

$$A \cup B = \begin{cases} \{\omega < a\} & c < a \\ \{\omega < c\} & b < a < c \\ \{\omega < a\} \cup \{b < \omega < c\} & a < b \end{cases}$$

$$A \cap B = \begin{cases} \{b < \omega < c\} & c < a \\ \{b < \omega < a\} & b < a < c \\ \phi & a < b \end{cases}$$

The union and intersection of an arbitrary collection of sets are defined by

$$\bigcup_{n\in N} A_n = \{\omega:\ \omega \in A_n \text{for some } n \in N\}$$

$$\bigcap_{n\in N} A_n = \{\omega:\ \omega \in A_n \text{ for all } n \in N\}$$

where N is an arbitrary index set which may be finite or countably infinite.

The union and intersection follow the reflexive, commutative, associative, and distributive laws.

The complements $(\bigcup_{n\in N} A_n)^c$ and $(\bigcap_{n\in N} A_n)^c$ are given by de-Morgan's laws and as follows:

$(\bigcup_{n\in N} A_n)^c = \{\omega:\ \omega$ does not belong to any $A_n, n \in N\}$

$= \{\omega: \ \omega \notin A_n \text{ for all } n \in N\}$

$= \bigcap_{n\in N} A_n^c$

$(\bigcap_{n\in N} A_n)^c = \{\omega: \ \omega \text{ does not belong to each and every } A_n, n \in N\}$

$= \{\omega: \ \omega \text{ does not belong to some } A_n, n \in N\}$

$= \bigcup_{n\in N} A_n^c$

Definition (1.1) Sequences [Krishnan, 84]:

A sequence of sets A_n, $n \in N$, is increasing if $A_{n+1} \supset A_n$ and decreasing if $A_{n+1} \subset A_n$ for every $n \in N$.

Remark (1.1) [Krishnan, 06]:

1. A sequence which is either increasing or decreasing is called a monotone sequence, we can write the limits (N countably infinite) of monotone sequences as:

$$\left.\begin{aligned} \lim_{n\to\infty} A_n &= \lim_{n} A_n \\ &= \bigcup_{n=1}^{\infty} A_n && \{A_n\} \text{ increasing} \\ \lim_{n\to\infty} A_n &= \lim_{n} A_n \\ &= \bigcap_{n=1}^{\infty} A_n && \{A_n\}\ decreasing \end{aligned}\right\} \qquad (2.1)$$

The limit of monotone sequences $\{A_n\}$ is written as $A_n \uparrow A$ when it is increasing and $A_n \downarrow A$ when it is decreasing.

2. We can define a ***superior limit*** and ***inferior limit*** for any sequence $\{A_n\}$ not necessarily monotone. We define subsequences $\{B_n\}$ and $\{C_n\}$ derived from $\{A_n\}$ as follows:

$$B_n = sup_{k\geq n}A_k = \bigcup_{k=n}^{\infty} A_k$$

$$= \{\omega: \quad \omega \ belongs\ to\ at\ least\ one\ of\ A_n, A_{n+1}, \dots\}$$

(1.3)

$$C_n = inf_{k\geq n}A_k = \bigcap_{k=n}^{\infty} A_k$$

$$= \{\omega: \quad \omega \ belongs\ to\ all\ A_k\ except\ A_1, A_2, \dots, A_{n-1}\}$$

(1.4)

Clearly the sequence $\{B_n\}$ and $\{C_n\}$ are monotone decreasing and increasing, respectively.

Example (1.2):

Let A_k be the set of points $(x, y)$of the Cartesian plane R^2 where $\{0 \leq x < k, 0 \leq y < 1/k\}$, that is

$$A_k = \{x, y \in R^2: 0 \leq x < k, 0 \leq y < \frac{1}{k}\}$$

Here $\{A_k\}$ does not belong to the monotone class, while the subsequence

$$B_n = \bigcup_{k=n}^{\infty} A_k = \{(x, y) \in R^2: \ 0 \leq x < \infty, \qquad 0 \leq y < \frac{1}{n}\}$$

is a decreasing sequence, and hence

$$B = lim_n B_n = \bigcap_{n=1}^{\infty} B_n = \{(x, y) \in R^2: 0 \leq x < \infty, y = 0\} = lim\ sup_n A_n$$

Similarly,

$$C_n = \bigcap_{k=n}^{\infty} A_k = \{(x,y) \in R^2 : 0 \le x < n, y = 0\}$$

is an increasing sequence, and hence

$$C = lim_n C_n = \bigcup_{n=1}^{\infty} C_n = \{(x,y) \in R^2 : 0 \le x < \infty, y = 0\} = lim\ inf_n A_n$$

Since $lim\ sup_n A_n = lim\ \ inf_n B_n = \{(x,y) \in R^2 : 0 \le x < \infty, y = 0\}$, we have

$$lim_n A_n = B = C = \{(x,y) \in R^2 : 0 \le x < \infty, y = 0\}.$$

Remark (1.2):

We define ϑ as the nonempty class of subsets drawn from the sample space Ω. We say that the class ϑ is a field or algebra of sets in Ω if it satisfies the following definition.

Definition (1.2) Field (Algebra) [Krishnan, 06]:

A class of a collection of subsets $A_j \subset \Omega$ denoted by ϑ is a field when the following condition are satisfied :

1. If $A_i \in \vartheta$, $then$ $A_i^c \in \vartheta, i = 1,2,\dots,n$
2. If $\{A_i ,i=1,2,\dots,n\} \in \vartheta$,then $\bigcup_{i=1}^{n} A_i \ \epsilon\ \vartheta$ *(1.5)*

Remark (1.3):

Given the above two conditions, de Morgan's law ensures that finite intersections also belong to the field. Thus a class of subsets is a field if and only if it is closed under all finite set operations like unions, intersection, and complementation. Since every Boolean algebra of sets is isomorphic to an algebra of subsets of Ω, we can also call the field a Boolean field or Boolean algebra. Every field contains as elements the sample space Ω and the empty set ϕ, *[Krishnan, 84]*

Example (1.3) [Krishnan, 84]:

Let $\Omega = R$ and consider a class ϑ of all intervals of the form $(a, b]$, that is $\{x \in R: a < x \leq b\}$:

$$(a,b] \cap (c,d] = \begin{cases} \phi & when \quad a < b < c < d \\ (c,b] & when \quad a < c < b < d \\ (a,d] & when \quad c < a < d < b \\ (c,d] & when \quad a < c < d < b \\ (a,b] & when \; c < a < b < d \end{cases}$$

Clearly the class ϑ is closed under intersection. However,

$$(a,b]^c = (-\infty, a] \cup (b, \infty) \notin \vartheta$$

$$(a,b] \cup (c,d] \notin \vartheta \qquad \text{if } a < b < c < d$$

The class ϑ is not a field.

Definition (1.3) σ-Field (σ-Algebra) [Krishnan, 06]:

A class of a countable infinite collection of subsets $A_j \subset \Omega$ denoted by $\mathcal{F}$ is a σ-field when the following conditions are satisfied:

1. If $A_i \in \mathcal{F}$, then $A_i^c \in \mathcal{F}$.
2. If $\{A_i, i=1,2,\ldots\} \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. *(1.6)*

In general a σ-field is a field, but a field may not be a σ-field.

Definition (1.4) Borel σ-Field [Krishnan, 84]:

The minimum σ-field generated by the collection of open sets of a topological space Ω is called the Borel σ-field or Borel field. Members of this σ-field are called Borel sets.

Remarks (1.4) [Krishnan, 84]:

1. Clearly the Borel σ-field is a σ-field, and hence each closed set is also a Borel set.
2. The important topological space with which we will be concerned is the real line R. The collection of Borel sets on the real line is denoted by $\mathcal{R}$.
3. Each open interval is a member of $\mathcal{R}$. From the relationships

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$$

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right) \quad (1.7)$$

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$

4. We find the intervals $(a, b], [a, b)$ and $[a, b]$ are Borel sets. Hence the Borel field $\mathcal{R}$ contains all subsets of the form given above and their complements, countable unions, and intersections, Each set $\{b\} = [b, b] = (-\infty, b) \cap (b, \infty)$ consisting of a single point b is in $\mathcal{R}$, and so are countable unions of single points.

Example (1.4):

Let $\Omega = R$ and ϑ be the class of all intervals of the form $(-\infty, a], (b, c]$, and (d, ∞):

$$(b, c]^c = (-\infty, b] \cup (c, \infty) \in \vartheta$$

$$(d, \infty)^c = (-\infty, d] \in \vartheta$$

$$(-\infty, a]^c = (a, \infty) \in \vartheta$$

From this example the class ϑ is closed under finite intersection. Similarly, it can be shown that ϑ is closed under finite union. Hence the class ϑ is a field. However, for infinite intersections of the form

$$\bigcap_{n=1}^{\infty}\left(b-\frac{1}{n},c\right) = [b,c) \notin \vartheta$$

The class ϑ is not a σ-field.

<u>Definition (1.5) Measurable space [Stirazker, 05] :</u>

A suitable model of the random experiment is therefore a sample space Ω and a σ-field $\mathcal{F}$of subsets of Ω. The space $(\Omega,\mathcal{F})$ thus created is called a measurable space.

<u>Remarks (1.5) [Krishnan, 84], [Stirazker, 05]:</u>

1. ***Events*** are defined as the subsets of Ω which are elements in the σ-field.
2. In particular, Ω is called the ***certain event.***
3. If two events A and B satisfy $A \cap B = \phi$, then they are said to be ***disjoint.***
4. The complement Ω^c is an event called the impossible event, which we denote by $\Omega^c = \phi$, the ***empty set.***
5. If $\{A_i$, i=1,2,…,n $\}$ is a class of disjoint sets of Ω such that $\bigcup_{i=1}^{n} A_i = \Omega$ then the $\{A_i\}$ ***collectively exhaust*** Ω.

<u>Definition (1.6) probability measure [Krishnan,06] :</u>

A ***probability measure*** is a set function P defined on a σ-field $\mathcal{F}$ of subsets of a sample space Ω such that it satisfies the following axioms of *Kolmogorov* for any $A \in \mathcal{F}$:

1. $P(A) \geq 0$ (nonnegativity)
2. $P(\Omega) = 1$ (normalization) *(1.8)*

3. $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ (σ-additivity)
   Whit $A_n \in \mathcal{F}$, and A_i and A_j being pairwise disjoint.

It is also called probability distribution.

Remarks (1.6):

1. Any set function μ defined on a measurable space $(\Omega, \mathcal{F})$ satisfying axioms 1 and 3 of definition (1.6) is called a measure, and a probability measure is a normed or scaled measure because of axiom 2.*[Krishnan, 84]*
2. Any bounded measure with suitable normalization can be converted into a probability measure.*[Krishnan, 84]*
3. If $\mu(A)$ is finite for each $A \in \mathcal{F}$, then μ is a finite measure. However, if $\mu(A)=\infty$ but if there exists a sequence $\{A_n\}$ of members of $\mathcal{F}$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n)$ is finite for each n, then μ is a σ-finite measure, The triplet $(\Omega, \mathcal{F}, \mu)$ is called a measure space, The measure space $(\Omega,\mathcal{F}, P)$ is called a probability space.*[Krishnan, 84]*

4. The probability space serves to describe any random experiment where:
   1. Ω is nonempty set called the sample space, whose elements are the element outcomes of a random experiment,
   2. $\mathcal{F}$ is an σ-field of subsets of Ω.
   3. P is a probability measure defined on the measurable space $(\Omega,\mathcal{F})$. *[Stirzaker, 05]*

Lemma (1.1) Sequential Monotone Continuity [Krishnan, 84]:

Let $\{A_n\}$ be a monotone decreasing sequence in $\mathcal{F}$ such that

$A_{n+1} \subset A_n$, and let $lim_{n\to\infty} A_n = \phi$. Then

$$lim_{n\to\infty} p(A_n) = 0 \qquad (1.9)$$

The probability measure is said to satisfy the sequential monotone continuity at ϕ.

Proposition (1.1) Sequential Continuity [Krishnan, 84]:

Let $\{A_n\}$ be a convergent sequence of events in $\mathcal{F}$, with

$lim_{n\to\infty} A_n = A$.

Then

$$lim_{n\to\infty} P(A_n) = P(lim_{n\to\infty} A_n) = P(A) \qquad (1.10)$$

The probability measure is sequentially continuous.

Proof :

1. If $A = \phi$, then this is exactly Lemma (1.1).
2. If A is nonempty set and $\{A_n\}$ is a monotone sequence, $A_n \downarrow A$, (figure 1.1)

$$P(A_n) = P(A_n - A + A) = P(A_n - A) + P(A)$$

Since $(A_n - A)$ and A are disjoint. If $A_n \uparrow A$ (figure 1.2)

$$P(A_n) = P(A - A + A_n) = P(A) - P(A - A_n)$$

In either case $lim_n(A_n - A)$ or $lim_n(A - A_n)$ decreases to ø, and by ***lemma 1.1*** the result follow.

3. If $\{A_n\}$ is not a monotone sequence, then $\{B_n = sup_{K\geq n} A_k\}$ and $\{C_n = inf_{k\geq n} A_k\}$ are monotone decreasing and increasing sequences, respectively, from equation (1.3) and (1.4). Therefore $B_n \supset A_n \supset C_n$ and $B_n \downarrow A$ and $C_n \uparrow A$.

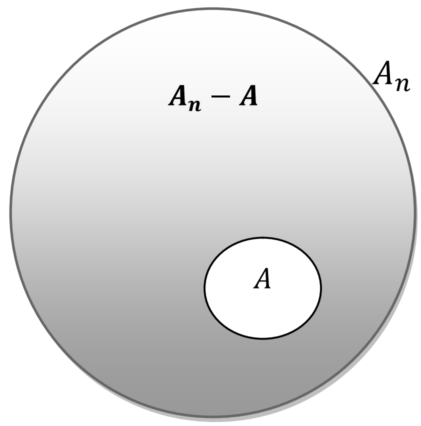

Figure (1.1) (in two dimension)

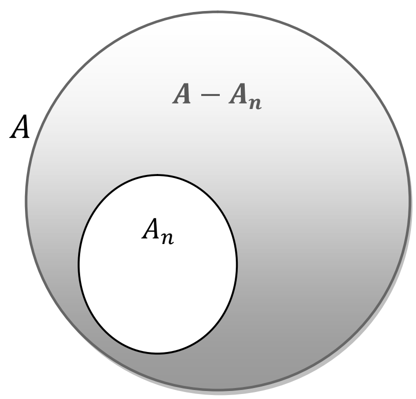

Figure (1.2) (in two dimensional space)

1.2 RANDOM VARIABLE [Krishnan, 84],[Stirzaker, 05]:

An important class of functions is the measurable functions which are different from the measure functions μ, whereas measure functions are set functions, measurable functions are invariably point functions.

Definition (1.7) Measurable Function [Krishnan, 84]:

Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let g be a function with domain $E_1 \subset \Omega_1$ and range $E_2 \subset \Omega_2$

$$g: \Omega_1 \to \Omega_2$$

g is called an $\mathcal{F}_1$-measurable function or an $\mathcal{F}_1$-measurable mapping if for every $E_2 \in \mathcal{F}_2$

$$g^{-1}(E_2) = \{\omega: \ g(\omega) \in E_2\} \triangleq E_1 \tag{1.11}$$

is in the σ-field $\mathcal{F}_1$.

Remarks (1.7) [Krishnan, 84]:

1. If g is measurable with respect to the σ-field $\mathcal{F}$ of sets that are P-measurable, then we might also say that g is P-measurable if there is no confusion.
2. The set E_1 given by $g^{-1}(E_2)$ is called the ***inverse image*** or ***inverse mapping of*** $\boldsymbol{E_2}$, and it is measurable set.
3. Inverse mappings preserve all set relations.

Definition (1.8) Random Variable [Stirzaker, 05]:

Measurable space consisting of the real line R and σ-field of Borel sets $\mathcal{R}$. Let the probability measure P be defined on $(\Omega, \mathcal{F})$. The measurable mapping X from $(\Omega, \mathcal{F})$ into (R, $\mathcal{R}$) is called a real-valued random variable.

Remarks (1.8) [Krishnan, 84], [Stirzaker, 05]:

1. Naturally, the probability measure P induces a probability measure P_X in the space (R, $\mathcal{R}$). If $E_2 \in \mathcal{R}$,then

   $$P_X(E_2) = P\big(X^{-1}(E_2)\big) = P(E_1) = P\{\omega : X(\omega) \in E_2\} \quad (1.12)$$

   Equation (1.12) related the probability measure P_X in (R, $\mathcal{R}$) to the probability measure P in (Ω, $\mathcal{F}$). Instead of writing $P\{\omega:\ X(\omega) \in E_2\}$, we shall have the abbreviated notation $P\{X \in E_2\}$.
2. If Ω is a metric topological space, then $\mathcal{F}$ is the σ-field of all Borel sets of Ω. Then a function g mapping $\Omega \to R$ is a Borel function if for every $E_2 \in \mathcal{R}$, $g^{-1}(E_2)$ is a Borel set of Ω. Since Borel sets of Ω are measurable by assumption, every Borel function is $\mathcal{F}$-measurable.

Example (1.6) [Evans, 06]:

Let $A \in \mathcal{F}$, then the *indicator function* of A,

$$\mathcal{X}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \ or\ \Omega - A \end{cases} \quad (1.13)$$

is a random variable.

1.2.1 Borel–Cantelli Lemma [Evans, 06], [Krishnan, 84] :

We introduce next a simple and very useful approach to check if some sequence $A_1, \dots, A_n, \dots$ of events "occurs infinitely often".

Definition (1.9):

Let $A_1, \dots, A_n, \dots$ be a sequence of events in a probability space (Ω,$\mathcal{F}$, P), Then the event

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{\omega \in \Omega | \omega \ belongs\ to\ infinitely\ many\ of\ the\ A_n\},$$

is called ***"A_n infinitely often"***, abbreviated "***A_n i.o.***".

Lemma (1.2) Borel–Cantelli:

If $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then*

$$P(A_n \ i.o.) = 0 \qquad (1.14)$$

Remark (1.9):

In applications, ***Borel-Cantelli Lemma*** is a very important technique; and will be needed to the guarantee the existence of unique solution of stochastic differential equations as one can see this fact later on chapter two. The following results are needed later in chapter two.

A sequence of random variables $\{X_k\}_{k=1}^{\infty}$ defined on some probability space *converges in probability* to a random variable X, provided

$$\lim_{k\to\infty} P(|X_k - X| > \epsilon) = 0 \qquad \text{for each } \epsilon > 0. \qquad (1.15)$$

Theorem (1.1)[Krishnan, 06]:

If $X_k \to X$ in probability, then there exists a subsequence

$$\{X_{k_j}\}_{j=1}^{\infty} \subset \{X_k\}_{k=1}^{\infty}$$

such that

$$X_{k_j}(\omega) \to X(\omega) \qquad \text{for almost every } \omega. \qquad (1.16)$$

1.2.2 Distribution Functions [Evans, 06], [Hsu,97] :

Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose $X: \Omega \to R^n$ random variable, in this section some additional concepts about basic statistical definitions and properties of the distribution function are considered.

Definition (1.10) distribution function:

(i) The ***distribution function*** of X is the function $F_X: R^n \to [0,1]$ defined by

$$F_X(x) \coloneqq P(X \leq x) \text{ for all } x \in R^n \tag{1.17}$$

(ii) If $X_1, X_2, \dots, X_m: \Omega \to R^n$ are random variables, their ***joint distribution function*** is $F_{X_1,\dots,X_m}: (R^n)^m \to [0,1]$,

$$F_{X_1,\dots,X_m}(x_1, \dots, x_m) \coloneqq P(X_1 \leq x_1, \dots, X_m \leq x_m) \tag{1.18}$$

for all $x_i \in R^n, i = 1, \dots, m$.

<u>Definition (1.11) density function:</u>

Suppose $X: \Omega \to R^n$ is a random variable and $F = F_X$ its distribution function. If there exists a nonnegative, integrable function *f*: $R^n \to R$ such that

$$F(x) = F(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} f(y_1, y_2, \dots, y_n) dy_n \dots dy_1 \tag{1.19}$$

Then *f* is called the ***density function*** for X.
It follows then that

$$P(X \in B) = \int_B f(x)dx \qquad \text{for all } B \in \mathcal{B} \tag{1.20}$$

This formula is important as the expression on the right hand side is an ordinary integral, and can often be explicitly calculated.

<u>Remark (1.10):</u>

If the probability distribution function is differentiable, then we obtain the probability density function $f(x)$

$$f(x) = \frac{dF(x)}{dx} \tag{1.21}$$

<u>Example (1.7)[Evans, 06]:</u>

If $X: \Omega \to R^n$ has density

$$f(x) = \frac{1}{((2\pi)^n det\ C)^{1/2}} e^{-\frac{1}{2}(x-m)C^{-1}(x-m)} \qquad (x \in R^n)$$

for some $m \in R^n$ and some positive definite, symmetric matrix C, we say X has a Gaussian (or normal) distribution, with mean m and covariance matrix C. We then write X is an $N(m,C)$ random variable.

Remark (1.11) Right continuous [Krishnan, 84]:

Functions are those functions for which $F(x) = lim_{\epsilon\downarrow 0}F(x+\epsilon)$ Similarly left continuous functions are those functions for which $F(x) = lim_{\epsilon\downarrow 0}F(x-\epsilon)$

We now show how the concept of distribution function is related to the concept of measure. Let the measure space be $(\Omega,\mathcal{F},\mu)$, where $\Omega = R$ the real line, $\mathcal{F}$ is the σ-field of Borel sets on the real line, and μ is a finite measure. Let k be any point in R, we define a function F_k as

$$F_k(x)=\begin{cases} -\mu(x,\ k] & ,x < k \\ 0 & ,x = k \\ \mu\ (k, x\ ] & ,x > k \end{cases} \qquad (1.22)$$

Clearly

$$F_k(b\ ) - F_k(a\ ) = \mu(a\ , b] \qquad for\ \ a \le b \qquad (1.23)$$

and since μ is a measure with $\mu(a, b]$ the function is increasing. Further

$$lim_{b\downarrow a+}[\ F_k(b) - F_k(a)] = lim_{b\downarrow a+}\ \mu(a, b] = \mu(\phi) = 0 \qquad (1.24)$$

1.2.3 Expectation of Random Variables:

Let $(\Omega, \mathcal{F}, P)$, be a probability space. The expectation of a random variable X is usually defined by the Stieltjes integral

$$EX = \int_{-\infty}^{\infty} X dF(x) \tag{1.25}$$

Definition (1.12) Expectation [Krishnan, 84]:

Let$(\Omega, \mathcal{F}, P)$, be a probability space, and let X be a real random variable. The expectation of X is defined by

$$EX = \int_{\Omega} X(w) dP(w) \qquad or \qquad \int_{\Omega} x\, dP \tag{1.26}$$

Remark (1.12):

There are some properties of expectation operation, such as:

1. Linearity: $E(aX + bY) = aEx + bEY$ for all constants a and b.
2. Homogeneity: $E(cX) = cEX$ for constant c.
3. Order preservation $X \geq Y$ implies $EX \geq EY$.

Lemma* (1.3) *[Evans, 06]:

Let $X: \Omega \to R^n$ be a random variable, and assume that its distribution function $F = F_X$ which has the density function. Suppose $g\!: R^n \to \boldsymbol{R}$, and $Y = g(X)$ is integrable. Then

$$E(Y) = \int_{R^n} g(x) f(x) dx.$$

In particular,

$$E(X) = \int_{R^n} x f(x) dx \tag{1.27}$$

and

$$V(X) = \int_{R^n} |x - E(X)|^2 f(x) dx. \tag{1.28}$$

Example (1.8) [Evans, 06]:

If X is $N(m, \sigma^2)$, then

$$E(X) = m$$

and

$$V(X) = \sigma^2$$

Lemma (1.4) (Chebyshev's inequality) [Krishnan, 06]:

If X is a random variable and $1 \leq p < \infty$, then

$$P(|X| \geq \lambda) \leq \frac{1}{\lambda^p} E(|X|^p) \qquad \text{for all } \lambda > 0. \tag{1.29}$$

Proof:

$$E(|X|^p) = \int_{\Omega} |X|^p \, dP \geq \int_{\{|X| \geq \lambda\}} |X|^p \, dP \geq \lambda^p P(|X| \geq \lambda).$$

1.2.4 Martingales :

Now suppose $Y_1, Y_2, \ldots$ are independent real-valued random variables, with

$$E(Y_i) = 0 \quad (i = 1, 2, \ldots).$$

Define the sum $S_n := Y_1 + \cdots + Y_n$.

The best guess of S_{n+k}, given the values of $S_1, \ldots, S_n$. Is coming from the following fact

$$E(S_{n+k}|S_1,\dots,S_n) = E(Y_1 + \cdots + Y_n|S_1,\dots,S_n) + E(Y_{n+1} + \cdots + Y_{n+k}|S_1,\dots,S_n)$$
$$= Y_1 + \cdots + Y_n + E(Y_{n+1} + \cdots + Y_{n+k})$$
$$= S_n.$$

Where $E(Y_{n+1} + \cdots + Y_{n+k}) = 0$.

Definition (1.13) [Evans, 06]:

Let $X(.)$be a real–valued stochastic process (as we define later in section (1.3)). Then

$$\mathcal{U}(t) \coloneqq \mathcal{U}(X(s)|0 \le s \le t), \qquad (1.30)$$

the σ-algebra generated by the random variables $X(s)$ for $0 \le s \le t$, is called the ***history*** of the process until (and including) time $t \ge 0$.

Definition (1.14) [Evans, 06]:

Let $X(.)$be a stochastic process (as define later in section(1.3)), such that

$E(|X(t)|) < \infty$ for all $t \ge 0$.

(i) If

$$X(s) = E(X(t) \mid U(s)) \quad a.s. \text{ for all } t \ge s \ge 0, \qquad (1.31)$$

then $X(.)$ is called a ***martingale.***

(ii) If

$$X(s) \le E(X(t)| U(s)) \quad a.s. \text{ for all } t \ge s \ge 0, \qquad (1.32)$$

$X(.)$ is a *submartingale*.

Remark (1.13):

Example of martingale can be found in section three when the Brownian motion is define.

Theorem (1.2) (Martingale inequalities) [Krishnan, 84]:

Let $X(.)$ be a stochastic process with continuous sample paths a.s.

(i) *If $X(.)$ is a submartingale, then*

$$P(max_{1\leq s\leq t}X(s)\geq\lambda)\leq\frac{1}{\lambda}E(X(t)^{+}) \text{ for all } \lambda>0, t\geq 0. \qquad (1.33)$$

(ii) *If $X(.)$ is a martingale and $1<p<\infty$, then*

$$E(max_{0\leq s\leq t}|X(s)|^{p})\leq\left(\frac{p}{p-1}\right)^{p}E(|X(t)|^{p}). \qquad (1.34)$$

.

1.2.5 Independence :

Definition (1.15) Independence [Hsu, 97]:

i. Let $(\Omega,\mathcal{F},P)$, be a probability space and let $A,B\in\mathcal{F}$. The events $A$and B are independent (denoted by $A\perp B$) if
$P(A\cap B)=P(A)P(B)$

ii. n events A_1, A_2,…,A_n are independent if for any subset $\{k_1,k_2,\ldots,k_r\}$,where r=1,2,…,n

$$P\left(\bigcap_{i=1}^{r}A_{k_i}\right)=\prod_{i=1}^{r}p(A_{k_i})$$

Remark (1.14) [Krishnan, 84]:

Let $(\Omega,\mathcal{F},P)$be a probability space, let $\mathcal{F}_1$be a sub σ-field of $\mathcal{F}$, and let X be an integrable real–valued random variable . The conditional expectation of X relative to $\mathcal{F}_1$is an integrable $\mathcal{F}_1$-measurable random variable $E(X\backslash\mathcal{F}_1)$ or $E^{\mathcal{F}_1}X$, such that for every$A \in \mathcal{F}_1$,

$$\int_A E(X\backslash \mathcal{F}_1)dP = \int_A E^{\mathcal{F}_1}XdP = \int_A XdP \qquad (1.35)$$

1.2.6 Convergence of Random Variable

The convergence of random variable and their kinds are of our interest and then submitted as follows:

Definition (1.16) Almost Surely Convergence [Krishnan, 06]:

A sequence of random variables $\{X_n\}$ converges almost surely *(a.s.),* or almost certainly, or strongly, to X if for every ω-point not belonging to the null event A,

$$lim_{n\to\infty}|X_n(\omega) - X(\omega)| = 0 \qquad (1.36)$$

This type of convergence is known as convergence with probability 1 and is denoted by

$$X_n(\omega)_{n\to\infty} \xrightarrow{a.s.} X(\omega)$$

or

$$X(\omega) = lim_{n\to\infty} X_n(\omega) \qquad (a.s.)$$

Remark (1.15) [Krishnan, 84]:

If the limit X is not known a priori, then we can define a mutual convergence *almost surely*. The sequence X_n converges *mutually almost surely* if

$$sup_{m\geq n}|X_m - X_n|_{n\to\infty} \xrightarrow{a.s.} 0$$

Both definitions are equivalent.

Definition (1.17) Convergence in probability [Krishnan, 84]:

A sequence of random variables $\{X_n\}$ converges in probability to X if for every $\epsilon > 0$, however small,

$$lim_{n\to\infty}\, p(|X_n - X| \geq \epsilon)=0, \quad or$$

$$lim_{n\to\infty}\, p(|X_n - X| < \epsilon)= 1,$$ It is denoted by

$$X_n(\omega)_{n\to\infty} \xrightarrow{l.i.p.} X(\omega), \; or$$

$$X(\omega)= l.i.p._{n\to\infty}\, X_n(\omega)$$

(where *l.i.p.* is standing for limit in probability)

Remarks (1.16) [Krihsnan 06]:

The concept of convergence in probability plays an important role in the consistency of estimators and the weak law of large numbers. We give next some results concerning this concept.

i. If a sequence of random variables $\{X_n\}$ converges almost surely to X, then it converges in probability to the same limit. The converse is not true. However, the following is true.
ii. If $\{X_n\}$ converges in probability to X, then there exist a subsequence $\{X_{nk}\}$ of $\{X_n\}$ which converges almost surely to the same limit.
iii. $\{X_n\}$ Converges in probability if and only if it converges mutually in probability.

1.3 STOCHASTIC PROCESSES:

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. Let T be an arbitrary indexed parameter set called the time set. T can be the real line R, the positive real line R^+, the set of positive integers N, or any semiclosed interval in R or R^+, unless otherwise specified. We shall assume that T is a semiclosed time interval in R^+. Sometime we will explicitly state that T is in R^+.

Definition (1.18) Stochastic Process [Krishnan, 84]:

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space and let T be any time set. Let $(R,\mathcal{R})$ be a measurable space, where R is the real line and $\mathcal{R}$ is the σ-field of Borel sets on the real line. A stochastic process $\{X_t, t \in T\}$ is a family of random variables defined on the probability space $(\Omega,\mathcal{F}, \mathcal{P})$ and taking values in the measurable space $(R,\mathcal{R})$.

Remarks (1.17) [Krishnan, 06], [Pritchard, 01]:

1. The probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called the base space and the measurable space $(R, \mathcal{R})$ the state space.
2. For each $t \in T$, the $\mathcal{F}$-measurable random variable X_t is called the state of the process at time t.
3. For each ω ∈ Ω the mapping $t \longmapsto X_t(t)$ define on T and taking values in R is called a ***sample function.***
4. If the time set T is N, then the stochastic process $\{X_t, t \in T\}$ becomes $\{X_n, n \in N\}$ and is called a discrete stochastic process.
5. If the time set T is R or R^+, then the stochastic process is a continuous one.
6. We are concerned with continuous-time, real-valued stochastic processes $(X_t)_{0 \le t < \infty}$. These may be thought of as random function for each outcomes of random element, we have a real-valued function of a

real variable t. These possible outcomes (functions) are called ***realizations*** or ***sample paths***.

Example (1.9) [Pritchard, 01]:

1. Random walk (one step up or down, with probability $1/2$ for each, at each integral time), with linear interpolation.
2. Brownian motion.

Proposition (1.2) [Krishnan,84]:

Let $\{F_{T_n}\}$ be a compatible family of finite dimensional distribution functions with all finite $T_n \subset T$. Then we can always construct a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a stochastic process $\{X_t, t \in T\}$ such that the stochastic process has the given finite dimensional distribution.

Definition (1.19) (covariance matrix) [Raphael,72]:

Consider a vector-valued stochastic process W(t). Then we call

$$m(t) = E\{W(t)\}$$

the mean of the process,

$$R_W(t_1, t_2) = E\{[W(t_1) - m(t_1)][W(t_2) - m(t_2)]^T\} \tag{1.37}$$

The covariance matrix, and

$$C_w(t_1, t_2) = E\{W(t_1)W^T(t_2)\}$$

(1.38)

Is the second-order joint moment matrix of $W(t)$. $R_W(t,t) = Q(t)$ is termed as the variance matrix, while

$$C_w(t,t) = \acute{Q}(t) \tag{1.39}$$

is the second-order moment matrix of the process.

Remarks (1.18) [Raphael, 72]:

1. The joint moment matrix written out more explicitly is

$$C_w(t_1,t_2) =$$

$$E\{W(t_1)W^T(t_2)\} = \begin{pmatrix} E\{w_1(t_1)w_2(t_2)\} & \cdots & E\{w_1(t_1)w_m(t_m)\} \\ E\{w_2(t_1)w_1(t_2)\} & \cdots & E\{w_2(t_1)w_m(t_2)\} \\ \vdots & \cdots & \vdots \\ E\{w_m(t_1)w_1(t_2)\} & \cdots & E\{w_m(t_1)w_m(t_2)\} \end{pmatrix} \quad (1.40)$$

2. Each element of $C_w(t_1,t_2)$ is a scalar joint moment function. Similarly, each element of $R_W(t_1,t_2)$ is a scalar covariance function.

1.3.1 Classes of Stochastic Processes:

In this subsection we shall consider several types of stochastic process and discuss their properties.

Definition (1.20) stationary process [Hsu, 97]:

Let $\{ X_t \ , t \in T\}$ be a stochastic process with time set T defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in the state space $(\mathrm{R}, \mathcal{R})$. Let $\mathrm{T_n} = \{t_1, t_2, \dots, t_n\}$ be any finite set of values belonging to T. Then the process is ***strictly stationary*** or ***stationary*** if for any Δt the joint distribution of the sequence $\{X(t_1), X(t_2), \dots, X(t_n)\}$is the same as the joint distribution of $\{X(t_1+\Delta t), X(t_2+\Delta t), \dots, X(t_n+\Delta t)\}$for any positive integer n.

Definition (1.21) Wide Sense Stationary [Krishnan, 84]:

A real stochastic process $X_t, t \in T$, is ***wide sense stationary*** or *covariance stationary* if :

1. $EX_t^2 < \infty$.
2. $\mu_x = EX_t$ a constant.

3. C_X (t – s) = $E\{(X_t - \mu)(X_s - \mu)\}$ depends only on the time difference t –s and not on either t or s.

Remark (1.19):

The strict sense ***stationary*** of definition (1.20) implies ***wide sense stationary*** of definition (1.21), but the converse is not true,*[Krishnan, 84]*.

Example (1.10) [Krishnan, 84]:

Let us consider a stochastic process consisting of a sequence $\{X_1, X_2, \ldots\}$ of independent identically distributed random variables with mean μ and variance σ^2. The autocovariance $\sigma_x(h) = \sigma^2 \delta_h$, where h is the lag and δ_h is the kronecker delta. Clearly this process is ***wide sense stationary*** according to the definition.

Example (1.11):

Let us define the random signal:

$$x(t) = \alpha \ \sin(0.5t + \theta)$$

Where α is a positive random variable with mean 0.63 and variance 0.11, θ is uniformly distributed between 0 and 2π, and α and θ are uncorrelated.
where the $p.d.f$ of uniformly distribution is

$$f(x) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & e.w \end{cases}$$

The mean of this random signal is calculated as:

$$E[x(t)] = E[\alpha \ \sin(0.5t + \theta)]$$

$$= E(\alpha) \int_0^{2\pi} \sin(0.5t + \theta) \frac{1}{2\pi} d\theta$$

$$= (0.63) \int_0^{2\pi} [\sin(0.5t) \cos(\theta) + \cos(0.5t) \sin(\theta)] \frac{1}{2\pi} d\theta$$

$$= (0.63)\left[\frac{\sin(0.5t)}{2\pi}\left[-sin\theta|_0^{2\pi} + \frac{\cos(0.5t)}{2\pi}[cos\theta|_0^{2\pi}\right]\right.$$

$$= 0.$$

The correlation function of this signal is calculated as:

$$R_x(t_1, t_2) = E[x(t_1)x(t_1)]$$

$$= \frac{1}{2}E(\alpha^2)\int_0^{2\pi}[sin\{0.5(t_2 - t_1) + 2\theta\} + sin\{0.5(t_2 - t_1)\}]\frac{1}{2\pi}d\theta$$

$$= \frac{1}{4\pi}E(\alpha^2)\int_0^{2\pi}[\{\sin 0.5\}(t_2 - t_1)\cos(2\theta)$$

$$+ cos\{0.5(t_2t_1)\sin(2\theta)\}\ sin\{0.5(t_2 - t_1)\}]\,d\theta$$

$$= \frac{1}{4\pi}E(\alpha^2)\left[\frac{\sin(0.5(t_2 - t_1))}{2\pi}\int_0^{2\pi}\cos(2\theta)\,d\theta\right.$$

$$+ \frac{\cos(0.5(t_2 - t_1))}{2\pi}\int_0^{2\pi}\sin(2\theta)\,d\theta + \frac{\sin(0.5(t_2 - t_1))}{2\pi}\int_0^{2\pi}d\theta$$

$$= 0.20\cos(0.5(t_2 - t_1))$$

The mean is independent of time, and the correlation function depends only on time difference $(t_2 - t_1)$, so this random signal is wide sense stationary. This result is reasonable since there is no preferred time if the phase is uniformly distributed from 0 to 2π.

Definition (1.22) Independent Increment Process [Krishnan, 06]:

A stochastic process $\{X_t, t \in T\}$ define on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is an independent increment process if for any collection $\{t_1, t_2, \ldots, t_n\} \subset T$ satisfying $t_1 < t_2 < \cdots < t_n$ the increment of the processX_t, $(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), \ldots, (X_{t_n} - X_{t_{n-1}})$ are a sequence of independent random variables.

1.3.2 White Noise:

The following definitions are needed to complete understanding white noise:

Definition (1.23) (Spectral Density) [Krishnan, 84]:

Let $\{X_t, t \in T, T = (-\infty, \infty)\}$ be a quadratic mean continuous wide sense stationary process defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with autocorrelation function $R(\tau) = EX_{t+\tau}X_\tau$ belonging to the space L^2. The power spectral density function $S(\upsilon)$ is defined as the Fourier transform of the autocorrelation function $R(\tau)$ given by

$$S(\upsilon) = \int_{-\infty}^{\infty} R(\tau)e^{-j2\pi\upsilon\tau}d\tau \qquad (1.41)$$

and the inversion of $S(\upsilon)$ given us

$$R(\tau) = \int_{-\infty}^{\infty} S(\upsilon)\, e^{-j2\pi\upsilon\tau}d\upsilon \qquad (1.42)$$

Since $R(\tau)$ is nonnegative definite, $S(\upsilon)$ is also nonnegative definite, and since $R(\tau)$ is square integrable, $S(\upsilon)$ is also square integrable. The average energy contained in the process X_t is given by $R(0) = EX_t^2$, and hence from equation (1.41)

$$EX_t^2 = \int_{-\infty}^{\infty} S(\upsilon)d\,\upsilon \qquad (1.43)$$

Definition (1.24) (Delta Function) [Krishnan, 84]:

A ***delta function*** belongs to a class of generalized functions whose effect on a continuous function of rapid decay $\phi(.)$ under an integral is given by

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)$$

The function $\phi(t)$ is said to belong to a class of test functions of rapid decay.

Delta functions are somewhat loosely defined by

$$lim_{a\to 0}\frac{1}{a}f\left(\frac{t}{a}\right) = lim_{b\to\infty} bf(bt) = \delta(t)$$

Where $f(t)$ is a function satisfying the requirement $\int_{-\infty}^{\infty} f(t)dt = 1$. The definition is indeed useless without defining the limiting operation since $lim_{a\to 0}(1/a)f(t/a)$ or $lim_{b\to\infty} bf(bt)$ does not converge in any accepted sense to the delta function. We define the limit in equation above to be the delta function in the sense

$$lim_{a\to 0}\int_{-\infty}^{\infty}\phi(t)\frac{1}{a}f\left(\frac{t}{a}\right)dt = \phi(0)$$

$$lim_{b\to\infty}\int_{-\infty}^{\infty}\phi(t)bf(bt)dt = \phi(0)$$

Where $\phi(t)$ belongs to the class of test functions of rapid decay. We are tacitly defining delta function only inside an integral and not out side it.

White noise [Raphel, 72]::

One frequently encounters in practice zero-mean scalar stochastic process X with the property that $X(t_1)$ and $X(t_2)$ are uncorrelated even for values of $|t_2 - t_1|$ that are quite small, that is,

$$R_X(t_2,t_1) \cong 0 \qquad for\ |t_2 - t_1| > \varepsilon \tag{1.44}$$

where ε is a small number. The covariance function of such stochastic processes can be idealized as follows:

$$R_X(t_2,t_1) = V(t_1)\delta(t_2 - t_1), \qquad V(t_1) \geq 0. \tag{1.45}$$

Here $\delta(t_2 - t_1)$ is the delta function and $V(t_1)$ is referred to as the *intensity* of the process at time t. Such processes are called ***white noise processes***. We can of course extend the notion of a white noise process to vector-valued process:

Definition (1.25) (white noise process) [Raphael, 72]:

Let $X(t)$ be a zero mean vector-valued stochastic process with covariance matrix

$$R_X(t_2, t_1) = V(t_1)\delta(t_2 - t_1) \tag{1.46}$$

where $V(t_1) \geq 0$.

The process $X(t)$ is then said to be a ***white noise stochastic process*** with intensity $V(t)$.

White noise differential equation [Krishnan, 84]:

We now investigate the problem of a differential equation driven by white noise. Suppose we are given the differential equation in the following form:

$$\frac{dY_t}{dt} = \alpha(t)Y_t + \beta(t)X_t \qquad t \in T, Y_a \tag{1.47}$$

Where Y_a is the initial condition and X_t is a white noise process, Y_a is the initial condition. Presented in the form (1.47) cannot be interpreted as an ordinary differential equation without making assumptions on differentiability and separability of Y_t and X_t, even if X_t is not white but some other quadratic mean continuous random process. Instead of interpreting this equation as a differential equation, we can interpret it as an integral equation without worrying about these assumptions. We interpret the stochastic process $\{Y_t, t \in [a, b)\}$ with $E|Y_t|^2 < \infty$ as the solution to the differential equation (1.47) if it satisfies the following integral equation:

$$Y_t = Y_a + \int_a^t \alpha(s)Y_s ds + \int_a^t \beta(s)dZ_s \qquad a \leq t \leq b \tag{1.48}$$

Where Z_t is the process of orthogonal increment associated with the white noise process X_t, Y_a is the initial condition satisfying $E|Y_a|^2 < \infty$, and $\alpha(t)$ and $\beta(t)$ belong to a class of square integrable functions.

The above integral equation can also be written as

$$dY_t = \alpha(t)Y_t dt + \beta(t)dZ_t \qquad a \le t < b, \qquad Y_a, \; E|Y_a|^2 < \infty$$

We have more to say about these differential equations when we discuss Itô stochastic differential equations.

1.3.3 BROWNIAN MOTION:

Next we define a Brownian motion process assuming that the time set $T = R^+$or any interval [0, a], a > 0.

Definition (1.26) Brownian motion [Krishnan, 06]:

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. The stochastic process $\{X_t, t \in T\}$ defined on $(\Omega, \mathcal{F}, P)$ is a Brownian motion process with parameter σ^2 if

1. $W_0(\omega) = 0$.
2. $\{X_t\}$ is a stationary independent increment process.
3. For every s and t, $s \le t$, belonging to T the increment $W_t - W_s$ are Gaussian distributed with mean zero and variance $\sigma^2(t - s)$.
4. For almost all $\omega \in \Omega$ the sample functions $t \to W_t(\omega)$ are uniformly continuous in the interval T.

With the definition given above we shall now drive the auto covariance function $C_w(t, s)$.

For $t > s$.

$$C_w(t,s) = EW_tW_s = E(W_t - W_s + W_s)W_s$$

$$= E(W_t - W_s)W_s + EW_s^2$$

$= EW_s^2$ from 2

$= \sigma^2 s$ from 3

Similarly, for $t < s$, $C_w(t,s) = \sigma^2 t$. Hence $C_w(t,s) = \sigma^2(t \wedge s)$, where $t \wedge s = \min\{t,s\}$.

Remark (1.20) [Stirzaker, 05]:

If $\sigma^2 = 1$, then $W(t)$ is said to be the ***standard Brownian process*** *(standard Wiener process)*.

Example (1.12) [Evans, 06]:

Let $W(.)$ be a 1-dimensional Brownian motion (wiener process), as defined later. Then

$W(.)$ is a martingale.

To see this, write $\mathcal{W}(t) \coloneqq \mathcal{U}(W(s)|0 \le s \le t)$, and let $t \ge s$. Then

$$E(W(t)|\mathcal{W}(s)) = E(W(t) - W(s)|\mathcal{W}(s)) + E(W(s)|\mathcal{W}(s))$$

$$= E\big(W(t) - W(s)\big) + W(s)$$

$$= W(s) \quad \text{a.s.}$$

***1.3.3.1 Computation of Joint Probabilities [Evans, 06]*:**

From the definition if $W(.)$is a Brownian motion, then for all $t > 0$ and $a \le b$,

$$P(a \le W(t) \le b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{x^2}{2t}}\, dx \qquad (1.49)$$

since $W(t)$ is $N(0, t)$.

for more details see [Evans, 2006].

Example (1.13) (Geometric Brownian Motion) [Ross, 83]:

If $\{X_t, t \ge 0\}$ is Brownian motion, then the process $\{Y_t, t \ge 0\}$, defined by

$$Y_t(t) = e^{x(t)},$$

is called ***geometric Brownian motion***.

Since $X(t)$is normal with mean 0 and variance t, its mean and variance are given by

$$E[Y_t(t)] = E\left[e^{X(t)}\right] = e^{t/2}$$

$$var\left(Y_t(t)\right) = E[Y^2(t)] - (E[Y_t(t)])^2$$

$$= E\left[e^{2x(t)}\right] - e^t$$

$$= e^{2t} - e^t.$$

Example (1.14) (Brownian Motion Reflected at the Origin) [Ross, 83]:

If $\{X(t), t \geq 0\}$ is Brownian motion, then the process $\{Z(t), t \geq 0\}$, where

$$\mathrm{Z(t)} = |\mathrm{X(t)}|, \qquad \mathrm{t} \geq 0$$

is called ***Brownian motion reflected at the origin***.

The distribution of $\mathrm{Z(t)}$ is easily obtained. For $y > 0$,

$$P\{\mathrm{Z(t)} \leq \mathrm{y}\} = P\{X(t) \leq y\} - P\{X(t) \leq -y\}$$

$$= 2P\{X(t) \leq y\} - 1$$

$$= \frac{2}{\sqrt{2\pi t}} \int_{-\infty}^{y} e^{-x^2/2t} dx - 1,$$

Where the last equality follows since $X(t)$is normal with mean 0.

The mean and variance of $Z(t)$ are easily computed and

$$E[Z(t)] = \sqrt{2t/\pi}$$

$$var[Z(t)] = \left(1 - \frac{2}{\pi}\right) t.$$

Remark (1.21) [Evans, 06]:

1. Fix a point $x_0 \in \mathcal{R}^n$ and consider then the ordinary differential equation:

$$\left.\begin{array}{l} \dot{X}(t) = b\big(X(t)\big) \quad (t > 0) \\ X(0) = x_0, \end{array}\right\} \quad \textit{(ODE)} \tag{1.50}$$

where $b: \mathcal{R}^n \longrightarrow \mathcal{R}^n$ is a given, smooth vector field and the solution is the trajectory $X(.): [0, \infty) \to \mathcal{R}^n$.

2. $X(t)$ is the state of the system at time $t \geq 0$,

$$\dot{X}(t) \coloneqq \frac{d}{dt} X(t) \tag{1.51}$$

In many applications, however, the experimentally measured trajectories of systems modeled by *(ODE)* do not in fact behave as predicted.

Hence it seems reasonable to modify (*ODE's*), in such away to include the possibility of random effects disturbing the system. A formal way to do so is to write:

$$\left.\begin{array}{l} \dot{X}(t) = b\big(X(t)\big) + B\big(X(t)\big)\xi(t) \quad (t > 0) \\ X(0) = x_0, \end{array}\right\} \tag{1.52}$$

Where $B: \mathcal{R}^n \to \mathcal{M}^{n \times m}$(=space of $n \times m$ matrices) and

$\xi(.) \coloneqq m$-dimensional "white noise".

This approach presents us with these ***mathematical problems*:**

1. Define the "white noise" $\xi(.)$ as we define.
2. Define what it means for $X(.)$ to solve (1.52).

3. Show (1.52) has a solution, discuss uniqueness, asymptotic behavior, dependence upon x_0, b, B, etc.

Some Heuristics:

Let us first study equation (1.52) in the case $m = n$, $x_0 = 0$, b ≡ 0, and B ≡ *I*. The solution of (1.52) in this setting turns out to be the *n*-dimensional *Wiener process*, or *Brownian motion*, denoted by $W(t)$. Thus we may symbolically write

$$\dot{W}(.) = \xi(.), \tag{1.53}$$

Thereby asserting that ***"white noise"*** is the time derivative of the Brownian motion.

Now return to the general case of the equation (1.52), write $\frac{d}{dt}$ instead of the dot, yielding:

$$\frac{dX(t)}{dt} = b\big(X(t)\big) + B\big(X(t)\big)\frac{dW(t)}{dt}, \tag{1.54}$$

and finally multiply by "*dt*":

$$\left.\begin{array}{l} dX(t) = b\big(X(t)\big) + B\big(X(t)\big)dW(t) \\ X(0) = x_0, \end{array}\right\} \quad \textit{(SDE)} \tag{1.55}$$

This expression, properly interpreted, is a stochastic differential equation (abbreviated by SDE). We say that $X(.)$ solves the (SDE) provided

$$X(t) = x_0 + \int_0^t b\big(X(s)\big)ds + \int_0^t B\big(X(s)\big)dW \quad \text{for all times } t > 0. \tag{1.56}$$

Now we must:

1. Construct $W(t)$.
2. Define the ***stochastic integral*** .
3. Show that equation (1.56) has a solution, etc.

2.1 Stochastic Integral:

It is well-known that stochastic integrals and Itô formula play a central role in modern probability theory and its applications in stochastic differential equation concerned by Brownian motion, etc.

This chapter concerning the most necessary mathematical principles discussing stochastic integration, Itô formula, Itô SDE, existence of a unique solution of Itô SDEs, as well as some solvable examples.

Now, we shall define the integral

$$I(T) = \int_0^T f(t)dw(t) \qquad (2.1)$$

Where $w(t)$ is a Brownian motion and $f(t)$ is a stochastic function, and study its basic properties. One may define

$$I(T) = f(T)w(T) - \int_0^T \acute{f}(t)w(t)dt$$

If f is absolutely continuous for each w. However, if f is only continuous, or just integrable, this definition does not make sense [Friedman, 75].

Remark (2.1) [Friedman, 75]:

Since $w(t)$ (the Brownian motion) is nowhere differentiable with probability 1, the integral (2.1) cannot be defined in the usual Lebesgue-Stieltjes sense.

The following definitions are needed to later on:

Definition (2.1) Separable Process [Krishnan, 84]:

Let $\{X_t, t \in T\}$ be stochastic process defined on $(\Omega, \mathcal{F}, P)$ with time set $T \in R$. Let K be any closed subset in R, and let I be an open interval in T. Then the process $\{X_t, t \in T\}$ is ***separable***, relative to the class of all closed sets K in R, if there exist a countable subset $S \subset T$ and an ω-set Λ of probability 0 such that the two ω-sets

$$\{\omega\!:\ X_t(\omega) \in K, t \in I \cap T\},$$

$$\{\omega\!:\ X_t(\omega) \in K, t \in I \cap S\}$$

differ by Λ.

Remark (2.2) [Krishnan, 06]:

The countable set $S \subset T$ is called a ***separating set*** or ***separant.*** What the definition implies is that if $\{X_t, t \in T\}$ is separable, then every set of the form $\{\omega\!:\ X_t(\omega) \in K, t \in I \cap T\}$ differs from the event $\{\omega\!:\ X_t(\omega) \in K, t \in I \cap S\}$ by the null set Λ and can be made an event by completing the underlying probability space.

Example (2.1):

The process X_t defined by

$$X_t(\omega) = \begin{cases} 1 & \omega = t, \quad t \in T \\ 0 & \omega \neq t \end{cases}$$

We cannot assert that $P\{X_t = 0,\ t \in T\ \} = 1$ because we cannot find a separating set.

Definition (2.2) Measurability, [Doob, 53]:

A stochastic process $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with a time set T is a ***measurable process*** if for all Lebesgue measurable sets B belonging to the σ-field $\mathcal{L}(T)$ generated by Lebesgue measurable sets the mapping $(t, \omega) \longrightarrow X_t(\omega)$ is a measurable on $T \times \Omega$ with respect to the product σ-field $\mathcal{L}(t) \otimes \mathcal{F}$, that is,

$$\{(t, \omega): \ X_t(\omega) \in B\} \in \mathcal{L}(t) \otimes \mathcal{F} \qquad (2.2)$$

Theorem (2.1) [Doob, 53]:

Let $\{X_t, t \in T\}$be a measurable stochastic process with respect to the product σ-field $\mathcal{L} \otimes \mathcal{F}$. Then

1. Almost all sample function of this process are Lebesgue measurable function of $t \in T$.
2. If $EX_t(\omega)$ exists for all $t \in T$, then it also defines a Lebesgue measurable function of $t \in T$.
3. If A is a Lebesgue time set in T and if $\int_A E|X_t| \, dt < \infty$, then almost all sample functions $X_t(\omega)$ are Lebesgue integrable on the set A, that is,

$$\int_A |X_t(\omega)| dt < \infty, \qquad \text{for almost all } \omega$$

Since the value of an absolutely convergent integral is independent of the order of integration, we have

$$\int_A EX_t(\omega) dt = E \int_A X_t(\omega) dt \qquad (2.3)$$

Definition (2.3) Increasing σ-field or Filtration σ-field [Krishnan, 06]:

Let$(\Omega, \mathcal{F})$ be a complete measurable space and let $\{\mathcal{F}_t, t \in T, T = R^+\}$ be a family of sub-σ-field of $\mathcal{F}$ such that for $s \le t$, $\mathcal{F}_s \subset \mathcal{F}_t$. Then $\{\mathcal{F}_t\}$ is called an ***increasing family*** of sub-σ-field on $(\Omega, \mathcal{F})$ or the ***filtration*** σ-field of $(\Omega, \mathcal{F})$.

Remark (2.3):

$\mathcal{F}_t$ is called the σ-field of events prior to t. If $\{X_t, t \in T\}$ is a stochastic process defined on $(\Omega, \mathcal{F}, P)$ then clearly $\mathcal{F}_t$ given by

$$\mathcal{F}_t = \sigma\{X_s, s \le t, t \in T\} \tag{2.4}$$

is increasing.

Definition (2.4) Adaptation of $\{X_t\}$, [Krishnan, 84]:

Let $\{X_t, t \in T, T = R^+\}$ be a stochastic process defined on probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t, t \in T, T = R^+\}$ be a filtration σ-field. The process $\{X_t\}$ is adapted to the family $\{\mathcal{F}_t\}$, if X_t is $\mathcal{F}_t$-measurable for every $t \in T$, or

$$E^{\mathcal{F}_t} X_t = X_t \qquad t \in T$$

Remarks (2.4) [Krishnan, 06]:

1. $E^{\mathcal{F}_t}$ represents the conditional expectation.
2. $\mathcal{F}_t$-adapted random processes are also $\mathcal{F}_t$-measurable and nonanticipative with respect to the σ-field $\mathcal{F}_t$.
3. If $\mathcal{F}_t$ is the σ-field by$\{X_t, s \le t\}$, then clearly the process $\{X_t, t \in T\}$ is adapted to the family $\{\mathcal{F}_t, t \in T\}$, which is called the ***natural family*** **or** ***natural filtration*** of the process $\{X_t\}$.

2.2 Approximation of functions by step functions:

We shall call a stochastic process also a stochastic function.

Let $w(t)$, $t \geq 0$ be Brownian motion on probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t (t \geq 0)$ be an increasing family of σ-fields, i.e., $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ if $t_1 < t_2$, such that $\mathcal{F}_t \subset \mathcal{F}$, $\mathcal{F}(w(s), 0 \leq s \leq t)$ is in $\mathcal{F}_t$, and

$$\mathcal{F}(w(\lambda + t) - w(t), \lambda \geq 0) \quad \text{is independent of } \mathcal{F}_t$$

for all $t \geq 0$. One can take, for instance, $\mathcal{F}_t = \mathcal{F}(w(s), 0 \leq s \leq t)$. Let $0 \leq \alpha < \beta < \infty$. A stochastic process $f(t)$ defined for $\alpha \leq t < \beta$ is called a ***nonanticipative*** function with respect to $\mathcal{F}_t$ if:

(i) $f(t)$ is a separable process; (see definition (2.1)).

(ii) $f(t)$ is a measurable process, i.e., the function $(t, \omega) \to f(t, \omega)$ from $[\alpha, \beta] \times \Omega$ into R^1 is a measurable; (as in definition (2.2)).

(iii) For each $t \in [\alpha, \beta]$, $f(t)$ is $\mathcal{F}_t$ measurable.

Remarks (2.5) [Friedman, 75]:

1. When (iii) holds we say that $f(t)$ is ***adapted*** to $\mathcal{F}_t$ (see definition (2.4)) .
2. Let us define $L^p_\omega[\alpha, \beta]$ $(1 \leq p \leq \infty)$ the class of all nonanticipative functions $f(t)$ satisfying:

$$P\left\{\int_\alpha^\beta |f(t)|^p dt < \infty\right\} = 1 \qquad (2.5)$$

3. We denote by $M^p_\omega[\alpha, \beta]$ the subset of $L^p_\omega[\alpha, \beta]$ consisting of all functions f with

$$E\int_{\alpha}^{\beta}|f(t)|^{p}dt < \infty \qquad (2.6)$$

Definition (2.5) step function [Evans, 05], [Strizaker, 05] :

A stochastic process $f(t)$ defined on $[\alpha,\beta]$ is called a ***step function*** if there exists a partition $\alpha = t_0 < t_1 < \cdots < t_r = \beta$ of $[\alpha,\beta]$ such that

$$f(t) = f(t_i) \text{ if } t_i \le t < t_{i+1},\ 0 \le i \le r-1. \qquad (2.7)$$

Lemma (2.1) [Friedman, 75]:

Let $f \in L_{\omega}^{2}[\alpha,\beta]$. Then :

(i) There exists a sequence of continuous functions g_n in $L_{\omega}^{2}[\alpha,\beta]$ such that

$$\lim_{n\to\infty}\int_{\alpha}^{\beta}|f(t)-g_n(t)|^{2}dt = 0 \quad a.s; \qquad (2.8)$$

(ii) There exists a sequence of step functions f_n in $L_{\omega}^{2}[\alpha,\beta]$ such that

$$\lim_{n\to\infty}\int_{\alpha}^{\beta}|f(t)-f_n(t)|^{2}dt = 0 \quad a.s; \qquad (2.9)$$

Lemma (2.2) [Friedman, 75]:

Let $f \in M_{\omega}^{2}[\alpha,\beta]$. Then :

(i) There exists a sequence of continuous functions K_n in $M_{\omega}^{2}[\alpha,\beta]$ such that

$$E\int_{\alpha}^{\beta}|f(t)-K_n(t)|^{2}dt \to 0 \qquad (2.10)$$

If $n \to \infty$;

(ii) There exists a sequence of bounded step functions l_n in $M_\omega^2[\alpha,\beta]$ such that

$$E\int_\alpha^\beta |f(t)-l_n(t)|^2 dt \to 0 \tag{2.11}$$

If $n \to \infty$;

Remark (2.6):

The following stochastic integral

$$\int_0^T W dW$$

where $W(.)$ is a 1-dimensional Brownian motion. A reasonable procedure is to construct a ***Riemann sum approximation***, and then–if possible–to pass to limits.

The following definitions are concerning:

Definitions (2.6) [Evans, 05]:

(i) If $[0,T]$ is an interval, a ***partition P*** of $[0,T]$ is a finite collection of points in $[0,T]$:

$$P := \{0 = t_0 < t_1 < \cdots < t_m = T\}.$$

(ii) Let the ***mesh size*** of P be $|P| := max_{0\le k\le m-1}|t_{k+1}-t_k|$.

(iii) For fixed $0 \le \lambda \le 1$ and P a given partition of $[0,T]$, set

$$\tau_k := (1-\lambda)t_k + \lambda t_{k+1} \qquad (k = 0,\dots,m-1).$$

For such a partition P and for $0 \le \lambda \le 1$, we define

$$R = R(P,\lambda) := \sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k)).$$

This is the corresponding Riemann sum approximation of $\int_0^T W dW$.

Lemma (2.3) (Quadratic variation) [Øksendal, 98]:

Let [α,β] be an interval in [0,∞), and suppose that:

$$P^n := \{\alpha = t_0 < t_1 < \cdots < t_m = \beta\}$$

be a partitions of [α,β], with $|P^n| \to 0$ as $n\to\infty$. Then

$$\sum_{k=0}^{m_n-1} (W(t_{k+1}^n) - W(t_k^n)) \to \beta - \alpha \qquad (2.12)$$

in $L_\omega^2[\alpha,\beta]$ as n→∞.

Definition (2.7) [Friedman, 75]:

Let $f(t)$ be a step function in $L_\omega^2[\alpha,\beta]$, say $f(t) = f_i$ if $t_i \le t < t_{i+1}$, $0 \le i \le r-1$ where $\alpha = t_0 < t_1 < \cdots < t_r = \beta$, the random variable:

$$\sum_{k=0}^{r-1} f(t_k)[W(t_{k+1}) - W(t_k)] \qquad (2.13)$$

where $max|t_{k+1} - t_k| \to 0$, $0 \le k \le r-1$; is denoted by

$$\int_\alpha^\beta f(t)dw(t) \qquad (2.14)$$

and is called the ***stochastic integral*** of f with respect to the Brownian motion w; it is also called the ***Itô integral.***

Lemma (2.4) [Evans, 05]:

We have for all constants $a, b \in$ R and for all step processes

$G, H \in L^2(0,T)$, G in $M_\omega^2[\alpha,\beta]$

(i) $\int_0^T (aG + bH)dW = a\int_0^T G\,dW + b\int_0^T H\,dW$,

(ii) $E\left(\int_0^T G\,dW\right) = 0$,

(iii) $E\left(\left(\int_0^T G\,dW\right)^2\right) = E\left(\int_0^T G^2\,dt\right)$.

Lemma (2.5) [Friedman, 75]:

If f is a step function in $M_\omega^2[\alpha,\beta]$, then

$$E\int_\alpha^\beta f(t)dw(t) = 0, \tag{2.15}$$

$$E\left|\int_\alpha^\beta f(t)dw(t)\right|^2 = E\int_\alpha^\beta f^2(t)dt. \tag{2.16}$$

Lemma (2.6) [Friedman, 75]:

Let f, g belong to $L_\omega^2[\alpha,\beta]$ and assume that $f(t) = g(t)$ for all $\alpha \le t \le \beta$, $\omega \in \Omega_0$. Then

$$\int_\alpha^\beta f(t)dw(t) = \int_\alpha^\beta g(t)dw(t) \quad for\ a.a.\,\omega \in \Omega_0. \tag{2.17}$$

Remark (2.7) [Øksendal, 98]:

1. Let $f \in L_\omega^2[0,T]$ and consider the integral

$$I(t) = \int_0^t f(s)dw(s), \qquad 0 \le t \le T \tag{2.18}$$

2. By definition, $\int_0^0 f(s)dw(s) = 0$, and we refer to $I(t)$ as the indefinite integral of f. Notice that $I(t)$ is $\mathcal{F}_t$ measurable.

If f is a step function, then clearly

$$\int_\alpha^\beta f(s)dw(s) + \int_\beta^\gamma f(s)dw(s) = \int_\alpha^\gamma f(s)dw(s)$$

if $0 \leq \alpha < \beta < \gamma \leq T$. (2.19)

By approximation we find that (2.19) holds for any f in $L^2_\omega[0,T]$.

Theorem (2.2) [Friedman, 75]:

Let $f \in M^2_\omega[0,T]$. Then

$$E\left\{sup_{0\leq t\leq T}\left|\int_0^t f(s)dw(s)\right|^2\right\} \leq 4E\left|\int_0^T f(t)dw(t)\right|^2$$

$$= 4E\int_0^T f^2(t)dt. \qquad (2.20)$$

2.3 Itô Formula:

Definition (2.8) [Evans, 05]:

Let $X(t)$ $(0 \leq t \leq T)$ be a stochastic process such that for any $0 \leq t_1 < t_2 \leq T$

$$X(t_2) - X(t_1) = \int_{t_1}^{t_2} a(t)dt + \int_{t_1}^{t_2} b(t)dw(t)$$

Where $a \in L^1_\omega[0,T], b \in L^2_\omega[0,T]$. Then we say that $X(t)$ has stochastic differential dX, on $[0,T]$, given by

$$dX(t) = a(t)dt + b(t)dw(t).$$

Observe that $X(t)$ is a nonanticipative function. It is also a continuous process. Hence, in particular, it belongs to $L_{\omega}^{\infty}[0,T]$.

Example (2.2) [Friedman, 75]:

By theorem (A.1) (see appendix)

$$\int_{t_1}^{t_2} tdw(t) = lim_{n\to\infty} \sum_{k=1}^{n-1} t_{n,k}\left[w\left(t_{n,k+1}\right) - w(t_{n,k})\right]$$

In probability.

Clearly

$$\int_{t_1}^{t_2} w(t)dt = lim_{n\to\infty} \sum_{k=1}^{n-1} w\left(t_{n,k+1}\right)\left(t_{n,k+1} - t_{n,k}\right)$$

for all ω for which $w(t,\omega)$ is continuous. The sum in the right-hand sides is equal to

$$lim_{n\to\infty} \sum_{k=1}^{n-1} \left[t_{n,k+1}w\left(t_{n,k+1}\right) - t_{n,k}w(t_{n,k})\right] = t_2w(t_2) - t_1w(t_1).$$

Hence

$$d\left(tw(t)\right) = w(t)dt + tdw(t). \tag{2.21}$$

Definition (2.9) [Friedman, 75]:

Let $X(t)$ be as in definition (2.8) and let $f(t)$ be a function in $L_{\omega}^{\infty}[0,T]$. We define

$$f(t)dX(t) = f(t)a(t)dt + f(t)b(t)dw(t).$$

Example (2.3) [Friedman, 75]:

$f(t)dX(t)$ is a stochastic differential $d\eta$, where

$$\eta(t) = \int_0^t f(s)d(s)ds + \int_0^t f(s)b(s)dw(s).$$

Theorem (2.3) [Friedman, 75]:

If

$$d\xi_i(t) = a_i(t)dt + b_i(t)dw(t) \;\; (i = 1,2),$$

Then

$$d(\xi_1(t)\xi_2(t)) = \xi_1(t)d\xi_2(t) + \xi_2(t)d\xi_1(t) + b_1(t)b_2(t)dt. \qquad (2.22)$$

The integrated form of (2.22) asserts that, for any $0 \leq t_1 < t_2 \leq T$,

$$\xi_1(t_2)\xi_2(t_2) - \xi_1(t_1)\xi_2(t_1)$$
$$= \int_{t_1}^{t_2} \xi_1(t)a_2(t)dt + \int_{t_1}^{t_2} \xi_1(t)b_2(t)dw(t) + \int_{t_1}^{t_2} \xi_2(t)a_1(t)dt$$
$$+ \int_{t_1}^{t_2} \xi_2(t)b_1(t)dw(t) + \int_{t_1}^{t_2} b_1(t)b_2(t)dt. \qquad (2.23)$$

Theorem (2.4) [Friedman, 75], [Øksendal, 98]:

Let $d\xi(t) = adt + bdw(t)$, and let $f(x,t)$ be a continuous function in $(x,t) \in R^1 \times [0,\infty)$ together with its partial derivatives f_x, f_{xx}, f_t. Then the process $f(\xi(t),t)$ has a stochastic differential, given by

$$df(\xi(t),t) = \left[f_t(\xi(t),t) + f_x(\xi(t),t)a(t) + \frac{1}{2} f_{xx}(\xi(t),t)b^2(t)\right]dt$$
$$+ f_x(\xi(t),t)b(t)dw(t). \qquad (2.24)$$

This is called the ***Itô formula***. Notice that if $w(t)$ were continuously differentiable in t, then (by the standard calculus formula for total derivatives the term $\frac{1}{2}f_{xx}b^2dt$ will not appear.

Proof: One can see the proof in appendix B.

Gronwall Lemma (2.7) [Evans, 05], [Øksendal, 98] :

Let φ and f be nonnegative, continuous functions defined for $0 \leq t \leq T$, and let $C_0 \geq 0$ be a constant. If:

$$\varphi(t) \leq C_0 + \int_0^t f\,\varphi\, ds \qquad for\ all\ \ 0 \leq t \leq T,$$

Then

$$\varphi(t) \leq C_0 e^{\int_0^t f\, ds} \qquad for\ all\ \ 0 \leq t \leq T.$$

2.4 Existence and Uniqueness Solution of Stochastic Differential Equations [Evans, 06],[Øksendal, 98]:

If $\sigma = (\sigma_{ij})$ is a matrix, we write $|\sigma|^2 = \sum_{ij}|\sigma_{ij}|^2$.

Let $b(x,t) = (b_1(x,t), \dots, b_n(x,t))$, $\sigma(x,t) = (\sigma(x,t))_{ij=1}^n$ and suppose the functions $b_i(x,t), \sigma_{ij}(x,t)$ are measurable in $(x,t) \in \mathrm{R}^n \times [0,\mathrm{T}]$. If $\xi(t)$ $(0 \leq t \leq T)$ is a stochastic process such that

$$d\xi(t) = b(\xi(t),\mathrm{t})dt + \sigma(\xi(t),\mathrm{t})dw(t), \qquad (2.25)$$

$$\xi(0) = \xi_0 \qquad (2.26)$$

Then we say that $\xi(t)$ satisfies the system of stochastic differential equations (2.25) and the initial condition (2.26). Note that it is implicitly assumed that $b(\xi(t),\mathrm{t})$ belongs to $L_\omega^1[0,T]$ and $\sigma(\xi(t),\mathrm{t})$ belongs to $L_\omega^2[0,T]$.

Theorem (2.5) [Friedman, 75] ,[Evans, 05] ,[Øksendal, 98] :

Suppose $b(x,\mathrm{t}), \sigma(x,\mathrm{t})$ are measurable in $(x,t) \in R^n \times [0,T]$ and

$$|b(x,t) - b(\tilde{x},t)| \leq K_\theta |x - \tilde{x}|, \qquad |\sigma(x,t) - \sigma(\tilde{x},t)| \leq K_\theta |x - \tilde{x}|,$$

$$|b(x,t)| \leq K(1 + |x|), \qquad |\sigma(x,t)| \leq K(1 + |x|) \qquad (2.27)$$

where K_θ, K are constants. Let $\xi_0$be any n-dimensional random vector independent of $\mathcal{F}(w(t), 0 \leq t \leq T)$, such that $E|\xi_0|^2 < \infty$. Then there exists a unique solution of (2.25) and (2.26) in $M_\omega^2[0,T]$.

The assertion of uniqueness means that if $\xi_1(t), \xi_2(t)$ are two solutions of (2.25), (2.26) and if they belong to $M_\omega^2[0,T]$, then

$$P\{\xi_1(t) = \xi_2(t) for\ all\ 0 \leq t \leq T\} = 1$$

Proof:

To prove the uniqueness, suppose $\xi_1(t)$ and $\xi_2(t)$ be two solutions belonging to $M_\omega^2[0,T]$. Then

$$\xi_1(t) - \xi_2(t) = \int_0^t [b(\xi_1(s), s) - b(\xi_2(s), s)]ds + \int_0^t \sigma(\xi_1(s), s)dw(s) - \int_0^t \sigma(\xi_2(s), s)dw(s). \quad (2.28)$$

Set $f_i(s) = \sigma(\xi_1(s), s)$ and note that the stochastic integral $\int_0^t f_i(s)dw(s)$ is defined with respect to an increasing family of σ-fields $\mathcal{F}_t$which may depend on i. If $f_i(s)$ is a step function, for $i = 1,2$, then using the definition of the stochastic integral we get (Lemma (A.2) and formula (a.1)) see the appendix.

$$E\left|\int_0^t f_1(s)dw(s) - \int_0^t f_2(s)dw(s)\right|^2$$

$$= E\int_0^t |f_1(s) - f_2(s)|^2 ds \qquad (2.29)$$

By approximation we find that (2.29) is true for any pair f_1, f_2 of nonanticipative functions with respect to $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ respectively, provided that $E\int_0^t |f_i(s)|^2 ds < \infty$ $(i = 1,2)$.

Taking the expectation of the squares of the absolute values on both sides of (2.28) and using (2.27) and (2.29) with $f_i(s) = \sigma(\xi_1(s), s)$, we find that

$$E|\xi_1(t) - \xi_2(t)|^2$$

$$\leq 2K_\theta^2 t \int_0^t E|\xi_1(s) - \xi_2(s)|^2 ds + 2K_\theta^2 \int_0^t E|\xi_1(s) - \xi_2(s)|^2 ds$$

Thus the function $\phi(t) = E|\xi_1(t) - \xi_2(t)|^2$ satisfies

$$\phi(t) \leq C\int_0^t \phi(s)ds, \qquad \phi(0) = 0,$$

Where C is a positive constant. Therefore $\phi(t) \equiv 0$, and the assertion of uniqueness is proved.

To prove the existence of a solution we introduce an increasing family of σ-fields $\mathcal{F}_t$ $(0 \leq t \leq T)$ such that ξ_0 is $\mathcal{F}_0$ measurable, and such that

$$\mathcal{F}(w(t+s) - w(t), 0 \leq s \leq T - t)$$

is independent of $\mathcal{F}_t$, for all $t \geq 0$. We can take for instance $\mathcal{F}_t$ to be σ-field generated by ξ_0 and $\mathcal{F}(w(s), s \leq t)$; here we use the assumption that $\xi_0$is independent of $\mathcal{F}(w(s), 0 \leq s \leq T)$.

Define $\xi_0(t) = \xi_0$ and

$$\xi_{m+1} = \xi_0 + \int_0^t b(\xi_m(s), s)ds + \int_0^t \sigma\,(\xi_m(s), s)dw(s) \qquad (2.30)$$

The inductive assumption is that $\xi_m \in M_\omega^2[0, T]$ and hence:

$$E|\xi_{k+1}(t) - \xi_k(t)|^2 \leq \frac{(Mt)^{K+1}}{(k+1)!}$$

for all $0 \leq k \leq m - 1$ (2.31)

Where M is some positive constant (depending only on K, K_θ,T).

Since $\xi_0 \in \mathcal{F}_0$, ξ_{m+1} is well defined if $m = 0$. Further

$$|\xi_1(t) - \xi_0|^2 \leq 2\left|\int_0^t b(\,\xi_0, s)ds\right|^2 + 2\left|\int_0^t \sigma(\xi_0, s)dw(s)\right|^2.$$

Taking the expectation and using (2.27), we get

$$E|\xi_1(t) - \xi_0|^2 \leq 2K^2t^2(1 + E|\xi_0|^2) + 2K^2t(1 + E|\xi_0|^2) \leq Mt$$

If $M \geq 2K^2(1 + T)(1 + E|\xi_0|^2)$. Thus implies that $\xi_1 \in M_\omega^2[0, T]$ and (2.31) holds for $m = 0$.

We now make the inductive assumption for any $m \geq 0$ and prove it for $m + 1$

Since $\xi_m \in M_\omega^2[0,T]$ it follows, using (2.27), that $b(\xi_m(t),t)$ and $\sigma(\xi_m(t),t)$ belong to $M_\omega^2[0,T]$, thus the integrals on the right-hand side of (2.30) are well defined.

Next,

$$|\xi_{m+1}(t)-\xi_m(t)|^2 \le 2\left|\int_0^t [b(\xi_m(s),s)-b(\xi_{m-1}(s),s)]\,ds\right|^2 +$$

$$2\left|\int_0^t [\sigma(\xi_m(s),s)-\sigma(\xi_{m-1}(s),s)]dw(s)\right|^2 \qquad (2.32)$$

Taking the expectation and using (2.32),

$$E|\xi_{m+1}(t)-\xi_m(t)|^2 \le 2K_0^2 tE\int_0^t |\xi_m(s)-\xi_{m-1}(s)|^2 ds +$$

$$2K_0^2 E\int_0^t |\xi_m(s)-\xi_{m-1}(s)|^2 ds.$$

Thus,

$$E|\xi_{m+1}(t)-\xi_m(t)|^2 \le M\int_0^t E|\xi_m(s)-\xi_{m-1}(s)|^2 ds$$

If $M \ge 2K_0^2(T+1)$. substituting (2.31) with $k=m-1$ into the right-hand side, we get

$$E|\xi_{m+1}(t) - \xi_m(t)|^2 \leq M \int_0^t \frac{(Ms)^m}{m!} ds = \frac{(Mt)^{m+1}}{(m+1)!}$$

Thus (2.31) holds for $k = m$. Since this implies that $\xi_{m+1} \in M_\omega^2[0,T]$, the proof of the inductive assumption for $m + 1$ is complete.

From (2.32) we also have

$$sup_{0\leq t\leq T}|\xi_{m+1}(t) - \xi_m(t)|^2 \leq 2TK_0^2 \int_0^T |\xi_m(s) - \xi_{m-1}(s)|^2 ds +$$

$$2sup_{0\leq t\leq T}\left|\int_0^t [\sigma(\xi_m(s), s) - \sigma(\xi_{m-1}(s), s)]dw(s)\right|^2$$

Taking the expectation and using theorem (A.1) (see appendix) and (2.31), we find that

$$Esup_{0\leq t\leq T}|\xi_{m+1}(t) - \xi_m(t)|^2 \leq 2TK_0^2 \int_0^T E|\xi_m(s) - \xi_{m-1}(s)|^2 ds +$$

$$8K_0^2 \int_0^T E|\xi_m(s) - \xi_{m-1}(s)|^2 ds \leq C\frac{(Mt)^m}{m!}$$

Where $C = 2K_0^2T^2 + 8K_0^2T$. Hence

$$P\left\{sup_{0\leq t\leq T}|\xi_{m+1}(t) - \xi_m(t)| > \frac{1}{2^m}\right\} \leq 2^{2m}C\frac{(Mt)^m}{m!}$$

Since $\sum[2^m(MT)^m/m!] < \infty$, the Borel cantelli lemma implies that

$$P\left\{sup_{0\leq t\leq T}|\xi_{m+1}(t) - \xi_m(t)| > \frac{1}{2^m} i.o.\right\} = 0.$$

Thus, for almost any ω there is a positive integer $m_0 = m_0(\omega)$ such that

$sup_{0\le t\le T}|\xi_{m+1}(t) - \xi_m(t)| \le \frac{1}{2^m}$ if $m \ge m_0(\omega)$

It follows that the partial sums

$$\xi_0 + \sum_{m=0}^{k-1}\left(\xi_{m+1}(t) - \xi_m(t)\right) = \xi_k(t)$$

are convergent uniformly in $t \in [0, T]$. Denoted the limit by $\xi(t)$. Then $\xi(t)$ is a continuous process. It is clearly also a nonanticipative function and it belongs to $L^2_\omega[0, T]$. Since for $a.a.\omega$,

$b(\xi_m(t), t) \to b(\xi(t), t)$ uniformly in $t \in [0, T]$,

$\sigma(\xi_m(t), t) \to \sigma(\xi(t), t)$ uniformly in $t \in [0, T]$,

and hence also

$$\int_0^T |\sigma(\xi_m(t), t) - \sigma(\xi(t), t)|^2 \xrightarrow{P} 0,$$

If we take $m \to \infty$ in (2.30) we obtain relation

$$\xi(t) = \xi_0 + \int_0^t b(\xi(s), s)ds + \int_0^t \sigma(\xi(s), s)dw(s). \tag{2.33}$$

Thus $\xi(t)$ is a solution of (2.25), (2.26).

From (2.30) we have

$$E|\xi_{m+1}(t)|^2 \leq 3E|\xi_0|^2 + 3E\left|\int_0^t b(\xi_m(s),s)ds\right|^2 + 3E\left|\int_0^t \sigma(\xi_m(s),s)dw\right|^2$$

$$\leq C(1 + E|\xi_0|^2) + C\int_0^t E\,|\xi_m(s)|^2 ds$$

Where C is some constant depending only on K, T. By induction we then get

$$E|\xi_{m+1}(t)|^2 \leq \left[C + C^2 t + C^3\frac{t^2}{2!} + \cdots + C^{m+2}\frac{t^{m+1}}{(m+1)!}\right][1 + E|\xi_0|^2].$$

Therefore

$$E|\xi_{m+1}(t)|^2 \leq C(1 + E|\xi_0|^2)e^{Ct}.$$

Taking $m \uparrow \infty$ and using Fatou's lemma; we conclude that

$$E|\xi(t)|^2 \leq C(1 + E|\xi_0|^2)e^{Ct}. \tag{2.34}$$

This implies that $\xi(t)$ belongs to $M_\omega^2[0,T]$.

Remarks (2.8) [Friedman, 75]:

1. The above method used to prove the existence of a solution $\xi(t)$ is called the ***method of successive approximation***; it is modeled after the corresponding proof for ordinary differential equations.

2. Very often we shall take the initial value ξ_0 to be a constant function $x\ a.s.$ notice that this random variable is independent of $\mathcal{F}(w(t), t \geq 0)$.

From (2.33) we obtain

$$sup_{0\le t\le T}|\xi(t)|^2 \le 3|\xi_0|^2 + 3\left[\int_0^t |b(\xi(s),s)|ds\right]^2 +$$

$$3sup_{0\le t\le T}\left|\int_0^t \sigma(\xi(s),s)dw(s)\right|^2$$

Taking the expectation and using (2.27) and theorem (A.1), we get

$$E sup_{0\le t\le T}|\xi(t)|^2 \le C_0(1 + E|\xi_0|^2) + C_0\int_0^T E\,|\xi(s)|^2 ds$$

Where C_0 is a constant depending only on K, T. Making use of (2.34), we obtain:

Corollary (2.1) [Friedman, 75]:

Under the assumptions of theorem (2.5)

$$E[sup_{0\le t\le T}|\xi(t)|^2] \le C^*(1 + E|\xi_0|^2) \qquad (2.35)$$

Where C^* is a constant depending only on K, T.

Theorem (2.6) Stronger Uniqueness and Existence Theorem :

Suppose $b_i(x,t), \sigma_i(x,t)$ are measurable functions in$(x,t) \in R^n \times [0,T]$, for $i = 1,2$, satisfying

$$|b_i(x,t) - b_i(\bar{x},t)| \le K_\theta|x - \bar{x}|, \qquad |\sigma_i(x,t) - \sigma_i(\bar{x},t)| \le K_\theta|x - \bar{x}|$$

$$|b_i(x,t)| \le K(1 + |x|), \qquad |\sigma_i(x,t)| \le K(1 + |x|).$$

Let D be a domain in $R^n$and suppose that

$$b_1(x,t) = b_2(x,t),$$

$$\sigma_1(x,t) = \sigma_2(x,t)$$

If $x \in D, 0 \leq t \leq T$. (2.36)

Let $\xi_i(t)$ $(i = 1,2)$ be the solution of

$$d\xi(t) = b_i(\xi_i(t),t)dt + \sigma_i(\xi_i(t),t), \qquad \xi_i(0) = \xi_{i0}$$

in $M_\omega^2[0,T]$ (with the same family of σ-fields $\mathcal{F}_t$) where $E|\xi_{i0}|^2 < \infty$. Assume finally that $\xi_{10} = \xi_{20}$ for $a.a.\omega$ for which either $\xi_{10} \in D$ or $\xi_{20} \in D$. Denote by τ_i the first time $\xi_i(t)$ intersects R^n/D if such time $t \leq T$ exists, and $\tau_i = T$ otherwise. Then

$$P(\tau_1 = \tau_2) = 1,$$

$$P\{sup_{0\leq t\leq \tau_1}|\xi_1(s) - \xi_2(s)| = 0\} = 1.$$

Thus if two stochastic equation have the same coefficients in a cylinder $Q = D \times [0,T]$ and if the initial condition coincide in D, then the corresponding solution agree until the first time they both leave D; they first leave D at the same time.

Remarks (2.9) [Friedman,75] :

1. This is local uniqueness theorem.
2. It remains true for the general domains Q.

2.5 Examples of Linear Stochastic Differential Equation:

Example (2.4) [Evans, 05]:

Let $m = n = 1$ and suppose g is a continuous function (not a random variable). Then the unique solution of

$$\left.\begin{array}{l} dX = gXdw \\ X(0) = 1 \end{array}\right\} \quad (2.37)$$

is

$$X(t) = e^{-\frac{1}{2}\int_0^t g^2 ds + \int_0^t g dw}$$

for $0 \leq t \leq T$. To verify this, note that

$$Y(t) := -\frac{1}{2}\int_0^t g^2 ds + \int_0^t g dw$$

Satisfies

$$dY = -\frac{1}{2} g^2 dt + g dw.$$

Thus Itô lemma for $u(x) = e^x$ gives

$$\begin{aligned} dX &= \frac{\partial u}{\partial x} dY + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} g^2 dt \\ &= e^Y\left(-\frac{1}{2} g^2 dt + g dw + \frac{1}{2} g^2 dt\right) \\ &= gXdw, \end{aligned}$$

As claimed.

Example (2.5)[Evans, 05] :

Similarly, the unique solution of

$$\left.\begin{array}{l} dX = fXdt + gXdw \\ X(0) = 1 \end{array}\right\} \quad (2.38)$$

Is

$$X(t) = e^{\int_0^t f - \frac{1}{2}g^2 ds + \int_0^t g dw}$$

for $0 \leq t \leq T$.

Example (2.6) (Stock prices) [Evans,05] :

Let $P(t)$ denote the price of a stock at time t. We can model the evolution of $P(t)$ in time by supposing that $\frac{dP}{P}$, the relative change of price, evolves according to the SDE

$$\frac{dP}{P} = \mu dt + \sigma dw$$

for certain constants $\mu > 0$ and σ, called the *drift* and the *volatility* of the stock. Hence

$$dP = \mu P dt + \sigma P dw; \tag{2.39}$$

and so

$$d(\log(P)) = \frac{dP}{P} - \frac{1}{2}\frac{\sigma^2 P^2 dt}{P^2}$$

by the Itô formula

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dw.$$

Consequently

$$P(t) = P_0 e^{\sigma w(t) + \left(\mu - \frac{\sigma^2}{2}\right)t},$$

similarly to Example (2.5). Observe that the price is always positive, assuming the initial price P_0 is positive.

Since (2.39) implies

$$P(t) = P_0 + \int_0^t \mu P ds + \int_0^t \sigma P dw$$

and $E\left(\int_0^t \sigma P dw\right) = 0$, we see that

$$E(P(t)) = P_0 + \int_0^t \mu E\big(P(s)\big)ds.$$

Hence

$$E(P(t)) = P_0 e^{\mu t}, \quad t \geq 0.$$

The expected value of the stock price consequently agrees with the deterministic solution of (2.39) corresponding to $\sigma = 0$.

Example (2.7) (Langevin's equation) [Evans, 05] :

A possible improvement of our mathematical model of the motion of a Brownian particle models *frictional forces* as follows for the one dimensional case:

$$\dot{X} = -bX + \sigma\xi,$$

where $\xi(.)$ is "white noise", $b > 0$ is a coefficient of friction, and σ is a diffusion coefficient.

In this interpretation $X(.)$ is the *velocity* of the Brownian particle: see Example 6 for the *position* process $Y(.)$. We interpret this to mean

$$\left.\begin{aligned} dX &= -bXdt + \sigma dw \\ X(0) &= X_\circ, \end{aligned}\right\} \qquad (2.40)$$

for some initial distribution $X0$, independent of the Brownian motion. This is the *Langevin equation.*

The solution is

$$X(t) = e^{-bt}X_\circ + \sigma \int_0^t e^{-b(t-s)}dw \qquad (t \geq 0)$$

as is straightforward to verify. Observe that

$$E(X(t)) = e^{-bt}E(X_\circ)$$

And

$$E\big(X^2(t)\big) = E\Bigg(e^{-2bt}X_\circ^2$$

$$+ 2\sigma e^{-bt}X_\circ \int_0^t e^{-b(t-s)}dw + \sigma^2\left(\int_0^t e^{-b(t-s)}dw\right)^2\Bigg)$$

$$= e^{-2bt}E(X_\circ^2) + 2\sigma e^{-bt}E(X_\circ)E\left(\int_0^t e^{-b(t-s)}dw\right)$$

$$+ \sigma^2\int_0^t e^{-2b(t-s)}ds = e^{-2bt}E(X_\circ^2) + \frac{\sigma^2}{2b}(1 - e^{-2bt}).$$

Thus the variance

$$V\big(X(t)\big) = E(X^2(t) - E(X(t))^2$$

Is given by

$$V(X(t)) = e^{-2bt}V(X_\circ) + \frac{\sigma^2}{2b}(1 - e^{-2bt}),$$

assuming, of course, $V\ (X_\circ) < \infty$. For any such initial condition X_0 we therefore have

$$\begin{cases} E(X(t)) \to 0 \\ V(X(t)) \to \dfrac{\sigma^2}{2b} \end{cases} \qquad as\ t \to \infty.$$

From the explicit form of the solution we see that the distribution of $X(t)$ approaches $N\left(0,\frac{\sigma^2}{2b}\right)$ as $t\to\infty$. We interpret this to mean that irrespective of the initial distribution, the solution of the SDE for large time "settles down" into a Gaussian distribution whose variance $\frac{\sigma^2}{2b}$ represents a balance between the random disturbing force $\sigma\,\xi(.)$ and the frictional damping force $-bX(.)$.

The output-feedback problem has received considerable attention in the recent nonlinear control literatures [Jankovic, 97], [Khalil, 96], [Krstic, 95], [Marino, 95], [Praly, 93], [Teel, 95].

In this chapter an output feedback (observer-based) backstepping control law which guarantees global asymptotic stability in probability has been proved. The stabilizing control laws which are also optimal with respect to meaningful cost functional have also been proved.

The necessary theorems for a certain nonlinear dynamic stochastic control system have been stated and proved.

Some concluding remarks are also included and discussed.

3.1 Preliminaries on Stability in Probability:

Consider the nonlinear stochastic system

$$dX = f(X)dt + g(X)dw \tag{3.1}$$

Where $X \in R^n$ is the state, w is an r-dimensional independent standard Brownian motion, and $f: R^n \to R^n$, $g: R^n \to R^{n\times r}$ are locally Lipschitz functions and satisfy $f(0) = 0, g(0) = 0$.

The following definitions are needed for complete understanding of the subject:

Remarks (3.1) [Hardy& Littlewood, 89]:

The following inequality is very important in the present work and is discussed as follows:

1. Young's inequality has the form:

$$XY \leq \frac{\epsilon^P}{P}|X|^P + \frac{1}{q\epsilon^P}|Y|^q$$

where $\epsilon > 0$, the constants $p > 1$ and $q > 1$ satisfy:

$$(p-1)(q-1) = 1$$

And $(X, Y) \in R^{2n}$.

2. Let γ is a positive function, then:

   $|\psi_i(\bar{x}_i)| \leq \gamma|\tilde{x}|^4$

   Where $\tilde{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$

3. Suppose a function $f(t,x)$ has domain D in (t,x)-space and suppose there exists K, such that if $(t,x_1),(t,x_2) \in D$, then

$$\|f(t,x_1) - f(t,x_2)\| \leq K\|x_2 - x_1\| \qquad (3.2)$$

   Then f satisfies a Lipschitz condition with respect to x in D, and K is a Lipschitz constant for f. [Brauer&Nohel, 73]

4. If $f(x), x \in R^n$ is differentiable function with bounded partial derivatives, then K is simply is the upper bound of the norm of the Jacobian matrix for the function $f(x)$, the upper bound taken over the entire R^n. However, in general, a Lipschitz function may not be differentiable. [Raghavan, 94]

5. Not necessary that all functions must satisfy the Lipschitz conditions; the following example shows a class of function not belongs to a class of Lipscitzian functions.

<u>Example (3.1) [Brauer& Nohel, 73]:</u>

If $f(t,y) = y^{1/3}$ in the rectangle $D = \{(t,y)\ |t| \leq 1, |y| \leq 2\}$, then f does not satisfy a Lipschitz condition in D.

To establish this, we only need to produce a suitable pair of points for which *(3.2)* fails to hold with any constant K. consider the points

$$(t,y_1),(t,0) \text{ With } -1 \leq t \leq 1, y_1 > 0.$$

Then

$$\frac{f(t,y_1)-f(t,0)}{y_1-0}=\frac{y_1^{1/3}}{y_1}=\frac{1}{y_1^{2/3}}.$$

Now choosing $y_1 > 0$ sufficiently small, it is clear that $K = \frac{1}{y_1^{2/3}}$ can be made larger than any preassigned constant. Therefore equation *(3.2)* fails to hold for any K.

Definition (3.2)[Deng&Krstic, 99]:

The equilibrium state x= 0 of *(3.1)* is said to be globally asymptotically stable in probability if for any $t_0 \geq 0$ and $\epsilon > 0$, $\lim_{x(t_0)\to 0}$, $P\{\sup_{t\geq t_0}|x(t)| > \epsilon\} = 0$, and for any initial condition $x(t_0)$, $P\{\lim_{t\to\infty} x(t) = 0\} = 1$.

Theorem (3.1) [Khas'minskii, 80], [Kushner, 67] , [Mao, 91]:

Consider the system

$$dX = F(X)dt + g(X)dw$$

and suppose there exists a positive definite, radially unbounded, twice continuously differentiable function $V(x)$ such that the infinitesimal generator

$$\mathcal{L}V = \frac{\partial V}{\partial X} F + \frac{1}{2}\, tr\left\{g^T \frac{\partial^2 V}{\partial X^2} g\right\} \tag{3.3}$$

is negative definite. Then the equilibrium state X= 0 of $dX = F(X)dt + g(X)dw$

is globally asymptotically stable in probability; where $tr(.)$ is standing for the trace operation.

3.2 Problem Formulation of Output-Feedback Stabilization in Probability:

The following problem formulation has been considered

In this section we deal with nonlinear ***output-feedback*** systems driven by Brownian motion. This class of systems is given by the following nonlinear stochastic differential equations:

$$dx_i = x_{i+1}dt + f_i(\bar{x}_i)dt + \varphi_i(y)^T dw + \psi_i(\bar{x}_i)^T dw, \quad i = 1, \dots, n-1$$

$$dx_n = u\,dt + f_n(\bar{x}_n)dt + \varphi_n(y)^T dw + \psi_n(\bar{x}_n)^T dw$$

$$y = x_1 \qquad (3.4)$$

Where

1. $x \in R^n$, is the state.
2. w is an r-dimensional independent standard Brownian motion.
3. $f = (f_1, f_2, \dots, f_n)^T, f$ is a vector valued function, satisfied:
   i. $f: R^n \to R^n, f(0) = 0.$
   ii. $f_i(\bar{x}_i) = f_i(x_1, \dots, x_i)$
   iii. $\|f(x)\| \le x^T Q x \le (x^T Q x)^2 \le (\lambda max(Q))^2 |x|^4$ (3.5)
   Where Q is a positive definite matrix, and $\lambda max(Q)$ is the largest eigan value of Q, and $|.|$ is standing for suitable norm.
4. $\varphi_i(y)$ are r-vector-valued smooth functions with $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T, \varphi: R^n \to R^{n \times r}$ and $\varphi_i(0) = 0$.
5. $\psi_i(\bar{x}_i)$ are r-vector-valued smooth functions with $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T, \psi: R^n \to R^{n \times r}$, with $\psi_i(0) = 0$, Where $\bar{x}_i = [x_1, \dots, x_i]^T$.
6. $u(0) = 0$.

7. Let f_i , φ_i , ψ_i are satisfied Lipchitz condition.
8. Since the states $x_2, \dots, x_n$ are not measured, need to be estimated by a dynamic observer which is suggested as

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + k_i(y - \hat{x}_1) \qquad i = 1, \cdots, n \tag{3.6}$$

Now, the entire system can be expressed as:

$d\tilde{x} = A_0\, \tilde{x}\, dt + f(\tilde{x})dt + \varphi(y)^T dw + \psi(\tilde{x})^T dw,$

Where $\tilde{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ and $\bar{x}_i = (x_1, x_2, \dots, x_i)$ $\quad i = 1, \dots, n$

Remark (3.3):

1. The observation errors $\tilde{x} = x - \hat{x}$ satisfy:

$$d\tilde{x} = \begin{bmatrix} -k_1 & & I & \\ \vdots & & & \\ -k_n & 0 & \dots & 0 \end{bmatrix} \tilde{x}dt + f(\tilde{x})dt + \varphi(y)^T dw + \psi(\tilde{x})^T dw,$$

$$d\tilde{x} = A_0\, \tilde{x}\, dt + f(\tilde{x})dt + \varphi(y)^T dw + \psi(\tilde{x})^T dw \tag{3.7}$$

A_0 is designed to be asymptotically stable. Now, the entire system can be expressed as

$$d\tilde{x} = A_0\, \tilde{x}dt + f(\tilde{x})dt + \varphi(y)^T dw + \psi(\tilde{x})^T dw$$

and

$$dy = (\hat{x}_2 + \tilde{x}_2)\, dt + \varphi_i(y)^T\, dw$$

$$d\hat{x}_2 = [\hat{x}_3 + k_2(y - \hat{x}_1)]dt$$

$$\vdots$$

$$d\hat{x}_n = [u + k_n(y - \hat{x}_1)]dt \tag{3.8}$$

2. Since $\varphi_i(0) = 0$, the α_i's will vanish at $\bar{\hat{x}}_{i-1} = 0$, y = 0, as well as at $\bar{z}_i$= 0, where $\bar{z}_i$=$(z_1, \dots, z_i)^T$. Thus, by the mean value theorem, and $\alpha_i(\bar{\hat{x}}_i, y)$ and $\varphi(y)$ can be expressed, respectively, as

$$\alpha_i(\bar{\hat{x}}_i, y) = \sum_{l=1}^{i} z_l \alpha_{il}(\bar{\hat{x}}_i, y) \quad (3.9)$$

$$\varphi(y) = y\psi(y) \quad (3.10)$$

Where $\alpha_i(\bar{\hat{x}}, y)$ and $\psi(y)$ are smooth functions.

Theorem (3.2):

Consider the system defined in problem formulation (3.2), and assuming that the dynamic observer is designed to be

$$d\tilde{x} = A_0\tilde{x}dt + f(\tilde{x})dt + \varphi(y)^T dw + \psi(\tilde{x})^T dw$$

a sequence of stabilizing functions $\alpha_i(\bar{\hat{x}}_i, y)$, where $\bar{\hat{x}}_i = [\hat{x}_2, \dots, \hat{x}_i]^T$, is constructed recursively to build a Lyapunov function of the form:

$$V(z, \tilde{x}) = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2$$

Define:

$$z_1 = y \quad (3.11)$$

$$z_i = \hat{x}_i - \alpha_i(\bar{\hat{x}}_{i-1}, y), \qquad i=2,\dots,n \quad (3.12)$$

and, if the following are satisfied:

$$\alpha_1 = -c_1 y - \frac{3}{2}y\psi_1\Bigg(y)^T\psi_1(y) - \frac{3}{4}\sigma_1^{4/3}y - \frac{3}{4}\epsilon_1^{4/3}y$$
$$- \frac{3}{4}\sum_{i=2}^{n}\xi_i^2\left(\psi_1(y)^T\psi_1(y)\right)^2 y - \frac{3bn\sqrt{n}}{\epsilon_2^2}|\psi(y)|^4 y + f(y)\Bigg]$$

and

$$\alpha_i = -c_i z_i - k_1\tilde{x}_1 + \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_1\tilde{x}_1) + \left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\hat{x}_2$$
$$+ \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) - \frac{3}{4}\sigma_i^{4/3}z_i - \frac{1}{4\sigma_{i-1}^4}z_i$$
$$- \frac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i - \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i$$

and

$$u = -c_n z_n - k_n \tilde{x}_1 + \sum_{l=2}^{n-1} \frac{\partial \alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1 \tilde{x}_1) + \left(\frac{\partial \alpha_{n-1}}{\partial y}\right)\hat{x}_2$$

$$+ \frac{1}{2}\left(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)\varphi_1(y)^T \varphi_1(y) - \frac{1}{2}\sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q}\varphi_p^T \varphi_q$$

$$- \frac{1}{4\sigma_{n-1}^4} z_n - \frac{3}{4}\eta_n^{4/3}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n + \frac{3}{4\xi_n^2}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4 z_n$$

and P is a positive definite matrix which satisfies:

$$A_0^T P + P A_0 = -I \tag{3.13}$$

Then the equilibrium point $x = 0$ at the origin of the closed-loop stochastic system (3.8), (3.29) is globally asymptotically stable in probability**.**

<u>Proof:</u>

We have $z_1 = y$ and $z_i = \hat{x}_i - \alpha_i(\bar{\bar{x}}_{i-1}, y)$

By using Itô's differentiation rule of the [*Øksendal, 95*], [*Fridman, 75*]

$$dz_1 = (\hat{x}_2 + \tilde{x}_2)dt + f_1(y)dt + \varphi_1(y)\, dw + \psi_1(y) \tag{3.14}$$

$$dz_i = \hat{x}_{i+1} + k_i \tilde{x}_1 - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\big(x_{l+1} + f_i(\bar{x}_l)\big) - \frac{\partial \alpha_{i-1}}{\partial y}\big((\hat{x}_2 + \tilde{x}_2) + f_i(\bar{x}_1)\big)$$

$$- \frac{1}{2}\sum_{p,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q}\varphi_p^T \varphi_q - \frac{1}{2}\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\varphi(y)^T \varphi(y)$$

$$i = 2, \dots, n \tag{3.15}$$

As we announced previously, we employ a Lyapunov function of the form:

$$V(z, \tilde{x}) = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P \tilde{x})^2 \tag{3.16}$$

Where P is a suitable positive definite matrix, Now we start the process of selecting the unknown functions $\alpha_i(\bar{\hat{x}}, y)$ to make $\mathcal{L}V$ negative definite. Along the solutions of *(3.7)*, *(3.14)*, and *(3.15)*, we have

$$\mathcal{L}V = y^3(\hat{x}_2 + \tilde{x}_2) + \frac{3}{2}y^2\varphi_1(y)^T\varphi_1(y)$$
$$+ \sum_{l=2}^{n} z_i^3 \Bigg[\hat{x}_{i+1} + k_i x_1^{\sim} - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\big(x_{l+1} + f_i(\bar{x}_l)\big)$$
$$- \frac{\partial \alpha_{i-1}}{\partial y}\big((\hat{x}_2 + \tilde{x}_2) + f_i(\bar{x}_1)\big) - \frac{1}{2}\sum_{p,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q}\varphi_p^T\varphi_q$$
$$- \frac{1}{2}\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\varphi(y)^T\varphi(y)\Bigg] + \frac{3}{2}\sum_{i=2}^{n} z_i^2 \,(\frac{\partial \alpha_{i-1}}{\partial y})^2\varphi_1(y)^T\varphi_1(y)$$
$$+ \frac{2}{2}b(\tilde{x}^T P\tilde{x})[\tilde{x}A_0Px + \tilde{x}^T PA_0x] + \frac{4}{2}tr\{\,\varphi(y)(2P\tilde{x}\,\tilde{x}^T P$$
$$+ \;\tilde{x}^T P\tilde{x}P)\varphi(y)^T\}$$

by using equation *(3.10)* we have:

$$\mathcal{L}V = y^3(z_2 + \alpha_1 + \tilde{x}_2) + y^3 f_1(y) + \frac{3}{2}y^2\varphi_1(y)^T\varphi_1(y)$$
$$+ \sum_{i=2}^{n} z_i^3 \Bigg[z_{i+1} + \alpha_i + k_i\tilde{x}_1 - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\big(x_{l+1} + f_i(\bar{x}_l)\big)$$
$$- \frac{\partial \alpha_{i-1}}{\partial y}\big((\hat{x}_2 + \tilde{x}_2) + f_i(\bar{x}_1)\big) - \frac{1}{2}\sum_{p,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q}\varphi_p^T\varphi_q$$
$$- \frac{1}{2}\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\varphi(y)^T\varphi(y)\Bigg] + \frac{3}{2}\sum_{i=2}^{n} z_i^2 \,(\frac{\partial \alpha_{i-1}}{\partial y})^2\varphi_1(y)^T\varphi_1(y)$$
$$+ \; b(\tilde{x}^T P\tilde{x})[\tilde{x}^T(A_0P + PA_0)\tilde{x}] + +2\;tr\{\,\varphi(y)(2P\tilde{x}\,\tilde{x}^T P$$
$$+ \;\tilde{x}^T P\tilde{x}P)\varphi(y)^T\}$$

$$\mathcal{L}V = y^3 z_2 + y^3\alpha_1 + y^3\tilde{x}_2 + y^3 f_1(y) + \frac{3}{2}y^2\varphi_1(y)^T\varphi_1(y) + \sum_{i=2}^{n} z_i^3\, z_{i+1}$$

$$+ \sum_{i=2}^{n} z_i^3\,\alpha_i - \sum_{i=2}^{n} z_i^3\, k_i\tilde{x}_1 - \sum_{i=2}^{n} z_i^3 \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\left(x_{l+1} + f_i(\bar{x}_l)\right)$$

$$- \sum_{i=2}^{n} z_i^3\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\hat{x}_2 - \sum_{i=2}^{n} z_i^3\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\tilde{x}_2 - \sum_{i=2}^{n} z_i^3\left(\frac{\partial\alpha_{i-1}}{\partial y}\right) f_1(y)$$

$$- \frac{1}{2}\sum_{i=2}^{n} z_i^3 \sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^T\varphi_q$$

$$- \frac{1}{2}\sum_{i=2}^{n} z_i^3\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y)$$

$$+ \frac{3}{2}\sum_{i=2}^{n} z_i^2\,\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2\varphi_1(y)^T\varphi_1(y) - b(\tilde{x}^T P\tilde{x})|\tilde{x}|^2$$

$$+ 2tr\{\varphi(y)(2P\tilde{x}\,\tilde{x}^T P + \tilde{x}^T P\tilde{x}P))\varphi(y)^T\} \qquad (3.17)$$

Now, by applying Young's inequality on equation (3.17):

The first term: take

$$\left(X = y^3, Y = z_2, p = \frac{4}{3}, q = \frac{1}{4}\right)$$

$$y^3 z_2 \le \frac{3}{4}\sigma_1^{4/3} y^4 + \frac{1}{4\sigma_1^4} \qquad (3.18)$$

The 3rd term: take

$$\left(X = y^3, Y = \tilde{x}_2, p = \frac{4}{3}, q = \frac{1}{4}\right)$$

$$y^3\tilde{x}_2 \le \frac{3}{4}\epsilon_1^{4/3}y^4 + \frac{1}{4\epsilon_1^4}\tilde{x}_2^4 \le \frac{3}{4}\epsilon_1^{4/3}y^4 + \frac{1}{4\epsilon_1^4}|\tilde{x}|^4 \qquad (3.19)$$

The 6th term: take

$$\left(X = y^3, Y = \tilde{x}_2, p = \frac{4}{3}, q = \frac{1}{4}\right)$$

$$\sum_{i=2}^{n} z_i^3 z_{i+1} \le \frac{3}{4}\sum_{i=2}^{n-1} \sigma_i^{4/3} z_i^4 + \frac{1}{4}\sum\nolimits_{i=3}^{n} \frac{1}{\sigma_{i-1}^4} z_i^4 \tag{3.20}$$

The 9th term: take

$$\left(X = -\sum_{i=2}^{n} z_i^3 \sum_{l=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_l}\right), Y = f(\tilde{x}_2), p = \frac{4}{3}, q = \frac{1}{4}\right)$$

$$-\sum_{i=2}^{n} z_i^3 \sum_{l=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_l}\right) f(\tilde{x}_2)$$

$$\le \quad \frac{3}{4}\sum_{i=2}^{n} \xi_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial x_l}\right)^{4/3} z_i^4 + \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}(\lambda_1 max(Q))^2|\tilde{x}|^4$$

(3.21)

The 11th term: take

$$\left(X = -\sum_{i=2}^{n} z_i^3 \frac{\partial\alpha_{i-1}}{\partial y}, Y = \tilde{x}_2, p = \frac{4}{3}, q = \frac{1}{4}\right)$$

$$-\sum_{i=2}^{n} z_i^3 \frac{\partial\alpha_{i-1}}{\partial y}\tilde{x}_2 \le \frac{3}{4}\sum_{i=2}^{n}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i^4 + \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4}\tilde{x}_2^4$$

$$\le \frac{3}{4}\sum_{i=2}^{n}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i^4 + \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4}|\tilde{x}|^4$$

(3.22)

The 15th term: take

$$\left(X = \frac{3}{2}\sum_{i=2}^{n} z_i^2 \,(\frac{\partial \alpha_{i-1}}{\partial y})^2, Y = \varphi_1(y)^T\varphi_1(y)\,, p = \frac{4}{3}\,, q = \frac{1}{4} \right)$$

$$\frac{3}{2}\sum_{i=2}^{n} z_i^2 \,(\frac{\partial \alpha_{i-1}}{\partial y})^2 \varphi_1(y)^T\varphi_1(y)$$

$$\leq \frac{3}{4}\sum\nolimits_{i=2}^{n} \frac{1}{\xi_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i^4 + \frac{3}{4}\sum_{i=2}^{n} \xi_i^2\big(\varphi_1(y)^T\varphi_1(y)\big)^2$$

(3.23)

The last term:

$$2btr\{\varphi(y)(\ 2P\tilde{x}\,\tilde{x}^T P + \ \tilde{x}^T P\tilde{x}P))\varphi(y)\}$$

$$2btr\{\varphi(y)(\ 2P\tilde{x}\,\tilde{x}^T P + \ \tilde{x}^T P\tilde{x}P))\varphi(y)\}$$

$$\leq 2bn|\varphi(y)(\ 2P\tilde{x}\,\tilde{x}^T P + \ \tilde{x}^T P\tilde{x}P))\varphi(y)^T|_\infty$$

$$\leq 2bn\sqrt{n}|\varphi(y)(\ 2P\tilde{x}\,\tilde{x}^T P + \ \tilde{x}^T P\tilde{x}P))\varphi(y)^T|$$

$$\leq 6bn\sqrt{n}y^2|\psi(y)|^2|P|^2|\tilde{x}|^2$$

$$\leq \frac{3bn\sqrt{n}}{\epsilon_2^2} y^4|\psi(y)|^4 + 3bn\sqrt{n}\epsilon_2^2|P|^4|\tilde{x}|^4 \qquad (3.24)$$

By substituting all the terms given in the equations *(3.18)* , *(3.19)* , *(3.20)* , *(3.21)* , *(3.22)* ,*(3.23)* and *(3.24)* in equation *(3.17)* then we have:

$$
\begin{aligned}
\mathcal{L}V \leq & -[b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}\left(\lambda_1 max(Q)\right)^2 \\
& - \left(\lambda_2 max\left(\frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}Q\right)\right)^2]|\tilde{x}|^4 \\
& + y^3\left[\alpha_1 y\psi_1\left(\psi_1(y) + \frac{3}{4}\sigma_1^{4/3}y + \frac{3}{4}\epsilon_1^{4/3}y\right.\right. \\
& \left. + \frac{3}{4}\sum_{i=2}^{n}\xi_i^2\left(\psi_1(y)^T\psi_1(y)\right)^2 y + \frac{3bn\sqrt{n}}{\epsilon_2^2}|\psi(y)|^4 y + f_1(y)\right] \\
& + \sum_{i=2}^{n-1} z_i^3[\alpha_i + k_1\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1\tilde{x}_1) - \left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\hat{x}_2 \\
& - \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^T\varphi_q - \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) + \frac{3}{4}\sigma_i^{4/3}z_i \\
& \left. + \frac{1}{4\sigma_{i-1}^4}z_i + \frac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i + \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i\right] \\
& + z_n^3\left[u + k_n\tilde{x}_1 - \sum_{l=2}^{n-1}\frac{\partial\alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1\tilde{x}_1) - \left(\frac{\partial\alpha_{n-1}}{\partial y}\right)\hat{x}_2\right. \\
& - \frac{1}{2}\left(\frac{\partial^2\alpha_{n-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) - \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\varphi_q \\
& \left. + \frac{1}{4\sigma_{n-1}^4}z_n + \frac{3}{4}\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} z_n + \frac{3}{4\xi_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4 z_n\right] \\
& + \frac{3}{4}\sum_{i=2}^{n}\xi_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial x_l}\right)^{4/3} + \frac{3}{4}\sum_{i=2}^{n}\xi_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} \qquad (3.25)
\end{aligned}
$$

Where $\lambda > 0$ is the smallest eigenvalue of P. The second equality comes from substituting $\hat{x}_i = z_i + \alpha_{i-1}$, and the inequality comes from Young's inequalities. At this point, we can see that all the terms can be cancelled by u and α_i. If we choose ϵ_1, ϵ_2, η_i and ξ_i to satisfy

$$b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}\left(\lambda_1 max(Q)\right)^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}\left(\lambda_2 max(Q)\right)^2 = \rho > 0 \tag{3.26}$$

And α_i and u as

$$\alpha_1 = -c_1 y\frac{3}{2}y\psi_1\Bigg(y)^T\psi_1(y) - \frac{3}{4}\sigma_1^{4/3}y - \frac{3}{4}\epsilon_1^{4/3}y - \frac{3}{4}\sum_{i=2}^{n}\xi_i^2\left(\psi_1(y)^T\psi_1(y)\right)^2 y - \frac{3bn\sqrt{n}}{\epsilon_2^2}|\psi(y)|^4 y + f(y)\Bigg] \tag{3.27}$$

$$\alpha_i = -c_i z_i - k_1\tilde{x}_1 + \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_1\tilde{x}_1) + \left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\hat{x}_2 + \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) - \frac{3}{4}\sigma_i^{4/3}z_i - \frac{1}{4\sigma_{i-1}^4}z_i - \frac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}z_i - \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i \tag{3.28}$$

$$u = -c_n z_n - k_n\tilde{x}_1 + \sum_{l=2}^{n-1}\frac{\partial\alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1\tilde{x}_1) + \left(\frac{\partial\alpha_{n-1}}{\partial y}\right)\hat{x}_2 + \frac{1}{2}\left(\frac{\partial^2\alpha_{n-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) - \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\varphi_q - \frac{1}{4\sigma_{n-1}^4}z_n - \frac{3}{4}\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3}z_n + \frac{3}{4\xi_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4 z_n \tag{3.29}$$

where $c_i > 0$, then the infinitesimal generator of the closed-loop system (3.5), (3.14), (3.15), and (3.29) is negative definite

$$\mathcal{L}V \leq -\sum_{i=1}^{n} c_i z_i^4 - \rho|\tilde{x}|^4 \tag{3.30}$$

With (3.30) and hence $\mathcal{L}V < 0$, from theorem (3.1) the critical point of (3.4) is global asymptotically stable in probability. That is complete the proof.

<u>Concluding remark (3. 4):</u>

If we put $f_i(\bar{x}_i) = 0$, and $\psi_i(\bar{x}_i) = 0,\ i = 1, \dots, n$, of system (3.4). Then we have the work in *[Deng & Krstic, 99]*.

<u>Remarks (3.5):</u>

1. Our output-feedback design will consist of applying a backstepping procedure to the system $(y, \hat{x}_2, \dots, \hat{x}_n)$, which also takes care of the feedback connection through the $\tilde{x}$-system.
2. The Lyapunov design for stochastic systems cannot be performed using the quadratic Lyapunov function *(3.8)* because of the term
$$\frac{1}{2} tr\left\{g^T\left(\frac{\partial^2 V}{\partial x^2}\right)g\right\}$$
in *(3.3)*. We instead employ ***quartic*** (fourth order) Lyapunov functions
$$V = \sum_{i=1}^{n} \frac{1}{4} z_i^4 + (\tilde{x}^T P \tilde{x})^2$$
3. Our presentation of the backstepping procedure here is very concise: instead of introducing the stabilizing functions α_i in a step-by-step fashion, we derive them simultaneously. The technique of back - stepping is referred to [***Krstic, 95***].

3.3 PROBLEM FORMULATION OF INVERSE OPTIMAL OUTPUT-FEEDBACK STABILIZATION:

This section first reviews some definitions and theorems established in *[Deng & Kristic, 97, b]*, which are then used in the design of an inverse optimal stabilizing control law.

Consider the system

$$dx = f(x)dt + g_1(x)dw + g_2(x)udt \qquad (3.31)$$

Where $f(0) = 0, g_1(0) = 0,$ and $u \in R^m$.

Definition (3.4) [Deng & Kristic, 97, b]:

A function $\alpha: \mathcal{R}_+ \longrightarrow \mathcal{R}_+$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, and $lim_{r\to\infty}\alpha(r) = \infty$.

Definition (3.5) [Deng & Kristic, 97, b]:

The problem of ***inverse optimal stabilization in probability*** for system (3.31) is solvable if there exist a class $\mathcal{K}_\infty$ function γ_2 whose derivative $\grave{\gamma_2}$ is also a class $\mathcal{K}_\infty$ function, a matrix-valued function $R_2(x)$ such that

$$R_2(x) = R_2(x)^T > 0 \qquad \textit{for all } x,$$

a positive definite radially unbounded function $\ell(x)$, and a feedback control law $u = \alpha(x)$ continuous away from the origin with $\alpha(0) = 0$, which guarantees global asymptotic stability in probability of the equilibrium $x = 0$ and minimizes the cost functional

$$J(u) = E\{\int_0^\infty \left[l(x) + \gamma_2(\left|R_2(x)^{1/2}u\right|)\right] d\tau\} \qquad (3.32)$$

Theorem (3.3) [Deng & Kristic, 97, b]::

Consider the control law

$$u = \alpha(x) = -R_2^{-1}(L_{g2}V)^T \frac{\ell\gamma_2\left(L_{g2}VR_2^{-1/2}\right)}{\left|L_{g2}VR_2^{-1/2}\right|^2} \tag{3.33}$$

where $V(x)$ is a Lyapunov function candidate, γ_2 is a class $\mathcal{K}_\infty$ function whose derivative is also a class $\mathcal{K}_\infty$ function, $R_2(x)$ is a matrix-valued function such that $R_2(x) = R_2(x)^T > 0$, and $\ell\gamma_2$ is the Legandre–Fenchel transform defined as

$$\ell\gamma_2 = \int (\grave{\gamma}_2)^{-1} \tag{3.34}$$

If the control law (3.33) achieves global asymptotic stability in probability for the system (3.31) with respect to $V(x)$, then the control law

$$u^* = \alpha^*$$

$$= -\frac{\beta}{2} R_2^{-1}(L_{g2}V)^T \frac{(\grave{\gamma}_2)^{-1}\left(\left|L_{g2}VR_2^{-1/2}\right|\right)}{\left|L_{g2}VR_2^{-1/2}\right|^2}, \quad \beta \geq 2 \tag{3.35}$$

Solves the problem of the inverse optimal stabilization in probability for the system (3.31) by minimizing the cost functional

$$J(u) = E\{\int_0^\infty \left[l(x) + \beta^2\gamma_2\left(\frac{2}{\beta}\left|R_2(x)^{1/2}u\right|\right)\right]d\tau\} \tag{3.36}$$

Where

$$l(x) = 2\beta\left[\ell\gamma_2\left(\left|L_{g2}VR_2^{-1/2}\right|\right) - L_fV - \frac{1}{2}tr\left\{g^T\frac{\partial^2 V}{\partial X^2}g\right\}\right]$$
$$+\beta(\beta-2)\ell\gamma_2\left(\left|L_{g2}VR_2^{-1/2}\right|\right) \tag{3.37}$$

<u>Remark (3.6) [Deng & Kristic, 99]:</u>

The function $l(x)$ is positive definite because, by assumption of the theorem, the bracketed term is positive definite, $\ell\gamma_2$ is in class $\mathcal{K}_\infty$, and $\beta \geq 2$.

Now we return to the output-feedback system defined in problem formulation in section (3.2) and redesign the control law (3.29) to make the problem inverse optimal. The following result is instrumental.

Theorem (3.4) (inverse optimal output – feedback stabilization):

Consider the system defined in problem formulation (3.2) assuming that the condition of theorem (3.1) are satisfied if there exist a continuous positive function $M(y,\hat{x})$ such that the control law of theorem (3.2) can be rewritten as:

$$u = \alpha(y,\hat{x}) = -M(y,\hat{x})z_n \tag{3.38}$$

Such that $\mathcal{L}V < 0$, when $V(z,\tilde{x}) = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2$

Then the control law

$$u^* = \alpha^*(y,\hat{x}) = \beta\alpha(y,\hat{x}), \qquad \beta \geq \frac{4}{3} \tag{3.39}$$

solve the problem of inverse optimal stabilization in probability.

Remark (3.7):

From theorem (3.4), if we can design a stabilizing control law that has z_n as a factor, we can easily find another control law which solves the problem of the inverse optimal stabilization in probability, as given by equation (3.39).

Proof:

If we consider carefully the last bracket of equation (3.25) where u is given as:

$$u = -c_n z_n - k_n \tilde{x}_1 + \sum_{l=2}^{n-1} \frac{\partial \alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1 \tilde{x}_1) + \left(\frac{\partial \alpha_{n-1}}{\partial y}\right)\hat{x}_2$$

$$+ \frac{1}{2}\left(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)\varphi_1(y)^T \varphi_1(y) - \frac{1}{2}\sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q}\varphi_p^T \varphi_q$$

$$- \frac{1}{4\sigma_{n-1}^4} z_n - \frac{3}{4}\eta_n^{4/3}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n + \frac{3}{4\xi_n^2}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4 z_n$$

Every term except the second, the third, the fourth, and the fifth has $z_n$as a factor. With the help of Young's inequalities, we have:

In the following inequalities, ϵ's are constants to be chosen:

The 2nd term: take $\left(X = z_n^3, Y = k_n \tilde{x}_1, p = \frac{3}{4}, q = \frac{1}{4}\right)$

$$z_n^3 k_n \tilde{x}_1 \le \frac{3}{4}\epsilon_3^{4/3} z_n^4 + \frac{1}{4\epsilon_3^4} k_n^4 \tilde{x}_1^4$$

$$\le \frac{3}{4}\epsilon_3^{4/3} z_n^4 + \frac{1}{4\epsilon_3^4} k_n^4 |\tilde{x}|^4 \tag{3.40}$$

The 3rd term: take $\left(X = -z_n^3 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} k_l, Y = \tilde{x}_1, p = \frac{3}{4}, q = \frac{1}{4}\right)$

$$-z_n^3 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} k_l \tilde{x}_l \le \frac{3}{4}\left(\epsilon_4 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} k_l\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_4^4}\tilde{x}_1^4$$

$$\le \frac{3}{4}\left(\epsilon_4 \sum_{l=2}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l} k_l\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_4^4}|\tilde{x}|^4 \tag{3.41}$$

The 6th term: take $\left(X = -\frac{1}{2} z_n^3 \frac{\partial^2 \alpha_{n-1}}{\partial y^2}\psi_1(y)^T \psi_1(y), Y = y^4, p = \frac{3}{4}, q = \frac{1}{4}\right)$

$$-\frac{1}{2} z_n^3 \frac{\partial^2 \alpha_{n-1}}{\partial y^2}\varphi_1(y)^T \varphi_1(y) = -\frac{1}{2} z_n^3 \frac{\partial^2 \alpha_{n-1}}{\partial y^2}\psi_1(y)^T \psi_1(y) y^2$$

$$\le \frac{3}{8}\left(\epsilon_5 \frac{\partial^2 \alpha_{n-1}}{\partial y^2}\psi_1(y)^T \psi_1(y)\right)^{4/3} z_n^4 + \frac{1}{8\epsilon_5^4} y^4 \tag{3.42}$$

The two terms:

$$-z_n^3\frac{\partial\alpha_{n-1}}{\partial y}\hat{x}_2 - z_n^3\sum_{l=2}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\hat{x}_{l+1}$$

$$= -z_n^3\sum_{l=2}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}z_{l+1} - z_n^3\frac{\partial\alpha_{n-1}}{\partial y}z_2 - z_n^3\sum_{l=1}^{n-1}\sum_{k=1}^{l}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}z_k$$

$$= -\sum_{l=2}^{n-1}z_n^3\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}z_{l+1} - z_n^3\frac{\partial\alpha_{n-1}}{\partial y}z_2 - \sum_{l=1}^{n-1}\sum_{k=1}^{l}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\alpha_{lk}z_n^3 z_k$$

$$\le \sum_{l=2}^{n-1}\left[\frac{3}{4}\left(\epsilon_6\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\right)^{4/3}z_n^4 + \frac{1}{4\epsilon_6^4}z_{l+1}^4\right] + \frac{3}{4}\left(\epsilon_6\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\right)^{4/3}z_n^4$$

$$+\frac{1}{4\epsilon_6^4}z_2^4 + \sum_{k=1}^{n-1}\left[\frac{3}{4}\left(\epsilon_7\sum_{l=k}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\alpha_{lk}\right)^{4/3}z_n^4 + \frac{1}{4\epsilon_7^4}z_k^4\right]$$

$$= z_n^4\left[\frac{3}{4}\sum_{l=2}^{n-1}\left(\epsilon_6\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\right)^{4/3} + \frac{3}{4}\left(\epsilon_6\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \frac{1}{4\epsilon_6^4}\right.$$

$$\left. + \frac{3}{4}\sum_{k=1}^{n-1}\left(\epsilon_7\sum_{l=k}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\alpha_{lk}\right)^{4/3}\right]$$

$$+\sum_{i=2}^{n-1}\frac{1}{4\epsilon_6^4}z_i^4 + \sum_{i=2}^{n-1}\frac{1}{4\epsilon_7^4}z_i^4 + \frac{1}{4\epsilon_7^4} \tag{3.43}$$

The 7th term: take $\left(X = -\frac{1}{2}z_n^3\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T, Y = \varphi_q, p = \frac{3}{4}, q = \frac{1}{4}\right)$

$$-\frac{1}{2}z_n^3\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\varphi_q \le \frac{3}{2}\left(\epsilon_8\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\right)^{4/3}z_n^4 + \frac{1}{4\epsilon_8^4}\left|\varphi_q\right|^4$$

$$\le \frac{3}{2}\left(\epsilon_8\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\right)^{4/3}z_n^4 + \frac{1}{4\epsilon_8^4}|\gamma|^4|\tilde{x}|^4 \tag{3.44}$$

Since $V(z,\tilde{x}) = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2$

And $\mathcal{L}V$ is given as:

$$
\begin{aligned}
\mathcal{L}V \le & -[b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}\big(\lambda_1 max(Q)\big)^2 \\
& - \left(\lambda_2 max\left(\frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}Q\right)\right)^2]|\tilde{x}|^4 \\
& + y^3\Bigg[\alpha_1 y\psi_1\Bigg(\psi_1(y) + \frac{3}{4}\sigma_1^{4/3}y + \frac{3}{4}\epsilon_1^{4/3}y \\
& + \frac{3}{4}\sum_{i=2}^{n}\xi_i^2\big(\psi_1(y)^T\psi_1(y)\big)^2 y + \frac{3bn\sqrt{n}}{\epsilon_2^2}|\psi(y)|^4 y + f_1(y)\Bigg] \\
& + \sum_{i=2}^{n-1} z_i^3[\alpha_i + k_1\tilde{x}_1 - \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1\tilde{x}_1) - \left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\hat{x}_2 \\
& - \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^T\varphi_q - \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) + \frac{3}{4}\sigma_i^{4/3}z_i \\
& + \frac{1}{4\sigma_{i-1}^4}z_i + \frac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}z_i + \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i\Bigg] \\
& + z_n^3\Bigg[u + k_n\tilde{x}_1 - \sum_{l=2}^{n-1}\frac{\partial\alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1\tilde{x}_1) - \left(\frac{\partial\alpha_{n-1}}{\partial y}\right)\hat{x}_2 \\
& - \frac{1}{2}\left(\frac{\partial^2\alpha_{n-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) - \frac{1}{2}\sum_{p,q=1}^{n-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\varphi_q \\
& + \frac{1}{4\sigma_{n-1}^4}z_n + \frac{3}{4}\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3}z_n + \frac{3}{4\xi_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4 z_n\Bigg] \\
& + \frac{3}{4}\sum_{i=2}^{n}\xi_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial x_l}\right)^{4/3} + \frac{3}{4}\sum_{i=2}^{n}\xi_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}
\end{aligned}
$$

Then back substation all the terms of u ,equations *(3.40), (3.41), (3.42), (3.43)* and *(3.44)* in $\mathcal{L}V$, to obtain:

$$
\begin{aligned}
\mathcal{L}V \leq & -\Bigg[b\lambda - 3bn\sqrt{n}\epsilon_2^2 |P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4\epsilon_4^4} - \frac{1}{4\epsilon_3^4} k_n^4 \\
& - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^2}(\lambda_1 \max(Q))^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^2}(\lambda_2 \max(Q))^2 \\
& - \frac{1}{4\epsilon_8^4}|\gamma|^4 \Bigg] |\tilde{x}|^4 \\
& + y^3 \Bigg[\alpha_1 + \frac{3}{2}\psi_1(y)^T \psi_1(y) y + \frac{3}{2}\delta_1^{4/3} + \frac{3}{4}\epsilon_1^{4/3} y \\
& + \frac{3}{4}\sum_{i=2}^{n}\xi_i^2 \left(\psi_1(y)^T \psi_1(y)\right)^2 y + \frac{3bn\sqrt{n}}{\epsilon_2^2}|\psi_1(y)|^4 y + \frac{1}{4\epsilon_7^4} y \\
& + \frac{1}{8\epsilon_5^4} y \Bigg] \\
& + \sum_{i=2}^{n-1} z_i^3 \Bigg[\alpha_i + k_i \tilde{x}_1 - \sum_{l=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_l}(\hat{x}_{l+1} + k_l \tilde{x}_1) - \frac{\partial \alpha_{i-1}}{\partial y}\tilde{x}_2 \\
& - \frac{1}{2}\sum_{p,q=1}^{i-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q}\varphi_p^T \varphi_q - \frac{1}{2}\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\varphi_1(y)^T \varphi_1(y) + \frac{3}{4}\delta_i^{4/3} z_i \\
& + \frac{1}{4\delta_{i-1}^4} z_i + \frac{3}{4}\eta_i^{4/3}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{4/3} z_i + \frac{3}{4\xi_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 z_i + \frac{1}{4\epsilon_6^4} z_i \\
& + \frac{1}{4\epsilon_7^4} z_i \Bigg] \\
& + z_n^3 \Bigg[u + \frac{3}{4}\sum_{l=2}^{n-1}\left(\epsilon_6 \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}\right)^{4/3} z_n + \frac{3}{4}\left(\epsilon_6 \frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n \\
& + \frac{1}{4\epsilon_6^4} z_n
\end{aligned}
$$

$$+\frac{3}{4}\sum_{k=1}^{n-1}\left(\epsilon_7\sum_{l=k}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\alpha_{lk}\right)^{4/3} z_n$$

$$+\frac{3}{8}\left(\epsilon_5\frac{\partial^2\alpha_{n-1}}{\partial y^2}\psi_1(y)^T\psi_1(y)\right)^{4/3} z_n$$

$$+\frac{3}{4}\left(\epsilon_4\sum_{l=2}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}k_l\right)^{4/3} z_n+\frac{3}{4}\epsilon_3^{4/3}z_n+\frac{1}{4\delta_{n-1}^4}z_n$$

$$+\frac{3}{4}\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} z_n+\frac{3}{4\xi_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4 z_n$$

$$\left.+\frac{3}{4}\left(\epsilon_8\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\right)^{4/3} z_n\right]$$

$$+\frac{3}{4}\sum_{i=2}^{n}\xi_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial x_l}\right)^{4/3}+\frac{3}{4}\sum_{i=2}^{n}\xi_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}$$

(3.45)

If $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6,\epsilon_{7,}\epsilon_8$ and $\eta_i$are chosen to satisfy

$$b\lambda-3bn\sqrt{n}\epsilon_2^2|P|^4-\frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4}-\frac{1}{4\epsilon_1^4}-\frac{1}{4\epsilon_4^4}-\frac{1}{4\epsilon_3^4}$$
$$-\frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^2}(\lambda_1\max(Q))^2-\frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^2}(\lambda_2\max(Q))^2-\frac{1}{4\epsilon_8^4}|\gamma|^4$$
$$=\rho>0 \quad (3.46)$$

$$\frac{1}{4\epsilon_7^4}+\frac{1}{8\epsilon_5^4}=\frac{c_1}{2} \quad (3.47)$$

$$\frac{1}{4\epsilon_6^4}+\frac{1}{4\epsilon_7^4}=\frac{c_i}{2} \quad (3.48)$$

Where c_1 and c_i are those in (3.27) and (3.28), and

From reconstruct a new controller then

$$u = -M(y,\hat{x})z_n$$

Where

$$M(y,\hat{x}) = c_n + \frac{3}{4}\sum_{l=2}^{n-1}\left(\epsilon_6\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\right)^{4/3} + \frac{3}{4}\left(\epsilon_6\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \frac{1}{4\epsilon_6^4}$$

$$+\frac{3}{4}\sum_{k=1}^{n-1}\left(\epsilon_7\sum_{l=k}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\alpha_{lk}\right)^{4/3} + \frac{3}{8}\left(\epsilon_5\frac{\partial^2\alpha_{n-1}}{\partial y^2}\right)^{4/3}$$

$$+\frac{3}{4}\left(\epsilon_4\sum_{l=2}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}k_l\right)^{4/3} + \frac{3}{4}\epsilon_3^{4/3} + \frac{1}{4\delta_{n-1}^4}$$

$$+\frac{3}{4}\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \frac{3}{4\xi_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4}$$

$$+\frac{3}{4}\left(\epsilon_8\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p\partial x_q}\varphi_p^T\right)^{4/3} \tag{3.49}$$

Then

$$u = \beta\alpha(y,\hat{x}), \qquad \beta \geq \frac{4}{3}$$

With (3.27), (3.28), and (3.48), we have also that:

$$\mathcal{L}V \leq -\frac{1}{2}\sum_{i=1}^{n}c_i z_i^4 - \rho|\tilde{x}|^4 < 0$$

Thus, according to theorem (3.4), we achieve not only global asymptotic stability in probability, but also inverse optimality.

That completes the proof.

Theorem (3.5) [Deng & Kristic, 99]:

The control law

$$u^* = -\beta M(y,\hat{x})z_n, \qquad \beta \geq \frac{4}{3} \tag{3.50}$$

guarantees that the equilibrium at the origin of the system defined in problem formulation in section (3.2) and equation (3.7) is globally asymptotically stable in probability and also minimizes the cost functional

$$J(u) = E\left\{\int_0^\infty \left[l(x,\tilde{x}) + \frac{27}{16\beta^2}M(y,\hat{x})^{-3}u^4\right]d\tau\right\} \tag{3.51}$$

For some positive definite radially unbounded function $l(x,\tilde{x})$ parameterized by β.

Proof:

Let

$$\gamma_2(r) = \frac{1}{4}r^4,\ R_2 = \left(\frac{4}{3}M\right)^{-(3/2)}$$

Applying Theorem 3.1, the result follows readily.

3.4 Algorithms and Examples:

Algorithm (3.1) A Robust Controller Stabilization in Probability:

Input: the dynamic control system described in problem formulation (3.2)

Output: robust control u, and the unknown design positive functions α_i, $i = 1,\dots,n$. and $V(x)$ is suitable stabilized *Lyapunov function.*

Step 1: consider problem formulation (3.2).

Step 2: check that Lipschitz condition for the following function: f, ϕ, ψ.

(See problem formulation (3.2)).

Step 3: if the Lipschitz condition is satisfied go to step 4 otherwise go to (step – stop).

Step 4: design suitable dynamic observer for the dynamic system of step 1:

$$d\hat{x}_1 = x_2 dt + k_1(y - \hat{x}_1)dt$$

$$d\hat{x}_2 = x_3 dt + k_2\tilde{x}_1 dt$$

$$\vdots$$

$$d\hat{x}_n = u dt + k_n\tilde{x}_n dt$$

Step 5: define the error vector :

$$e = x_i - \hat{x}_i$$

and hence

$$d\tilde{x}_i = x_{i+1}dt + f(\tilde{x}_i)dt + \phi_i^T(y)dw + \psi_i(\tilde{x}_i)dw - \hat{x}_{i+1}dt - k_i(y - \hat{x}_1)dt$$

$$d\tilde{x}_i = \tilde{x}_{i+1}dt - k_i\tilde{x}_1 + f(\tilde{x}_i)dt + \phi_i^T(y)dw + \psi_i(\tilde{x}_i)dw$$

$$d\tilde{x}_i = A_0\tilde{x}dt + f(\tilde{x}_i)dt + \phi_i^T(y)dw + \psi_i(\tilde{x}_i)dw$$

Where

$$A_0 = \begin{bmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -k_{n-1} & 0 & 0 & \cdots & 1 \\ -k_n & 0 & 0 & & 0 \end{bmatrix}$$

Step 6: find k_i , $i = 1, \dots, n$. so that A_0 is stable matrix. One can use many methods such as pole placement, …, etc.see [Ogata, 02].

Step 7: find the positive definite matrix P of the system:

$$A_0^T P + PA_0 = -I$$

where A_0 is defined in step 5, I is stands for n-identity matrix.

Step 8: suggest the Lyapunov function:

$$V(z,\tilde{x}) = \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2$$

Where $z_1 = y$ and $z_i = \hat{x}_i - \alpha_{\ i}(\bar{\hat{x}}_{i-1}, y)$, And P is the solution of step 7.

Step 9: select a suitable $\epsilon_1, \epsilon_2, \eta_i\ and\ \xi_i$ to satisfy :

$$b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}\left(\lambda_1 max(Q)\right)^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}\left(\lambda_2 max(Q)\right)^2 = \rho > 0$$

so that $\mathcal{L}V$ is negative

Step 10: choose

$$\alpha_1 = -c_1 y\frac{3}{2}y\psi_1\left(y)^T\psi_1(y) - \frac{3}{4}\sigma_1^{4/3}y - \frac{3}{4}\epsilon_1^{4/3}y - \frac{3}{4}\sum_{i=2}^{n}\xi_i^2\left(\psi_1(y)^T\psi_1(y)\right)^2 y - \frac{3bn\sqrt{n}}{\epsilon_2^2}|\psi(y)|^4 y + f(y)\right]$$

and

$$\alpha_i = -c_i z_i - k_1\tilde{x}_1 + \sum_{l=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}_l}(\hat{x}_{l+1} + k_1\tilde{x}_1) + \left(\frac{\partial\alpha_{i-1}}{\partial y}\right)\hat{x}_2 + \frac{1}{2}\left(\frac{\partial^2\alpha_{i-1}}{\partial y^2}\right)\varphi_1(y)^T\varphi_1(y) - \frac{3}{4}\sigma_i^{4/3}z_i - \frac{1}{4\sigma_{i-1}^4}z_i - \frac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3} z_i - \frac{3}{4\xi_i^2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i$$

and

$$u = -c_n z_n - k_n \tilde{x}_1 + \sum_{l=2}^{n-1} \frac{\partial \alpha_{i-1}}{\partial x_l}(x_{l+1} + k_1 \tilde{x}_1) + \left(\frac{\partial \alpha_{n-1}}{\partial y}\right) \hat{x}_2$$

$$+ \frac{1}{2}\left(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right) \varphi_1(y)^T \varphi_1(y) - \frac{1}{2} \sum_{p,q=1}^{n-1} \frac{\partial^2 \alpha_{n-1}}{\partial x_p \partial x_q} \varphi_p^T \varphi_q$$

$$- \frac{1}{4\sigma_{n-1}^4} z_n - \frac{3}{4} \eta_n^{4/3} \left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n + \frac{3}{4\xi_n^2} \left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4 z_n$$

Step 11: Then to be grantee that $\mathcal{L}V$ is negative definite and hence

$\mathcal{L}V \leq -\sum_{i=1}^{n} c_i z_i^4 - \rho|\tilde{x}|^4 < 0$, where c_i , $i = 1, \ldots, n$ are suitable

Step 12: the Lypunov function V of step 8 is constructed.

Step 13: stop.

<u>Algorithm (3.2) (inverse optimal stabilization):</u>

Input: the control u in algorithm (3.1) and the dynamic system of problem formulation (3.2).

Output: optimal control $u^* = \alpha(y, \hat{x}) = -M(y, \hat{x}) z_n$

Step 1: consider the steps of algorithum (3.1) from (step 1 – step 10).

Step 2: redesign the suggested controller u in algorithm (3.1), so that:

$$u = -M(y, \hat{x}) z_n$$

Step 3: $M(y, \hat{x})$ of step 2 can be derived to be:

$$M(y,\hat{x}) = c_n + \frac{3}{4}\sum_{l=2}^{n-1}\left(\epsilon_6 \frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\right)^{4/3} + \frac{3}{4}\left(\epsilon_6 \frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \frac{1}{4\epsilon_6^4}$$

$$+ \frac{3}{4}\sum_{k=1}^{n-1}\left(\epsilon_7 \sum_{l=k}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}\alpha_{lk}\right)^{4/3} + \frac{3}{8}\left(\epsilon_5 \frac{\partial^2\alpha_{n-1}}{\partial y^2}\right)^{4/3}$$

$$+ \frac{3}{4}\left(\epsilon_4 \sum_{l=2}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_l}k_l\right)^{4/3} + \frac{3}{4}\epsilon_3^{4/3} + \frac{1}{4\delta_{n-1}^4}$$

$$+ \frac{3}{4}\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \frac{3}{4\xi_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4}$$

$$+ \frac{3}{4}\left(\epsilon_8 \sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{n-1}}{\partial x_p \partial x_q}\varphi_p^T\right)^{4/3}$$

Then the controller of step 2 is designed.

Step 4: select $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_{7,}\epsilon_8$ and $\eta_i$are chosen to satisfy:

$$b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4\epsilon_4^4} - \frac{1}{4\epsilon_3^4}$$

$$- \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^2}(\lambda_1 \max(Q))^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^2}(\lambda_2 \max(Q))^2 - \frac{1}{4\epsilon_8^4}|\gamma|^4$$

$$= \rho > 0$$

Step 5: on using $= \frac{1}{4}y^4 + \frac{1}{4}\sum_{i=2}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P \tilde{x})^2$, $\mathcal{L}V$,controller of step 2

and α_1 , α_i of step 10 algorithm(3.1) to obtain $\mathcal{L}V < 0$.

Step 6: the controller $u^*$is the optimal stably control.

Step 7: stop.

Illustration (3.1):

Step1: Consider the system:

$$dx_1 = x_2 dt + \frac{1}{2} x_1^2 dw$$
$$dx_2 = u\, dt + \sin x_2 dw$$
$$y = x_1$$

and $|x_2| < 1$.

Step 2: Check Lipschiz condition:

Since $f(x) = x_2,\ |x_2| < 1$ we found that $k = 1$.

To check that ψ satisfy Lipschiz condition, first we must find the Jacobian matrix for ψ:

Where $\psi_1 = \frac{1}{2} x_1^2, \psi_2 = \sin x_2$

$$J = \frac{\partial\psi}{\partial x} = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_1} & \frac{\partial\psi_1}{\partial x_2} \\ \frac{\partial\psi_2}{\partial x_1} & \frac{\partial\psi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_1 & 0 \\ 0 & cosx_2 \end{bmatrix}$$

$$\left\|\frac{\partial\psi}{\partial x}\right\| = |x_1| + |cosx_2| < 1 + 1 = 2 = k$$

Then $\psi = (\psi_1, \psi_2)^T$ satisfy Lipshiz condition, and

$$\|\psi(x_1, x_2) - \psi(\check{x}_1, \check{x}_2)\| < \left\|\frac{\partial\psi}{\partial x}\right\| \|x - \check{x}\| = 2\|x - \check{x}\|$$

By equation (3.2).

Step 3: the observer system is:

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y - \hat{x}_1)$$
$$\dot{\hat{x}}_2 = u + k_2(y - \hat{x}_1)$$

Step 4: consider the error system:

$$d\tilde{x} = A_0\tilde{x}dt + x_2dt + \frac{1}{2}x_1^2dw + \sin x_2\, dw$$

Where $A_0 = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$

One can choose, for simplicity $k_1 = 4$, $k_2 = 6.25$ so that the matrix A_0 be a stable matrix.

Step 5: To find the positive definite matrix P ,the following equation should be solved:

$$A_0^TP + PA_0 = -I$$

$$\begin{bmatrix} -4 & -6.25 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} -4 & 1 \\ -6.25 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now, by using matlab program, P is found to be:

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.1452 & 0.0806 \\ 0.0806 & 1.2226 \end{bmatrix}$$

It's clear that P is symmetric matrix:

The eigen values of P are:

$$\lambda_1 = 0.1392,$$

$$\lambda_2 = 1.2286$$

Since λ_1, λ_2 are positive value, then P is positive definite matrix.

Step 6: To find the Lyapunov function:

$$V = \frac{1}{4}y^4 + \frac{1}{4}z_2^4 + \frac{b}{2}(\tilde{x}^TP\tilde{x})^2$$

$$V = \frac{1}{4}y^4 + \frac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\bar{x}}_1, y)\right)^4 + \frac{b}{2}\left([\tilde{x}_1 \quad \tilde{x}_2]^T \begin{bmatrix} 0.1452 & 0.0806 \\ 0.0806 & 1.2226 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}\right)^2$$

$$V = \frac{1}{4}y^4 + \frac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\bar{x}}_1, y)\right)^4 + \frac{b}{2}(0.1452\tilde{x}_1^2 + 1.2226\tilde{x}_2^2 + 0.1612\tilde{x}_1\tilde{x}_2)^2$$

Where b is a positive constant.

Step 7: to find ρ_1:

$$b\lambda - 6\sqrt{2}b\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{2}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{2}\frac{1}{\xi_i^4}\left(\lambda_1 max(P)\right)^2 - \frac{1}{4}\sum_{i=2}^{2}\frac{1}{\xi_i^4}\left(\lambda_2 max(P)\right)^2$$

$$= b\lambda - 3b2\sqrt{2}\epsilon_2^2|P|^4 - \frac{1}{4}\frac{1}{\eta_2^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4\xi_2^4}\left(\lambda_1 max(P)\right)^2$$

Where λ is the smallest eigenvalue of P, $\lambda = 0.1392$

$\lambda_1 max(P)$ is the largest eigenvalue of P. $\lambda_1 max(P) = 1.2286$

and if we choose $b = 0.1, \epsilon_1 = 0.01, \epsilon_2 = 50, \eta_2 = 0.1, \xi_2 = 0.8$

$|P| \equiv$ the determinant of the matrix P, $|P| \equiv 0.17102516$

So that $\rho_1 > 0$.

Step 8:

find $\alpha_1$and u :

$$\alpha_1 = -c_1 y - \frac{3}{8}y^3 - \frac{3}{4}\sigma_1^{4/3}y - \frac{3}{4}\epsilon_1{}^{4/3}y - \frac{3}{64}\xi_2^2 y^5 - \frac{6\sqrt{2}b}{16\epsilon_2^2}y^5$$

and

$$u = -c_2 z_2 - k_2 \tilde{x}_1 + \left(\frac{\partial \alpha_1}{\partial y}\right) \hat{x}_2 + \frac{1}{8}\left(\frac{\partial^2 \alpha_1}{\partial y^2}\right) y^4 - \frac{1}{4\sigma_1^4} z_2$$
$$- \frac{3}{4}\eta_2^{4/3}\left(\frac{\partial \alpha_1}{\partial y}\right)^{4/3} z_2 + \frac{3}{4\xi_2^2}\left(\frac{\partial \alpha_1}{\partial y}\right)^4 z_2$$

Hence, on using theorem (3.2), the solution is globally asymptotically stable in probability.

<u>Illustration (3.2):</u>

Step1: Consider the system:

$$dx_1 = x_2 dt + 2\sin(x_1) dw$$
$$dx_2 = u\, dt + 4\cos x_2\, dw$$

where $y = x_1$ and $|x_2| < 1$.

Step 2: check Lipchitz condition:

Since $f(x) = x_2$, $|x_2| < 1$, we found that $k = 1$.

To check that ψ satisfy Lipschiz condition, first we must find the Jacobian matrix for ψ:

Where $\psi_1 = 2\sin x_1$, $\psi_2 = 4\cos x_2$

$$J = \frac{\partial \psi}{\partial x} = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} \\ \frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2cosx_1 & 0 \\ 0 & -4sinx_2 \end{bmatrix}$$

$$\left\|\frac{\partial \psi}{\partial x}\right\| = |2cosx_1| + |-4sinx_2| < 2 + 4 = 6 = k$$

Then $\psi = (\psi_1, \psi_2)^T$ satisfy Lipshiz condition, and

$$\|\psi(x_1, x_2) - \psi(\check{x}_1, \check{x}_2)\| < \left\|\frac{\partial \psi}{\partial x}\right\| \|x - \check{x}\| = 6\|x - \check{x}\|$$

By equation (3.2).

Step 3: the observer system is:

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y - \hat{x}_1)$$
$$\dot{\hat{x}}_2 = u + k_2(y - \hat{x}_1)$$

Step 4: consider the error system:

$$d\tilde{x} = A_0\tilde{x}dt + x_2dt + 2\sin(y)\,dw + 4\cos x_2\,dw$$

Where $A_0 = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$

One can choose, for simplicity $k_1 = 4$, $k_2 = 6.25$ so that the matrix A_0 be a stable matrix.

Step 5: To find the positive definite matrix P ,the following equation should be solved:

$$A_0^T P + PA_0 = -I$$

$$\begin{bmatrix} -3 & -4.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} -3 & 1 \\ -4.5 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now, by using matlab program, we found P as:

$$P = \begin{bmatrix} 0.9167 & -0.5 \\ -0.5 & 0.537 \end{bmatrix}$$

And λ_1, λ_2 of P are:

$$\lambda_1 = 0.1920, \lambda_2 = 1.2617$$

It's clear that P is symmetric and positive definite matrix.

Step 6: To find Lypunov function:

$$V = \frac{1}{4}y^4 + \frac{1}{4}z_2^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2$$

$$V = \frac{1}{4}y^4 + \frac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\hat{x}}_1, y)\right)^4 + \frac{b}{2}\left([\tilde{x}_1 \quad \tilde{x}_2]^T \begin{bmatrix} 0.9167 & -0.5 \\ -0.5 & 0.537 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}\right)^2$$

$$V = \frac{1}{4}y^4 + \frac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\hat{x}}_1, y)\right)^4 + \frac{b}{2}(0.9167\tilde{x}_1^2 - \tilde{x}_1\tilde{x}_2 + 0.537\tilde{x}_2^2)^2$$

Where b is a positive constant.

Step 7:

To find ρ_1:

$$b\lambda - 6b\sqrt{2}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{2}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{2}\frac{1}{\xi_i^4}\left(\lambda_1 max(P)\right)^2 - \frac{1}{4}\sum_{i=2}^{n}\frac{1}{\xi_i^4}\left(\lambda_2 max(P)\right)^2 = b\lambda - 3b2\sqrt{2}\epsilon_2^2|P|^4 - \frac{1}{4}\frac{1}{\eta_2^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4\xi_2^4}\left(\lambda_1 max(P)\right)^2$$

Where λ is the smallest eigenvalue of P, $\lambda = 0.1920$

$\lambda_1 max(P)$ is the largest eigenvalue of P. $\lambda_1 max(P) = 1.2617$

and if we choose $b = 0.1, \epsilon_1 = 0.01, \epsilon_2 = 50, \eta_2 = 0.1, \xi_2 = 0.8$

$|P| \equiv$ the determinate of the matrix P, $|P| \equiv 0.2422679$

So that $\rho_1 > 0$.

Step 8: find $\alpha_1$and u :

$$\alpha_1 = -c_1 y - 6ysiny - \frac{3}{4}\sigma_1^{4/3}y - \frac{3}{4}{\epsilon_1}^{4/3}y - 12\xi_2^2 y(siny)^4 - \frac{96\sqrt{2}b}{\epsilon_2^2}y|siny|^4$$

$$u = -c_2 z_2 - k_2 \tilde{x}_1 + \left(\frac{\partial \alpha_1}{\partial y}\right)\hat{x}_2 + 2\left(\frac{\partial^2 \alpha_1}{\partial y^2}\right) y^2 (siny)^2 - \frac{1}{4\sigma_1^4} z_2$$
$$- \frac{3}{4}\eta_2^{4/3}\left(\frac{\partial \alpha_1}{\partial y}\right)^{4/3} z_2 - \frac{3}{4\xi_2^2}\left(\frac{\partial \alpha_1}{\partial y}\right)^4 z_2$$

Hence, on using theorem (3.2), the solution is globally asymptotically stable in probability.

Illustration (3.3):

Step1: Consider the system:

$$dx_1 = x_2 dt + \sin(x_1) dw$$
$$dx_2 = x_3\, dt + \cos(x_1 x_2)\, dw$$
$$dx_3 = u\, dt + \sin x_2 x_3\, dw$$
$$y = x_1$$

With $|x_2| < 1, |x_3| < 1$.

Step 2: check Lipschiz condition:

1. First, check $f(x_2, x_3) = (f_1, f_2) = (x_2, x_3)^T$

$$J_f = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\|\frac{\partial f}{\partial x}\right\| < 1 + 1 = 2 = k$$

Then $f = (x_2, x_3)^T$satisfy Lipshiz condition, and

$$\|f(x_1, x_2) - f(\check{x}_1, \check{x}_2)\| < \left\|\frac{\partial f}{\partial x}\right\| \|x - \check{x}\| = 2\|x - \check{x}\|$$

By equation (3.2).

2. Check $\psi(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ satisfied Lipschiz condition:

$$J = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_1} & \frac{\partial\psi_1}{\partial x_2} & \frac{\partial\psi_1}{\partial x_3} \\ \frac{\partial\psi_2}{\partial x_1} & \frac{\partial\psi_2}{\partial x_2} & \frac{\partial\psi_2}{\partial x_3} \\ \frac{\partial\psi_3}{\partial x_1} & \frac{\partial\psi_3}{\partial x_2} & \frac{\partial\psi_3}{\partial x_3} \end{bmatrix} =$$

$$\begin{bmatrix} \cos x_1 & 0 & 0 \\ -x_2 \sin x_1 x_2 & -x_1 \sin x_1 x_2 & 0 \\ x_3 \cos x_1 x_3 & 0 & x_1 \cos x_1 x_3 \end{bmatrix}$$

$$\left\|\frac{\partial\psi}{\partial x}\right\| = |\cos x_1| + |-x_2 \sin x_1 x_2| + |-x_1 \sin x_1 x_2| + |x_3 \cos x_1 x_3| + |x_1 \cos x_1 x_3|$$

$$\leq |\cos x_1| + |x_2||\sin x_1 x_2| + |x_1||\sin x_1 x_2| + |x_3||\cos x_1 x_3| + |x_1||\cos x_1 x_3|$$

$$\leq 1 + 1 + 1 + 1 + 1$$

$$\leq 5 = K$$

Then $\psi = (\psi_1, \psi_2, \psi_3)^T$ satisfy Lipshiz condition, and

$$\|\psi(x_1, x_2, x_3) - \psi(\check{x}_1, \check{x}_2, \check{x}_3)\| < \left\|\frac{\partial\psi}{\partial x}\right\| \|x - \check{x}\| = 5\|x - \check{x}\|$$

By equation (3.2).

Step 3: the observer system is:

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y - \hat{x}_1)$$

$$\dot{\hat{x}}_2 = \hat{x}_3 + k_2(y - \hat{x}_1)$$

$$\dot{\hat{x}}_3 = u + k_3(y - \hat{x}_1)$$

Step 4: consider the error system:

$$d\tilde{x} = A_0\tilde{x}dt + (x_2 + x_3)dt + \sin(y)dw + \cos(x_1 x_2)\,dw + \sin x_1 x_3\,dw$$

Where $A_0 = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix}$

If we chosen $k_1 = 3, k_2 = 4.5, k_3 = 6.25$ so that the matrix A_0 be stable.

Step 5: To find P, we must solve:

$$A_0^T P + P A_0 = -I$$

So that P obtains as:

$$P = \begin{bmatrix} 5.6940 & -0.5000 & -2.2931 \\ -0.5000 & 2.2931 & -0.5000 \\ -2.2931 & -0.5000 & 1.3807 \end{bmatrix}$$

Where P is positive definite matrix because the eigen value of P are:

$$\lambda_1 = 0.1836, \lambda_2 = 2.4833, \lambda_3 = 6.7009$$

Step 6: To find Lypunov function:

$$V = \frac{1}{4}y^4 + \frac{1}{4}z_2^4 + \frac{1}{4}z_3^4 + \frac{b}{2}(\tilde{x}^T P \tilde{x})^2$$

$$V = \frac{1}{4}y^4 + \frac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\hat{x}}_1, y)\right)^4 + \frac{1}{4}\left(\hat{x}_3 - \alpha_2(\bar{\hat{x}}_2, y)\right)^4$$
$$+ \frac{b}{2}\left([\tilde{x}_1 \quad \tilde{x}_2 \quad \tilde{x}_3]^T \begin{bmatrix} 5.6940 & -0.5000 & -2.2931 \\ -0.5000 & 2.2931 & -0.5000 \\ -2.2931 & -0.5000 & 1.3807 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \right)^2$$

$$V = \frac{1}{4}y^4 + \frac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\hat{x}}_1, y)\right)^4 + \frac{1}{4}\left(\hat{x}_3 - \alpha_2(\bar{\hat{x}}_2, y)\right)^4$$
$$+ \frac{b}{2}(5.6940\tilde{x}_1^2 + 2.2931\tilde{x}_2^2 + 1.3807\tilde{x}_3^2 - \tilde{x}_1\tilde{x}_2 - 4.5862\tilde{x}_1\tilde{x}_3$$
$$- \tilde{x}_2\tilde{x}_3)^2$$

Where b is a positive constant.

Step 7: to find ρ_1

$$b\lambda - 9b\sqrt{3}\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{3}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{3}\frac{1}{\xi_i^4}\left(\lambda_1 max(P)\right)^2$$
$$- \frac{1}{4}\sum_{i=2}^{3}\frac{1}{\xi_i^4}\left(\lambda_2 max(P)\right)^2$$

$$= b\lambda - 3bn\sqrt{n}\epsilon_2^2|P|^4 - \frac{1}{4}\frac{1}{\eta_2^4} - \frac{1}{4}\frac{1}{\eta_3^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4\xi_2^4}\left(\lambda_1 max(P)\right)^2$$
$$- \frac{1}{2\xi_3^4}\left(\lambda max(P)\right)^2$$

Where λ is the smallest eigenvalue of P, $\lambda = 0.1836$

$\lambda_1 max(P)$ is the largest eigenvalue of P. $\lambda_1 max(P) = 6.7009$

and if we choose $b = 0.1, \epsilon_1 = 0.01, \epsilon_2 = 50, \eta_2 = 0.1,\ \eta_2 = 0.09, \xi_2 = 0.8, \xi_2 = 0.5$.

$|P| \equiv$ the determinate of the matrix P, $|P| \equiv 3.0546$

So that $\rho_1 > 0$.

Step 8: find $\alpha_1,\ \alpha_2$ and u:

$$\alpha_1 = -c_1 y - \frac{3}{2}[(siny)^2 + (cosx_1x_2)^2 + (sinx_2x_3)^2] - \frac{3}{4}\sigma_1^{4/3}y - \frac{3}{4}{\epsilon_1}^{4/3}y$$
$$- \frac{3}{4}\xi_2^2\xi_3^2[(siny)^2 + (cosx_1x_2)^2 + (sinx_2x_3)^2]$$
$$- \frac{9\sqrt{3}b}{\epsilon_2^2}y|[siny\ cosx_1x_2\ sinx_2x_3]|^4$$

$$\alpha_2 = -c_2z_2 - k_2\tilde{x}_1 + \frac{\partial\alpha_1}{\partial\hat{x}_2}(\hat{x}_3 + k_2\tilde{x}_1) + \left(\frac{\partial\alpha_1}{\partial y}\right)\hat{x}_2$$
$$+ \frac{1}{2}y^6\left(\frac{\partial^2\alpha_1}{\partial y^2}\right)[(siny)^2 + (cosx_1x_2)^2 + (sinx_2x_3)^2] - \frac{3}{4}\sigma_2^{4/3}z_2$$
$$- \frac{1}{4\sigma_1^4}z_2 - \frac{3}{4}\eta_2^{4/3}\left(\frac{\partial\alpha_1}{\partial y}\right)^{4/3}z_2 - \frac{3}{4\xi_2^2}\left(\frac{\partial\alpha_1}{\partial y}\right)^4 z_2$$

$$u = -c_3 z_3 - k_3\tilde{x}_1 + \frac{\partial\alpha_2}{\partial x_2}(\hat{x}_3 + k_3\tilde{x}_1) + \left(\frac{\partial\alpha_2}{\partial y}\right)\hat{x}_2$$
$$+\frac{1}{2}y^6\left(\frac{\partial^2\alpha_2}{\partial y^2}\right)[(siny)^2+(cosx_1x_2)^2+(sinx_2x_3)^2]$$
$$-\frac{1}{2}\sum_{p,q=1}^{2}\frac{\partial^2\alpha_2}{\partial x_p\partial x_q}[(ysiny)^2 + (ycosx_1x_2)^2+(ysinx_2x_3)^2]$$
$$-\frac{1}{4\sigma_2^4}z_3 - \frac{3}{4}\eta_3^{4/3}\left(\frac{\partial\alpha_2}{\partial y}\right)^{4/3} z_3 + \frac{3}{4\xi_3^2}\left(\frac{\partial\alpha_2}{\partial y}\right)^4 z_3$$

Hence, on using theorem (3.2), the solution is globally asymptotically stable in probability.

<u>Illustration (3.4):</u>

Step 1: consider the system:

$$dx_1 = x_2 dt + x_1^3 dw$$
$$dx_2 = x_3\, dt + \cos(x_1)\sin(x_2)\, dw$$
$$dx_3 = x_4\, dt + sin^2(x_2x_3)dw$$
$$dx_4 = u\, dt + \cos(x_3x_4)dw$$
$$y = x_1$$

Step 2: check Lipschiz condition:

1. First, check $f(x_2, x_3, x_4) = (f_1, f_2, f_3) = (x_2, x_3, x_4)^T$

$$J_f = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\|\frac{\partial f}{\partial x}\right\| = |0| + |1| + |0| + |0| + |0| + |0| + |1| + |0| + |0| + |0| + |0|$$
$$+ |1| + |0| + |0| + |0| + |0| < 1 + 1 + 1 = 3 = k$$

Then $f = (x_2, x_3, x_4)^T$satisfy Lipshiz condition, and

$$\|f(x_1, x_2, x_3) - f(\breve{x}_1, \breve{x}_2, \breve{x}_3)\| < \left\|\frac{\partial f}{\partial x}\right\| \|x - \breve{x}\| = 3\|x - \breve{x}\|$$

By equation (3.2).

2. Check $\psi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ satisfied Lipschiz condition:

   we must find jacobian J :

$$J = \begin{bmatrix} \frac{\partial\psi_1}{\partial x_1} & \frac{\partial\psi_1}{\partial x_2} & \frac{\partial\psi_1}{\partial x_3} & \frac{\partial\psi_1}{\partial x_4} \\ \frac{\partial\psi_2}{\partial x_1} & \frac{\partial\psi_2}{\partial x_2} & \frac{\partial\psi_2}{\partial x_3} & \frac{\partial\psi_2}{\partial x_4} \\ \frac{\partial\psi_3}{\partial x_1} & \frac{\partial\psi_3}{\partial x_2} & \frac{\partial\psi_3}{\partial x_3} & \frac{\partial\psi_3}{\partial x_4} \\ \frac{\partial\psi_4}{\partial x_1} & \frac{\partial\psi_4}{\partial x_2} & \frac{\partial\psi_4}{\partial x_3} & \frac{\partial\psi_4}{\partial x_4} \end{bmatrix}$$

Where

$$\frac{\partial\psi_1}{\partial x_1} = 3x_1^2, \frac{\partial\psi_1}{\partial x_2} = 0, \frac{\partial\psi_1}{\partial x_3} = 0, \frac{\partial\psi_1}{\partial x_4} = 0,$$

$$\frac{\partial\psi_2}{\partial x_1} = -\sin(x_1)\sin(x_2), \qquad \frac{\partial\psi_2}{\partial x_2} = \cos(x_1)\cos(x_2), \frac{\partial\psi_2}{\partial x_3} = 0, \frac{\partial\psi_2}{\partial x_4} = 0,$$

$$\frac{\partial\psi_3}{\partial x_1} = 0, \frac{\partial\psi_3}{\partial x_2} = 2x_3\sin(x_2x_3)\cos(x_2x_3),$$

$$\frac{\partial\psi_3}{\partial x_3} = 2x_2\sin(x_2x_3)\cos(x_2x_3), \frac{\partial\psi_3}{\partial x_4} = 0,$$

$$\frac{\partial\psi_4}{\partial x_1} = 0, \frac{\partial\psi_4}{\partial x_2} = 0, \frac{\partial\psi_4}{\partial x_3} = -x_4\sin(x_3x_4), \frac{\partial\psi_4}{\partial x_4} = -x_3\sin(x_3x_4)$$

$$\left\|\frac{\partial\psi}{\partial x}\right\| = |3x_1^2| + |0| + |0| + |0| + |-\sin(x_1)\sin(x_2)| + |\cos(x_1)\cos(x_2)| + |0| + |0| + |0| + |2x_3\sin(x_2x_3)\cos(x_2x_3)| + |2x_2\sin(x_2x_3)\cos(x_2x_3)| + |0| + |0| + |0| + |-x_4\sin(x_3x_4)| + |-x_3\sin(x_3x_4)|$$

$$\left\|\frac{\partial\psi}{\partial x}\right\| \leq 3|x_1^2| + |\sin(x_1)||\sin(x_2)| + |\cos(x_1)| + |\cos(x_2)| + 2|x_3||\sin(x_2x_3)||\cos(x_2x_3)| + 2|x_2||\sin(x_2x_3)||\cos(x_2x_3)| + |x_4||\sin(x_3x_4)| + |x_3||\sin(x_3x_4)|$$

$$\left\|\frac{\partial\psi}{\partial x}\right\| \leq 3 + 1 + 1 + 1 + 2 + 2 + 1 + 1 = 12 = K$$

Then $\psi = (\psi_1, \psi_2, \psi_3, \psi_3)^T$ satisfy Lipshiz condition, and

$$\|\psi(x_1, x_2, x_3, x_4) - \psi(\check{x}_1, \check{x}_2, \check{x}_3, \check{x}_4)\| < \left\|\frac{\partial\psi}{\partial x}\right\| \|x - \check{x}\| \leq 12\|x - \check{x}\|$$

By equation (3.2).

Step 3: the observer system is:

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y - \hat{x}_1)$$
$$\dot{\hat{x}}_2 = \hat{x}_3 + k_2(y - \hat{x}_1)$$
$$\dot{\hat{x}}_3 = \hat{x}_4 + k_3(y - \hat{x}_1)$$
$$\dot{\hat{x}}_4 = u + k_4(y - \hat{x}_1)$$

Step 4: consider the error system:

$$d\tilde{x} = A_0\tilde{x}dt + (x_2 + x_3 + x_4)dt + (x_1^3 + \cos(x_1)\sin(x_2) + sin^2(x_2x_3))dw$$

Where

$$A_0 = \begin{bmatrix} -k_1 & 1 & 0 & 0 \\ -k_2 & 0 & 1 & 0 \\ -k_3 & 0 & 0 & 1 \\ -k_4 & 0 & 0 & 0 \end{bmatrix}$$

If we choose $k_1 = 2, k_2 = 6.25, k_3 = 7.5, k_4 = 9$ so that A_0 be stable matrix. Then the matrix P is written as:

$$P = \begin{bmatrix} 481.9063 & -0.5000 & -128.6250 & 0.5000 \\ -0.5000 & 128.6250 & -0.5000 & -35.2500 \\ -128.6250 & -0.5000 & 35.2500 & -0.5000 \\ 0.5000 & -35.2500 & -0.5000 & 10.4931 \end{bmatrix}$$

Where the eigen values of P are given as:

$$\lambda_1 = 0.3093, \lambda_2 = 1.3203, \lambda_3 = 138.3450, \lambda_4 = 516.2997$$

It's clear that P is positive definite matrix.

Step 6: To find Lypunov function:

$$V = \frac{1}{4}y^4 + \frac{1}{4}z_2^4 + \frac{1}{4}z_3^4 + \frac{1}{4}z_4^4 + \frac{b}{2}(\tilde{x}^T P \tilde{x})^2$$

$$V = \tfrac{1}{4}y^4 + \tfrac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\tilde{x}}_1, y)\right)^4 + \tfrac{1}{4}\left(\hat{x}_3 - \alpha_2(\bar{\tilde{x}}_2, y)\right)^4 + \tfrac{1}{4}\left(\hat{x}_4 - \alpha_3(\bar{\tilde{x}}_3, y)\right)^4$$

$$+ \frac{b}{2}\left([\tilde{x}_1 \quad \tilde{x}_2 \quad \tilde{x}_3 \quad \tilde{x}_4]^T \begin{bmatrix} 481.9063 & -0.5000 & -128.6250 & 0.5000 \\ -0.5000 & 128.6250 & -0.5000 & -35.2500 \\ -128.6250 & -0.5000 & 35.2500 & -0.5000 \\ 0.5000 & -35.2500 & -0.5000 & 10.4931 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_3 \end{bmatrix} \right)^2$$

$$\begin{aligned} V = {} & \frac{1}{4}y^4 + \frac{1}{4}\left(\hat{x}_2 - \alpha_1(\bar{\tilde{x}}_1, y)\right)^4 + \frac{1}{4}\left(\hat{x}_3 - \alpha_2(\bar{\tilde{x}}_2, y)\right)^4 \\ & + \frac{1}{4}\left(\hat{x}_4 - \alpha_3(\bar{\tilde{x}}_3, y)\right)^4 \\ & + \frac{b}{2}(481.9063\tilde{x}_1^2 + 128.625\tilde{x}_2^2 + 35.25\tilde{x}_3^2 + 10.4931\tilde{x}_4^2 \\ & - \tilde{x}_1\tilde{x}_2 - 257.25\tilde{x}_1\tilde{x}_3 + \tilde{x}_1\tilde{x}_4 - \tilde{x}_2\tilde{x}_3 - 70.5\tilde{x}_2\tilde{x}_4 - \tilde{x}_3\tilde{x}_4)^2 \end{aligned}$$

Where b is a positive constant.

Step 7:

To find ρ_1:

$$= b\lambda - 24b\epsilon_2^2|P|^4 - \frac{1}{4}\sum_{i=2}^{4}\frac{1}{\eta_i^4} - \frac{1}{4\epsilon_1^4} - \frac{1}{4}\sum_{i=2}^{4}\frac{1}{\xi_i^4}\left(\lambda_1 max(P)\right)^2 - \frac{1}{4}\sum_{i=2}^{4}\frac{1}{\xi_i^4}\left(\lambda_2 max(P)\right)^2$$

$$= b\lambda - 24b\epsilon_2^2|P|^4 - \frac{1}{4}\left[\frac{1}{\eta_2^4} + \frac{1}{\eta_3^4} + \frac{1}{\eta_4^4}\right] - \frac{1}{4\epsilon_1^4} - \frac{1}{2}\left[\frac{1}{\xi_2^4} + \frac{1}{\xi_3^4} + \frac{1}{\xi_4^4}\right]\left(\lambda max(P)\right)^2$$

Where λ is the smallest eigenvalue of P, $\lambda = 0.3093$

$\lambda_1 max(P)$ is the largest eigenvalue of P. $\lambda_1 max(P) = 516.2997$

and if we choose $b = 0.1, \epsilon_1 = 0.01, \epsilon_2 = 50, \eta_2 = 0.1,\ \eta_3 = 0.09,\ \eta_3 = 0.07, \xi_2 = 0.8, \xi_3 = 0.5,\ \xi_4 = 0.4$.

$|P| \equiv$ the determinate of the matrix P, $\quad |P| \equiv 2.9169 \times 10^4$

So that $\rho_1 > 0$.

Step 8:

$$\alpha_1 = -c_1 y - \frac{3}{2}y[(y^3)^2 + (cosx_1 sinx_2)^2 + (sin^2x_2x_3)^2 + (cosx_3x_4)^2] - \frac{3}{4}\sigma_1^{4/3}y - \frac{3}{4}\epsilon_1^{4/3}y - \frac{3}{4}y^{13}\sum_{i=2}^{4}\xi_i^2\, y((y^3)^2 + (cosx_1 sinx_2)^2 + (sin^2x_2x_3)^2 + (cosx_3x_4)^2)^2 - \frac{24b}{\epsilon_2^2}y|y^3 + cosx_1 sinx_2 + sin^2x_2x_3 + cosx_3x_4|^4$$

$$\alpha_2 = -c_2 z_2 - k_2 \tilde{x}_1 + \left(\frac{\partial \alpha_1}{\partial y}\right) \hat{x}_2$$
$$+ \frac{1}{2}\left(\frac{\partial^2 \alpha_1}{\partial y^2}\right)[(y^4)^2$$
$$+ (y cos x_1 sin x_2)^2 + (y sin^2 x_2 x_3)^2 + (y cos x_3 x_4)^2] - \frac{3}{4}\sigma_2^{4/3} z_2$$
$$- \frac{1}{4\sigma_1^4} z_2 - \frac{3}{4}\eta_2^{4/3}\left(\frac{\partial \alpha_1}{\partial y}\right)^{4/3} z_2 - \frac{3}{4\xi_2^2}\left(\frac{\partial \alpha_1}{\partial y}\right)^4 z_2$$

$$\alpha_3 = -c_3 z_3 - k_3 \tilde{x}_1 + \frac{\partial \alpha_2}{\partial \hat{x}_2}(\hat{x}_3 + k_3 \tilde{x}_1) + \left(\frac{\partial \alpha_2}{\partial y}\right) \hat{x}_2$$
$$+ \frac{1}{2}\left(\frac{\partial^2 \alpha_2}{\partial y^2}\right)[(y^4)^2$$
$$+ (y cos x_1 sin x_2)^2 + (y sin^2 x_2 x_3)^2 + (y cos x_3 x_4)^2] - \frac{3}{4}\sigma_3^{4/3} z_3$$
$$- \frac{1}{4\sigma_2^4} z_3 - \frac{3}{4}\eta_3^{4/3}\left(\frac{\partial \alpha_2}{\partial y}\right)^{4/3} z_3 - \frac{3}{4\xi_3^2}\left(\frac{\partial \alpha_2}{\partial y}\right)^4 z_3$$

And

$$u = -c_4 z_4 - k_4 \tilde{x}_1 + \sum_{l=2}^{3} \frac{\partial \alpha_3}{\partial \hat{x}_l} (\hat{x}_{l+1} + k_4 \tilde{x}_1) + \left(\frac{\partial \alpha_3}{\partial y}\right) \hat{x}_2$$
$$+ \frac{1}{2}\left(\frac{\partial^2 \alpha_3}{\partial y^2}\right)[(y^4)^2$$
$$+ (y cos x_1 sin x_2)^2 + (y sin^2 x_2 x_3)^2 + (y cos x_3 x_4)^2]$$
$$- \frac{1}{2}\sum_{p,q=1}^{3} \frac{\partial^2 \alpha_3}{\partial x_p \partial x_q}[(y^4)^2$$
$$+ (y cos x_1 sin x_2)^2 + (y sin^2 x_2 x_3)^2 + (y cos x_3 x_4)^2] - \frac{1}{4\sigma_3^4} z_4$$
$$- \frac{3}{4}\eta_4^{4/3}\left(\frac{\partial \alpha_3}{\partial y}\right)^{4/3} z_4 + \frac{3}{4\xi_4^2}\left(\frac{\partial \alpha_3}{\partial y}\right)^4 z_4$$

Hence, on using theorem (3.2), the solution is globally asymptotically stable in probability.

Conclusions

1. From the present study sufficient conditions for finding robust and inverse optimal stabilizing controller in probability for some stochastic dynamic system characterized by nonlinear functions which are Lipschitz in nature and presented by Brownian motion noise are discussed and presented.
2. Some computational algorithms to justify the work, and make the computational regiments for designer based on the present work is easily are presented.
3. The numerical solution for stochastic differential equation is not an easy task, so the graph and figures are omitted and left to the future.
4. The Ito – stochastic integral play a central role in modern probability theory and its applications to stochastic differential equation concerned by Brownian motion. So the difficulties for solution stochastic differential equations are resulted from this type of integrals and its computation.

   The numerical difficulties of stochastic differential equations are also resulting from the difficulties of evaluating Ito – stochastic integrals due to present Brownian motion (time – random variable) noise.

5. The present work is not an easy task and need some good backgrounds of probability theory, stochastic process, the dynamic system in the present of stochastic noise as well as stochastic

differential control system. And due to all necessary background, the task becomes difficult for many, so the work of this field becomes an interesting for us.

Future works

The future works that may be considered are the following:

1. Due to the complexity of the analytic and numerical solution of SDE, some numerical methods of solution of stochastic dynamic system (presented in the work) may be considered and developed.
2. The exponential stability in probability using Lyapunov function approach for some stochastic differential equation may also be taken, as well as the statistical properties of the solution of SDE may also been studied.
3. Full order observability or reduced order of some stochastic dynamic system and its stabilization using Lyapunov function approach may be considered.

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Appendix A

Theorem (A.1):

If $f \in L_w^2[\alpha,\beta]$ and f is continuous, then, for any sequence Π_n of a partitions $\alpha = t_{n,0} < t_{n,1} < \cdots < t_{n,m_n} = \beta$ of $[\alpha,\beta]$ with mesh $|\Pi_n| \to 0$,

$$\sum_{k=0}^{m_n-1} f(t_{n,k})\left[w(t_{n,k+1}) - w(t_{n,k})\right] \xrightarrow{P} \int_{\alpha}^{\beta} f(t)dw(t) \quad as\ n \to \infty.$$

Lemma (A.2):

Let $f \in M_\omega^2[0,T]$ and let ζ_1, ζ_2 be stopping times, $0 \le \zeta_1 \le \zeta_2 \le T$. Then

$$E\int_{\zeta_1}^{\zeta_2} f(t)dw(t) = 0,$$

$$E\left[\int_{\zeta_1}^{\zeta_2} f(t)dw(t)\right]^2 = E\int_{\zeta_1}^{\zeta_2} f^2(t)dt.$$

Theorem (A.3):

Let $f \in M_\omega^2[0,T]$ and let ζ_1, ζ_2 be stopping times (with respect to $\mathcal{F}_t$), $0 \le \zeta_1 \le \zeta_2 \le T$. Then

$$E\left\{\int_{\zeta_1}^{\zeta_2} f(t)dw(t)\middle|\mathcal{F}_{\zeta_1}\right\} = 0,$$

$$E\left\{\left|\int_{\zeta_1}^{\zeta_2} f(t)dw(t)\right|^2 \middle|\mathcal{F}_{\zeta_1}\right\} = E\left\{\int_{\zeta_1}^{\zeta_2} f^2(t)dt \middle|\mathcal{F}_{\zeta_1}\right\}. \qquad (a.1)$$

Appendix B

The proof of theorem (2.4)

Proof [Friedman, 75], [Øksendal, 98]:

The proof will be divided into 6 steps,

Step 1:

For any integer $m \geq 2$,

$$d\big(w(t)\big)^{m} = m\big(w(t)\big)^{m-1} + \frac{1}{2}m(m-1)\big(w(t)\big)^{m-2}dt. \qquad (b.1)$$

Indeed, this following by induction, using theorem (2.3).

By linearity of the stochastic differential we then get

$$dQ\big(w(t)\big) = \acute{Q}\big(w(t)\big)dw(t) + \frac{1}{2}\acute{\acute{Q}}\big(w(t)\big)dt \qquad (b.2)$$

For any polynomial Q.

Step 2:

Let $G(x,t) = Q(x)g(t)$ where $Q(x)$ is a polynomial and $g(t)$ is continuously differentiable for $t \geq 0$. By theorem (2.3) and (b.2),

$$\begin{aligned} dG(w(t),t) &= f\big(w(t)\big)dg(t) + g(t)df\big(w(t)\big) \\ &= \left[f\big(w(t)\big)\acute{g}(t) + \frac{1}{2}g(t)\acute{\acute{f}}(w(t))\right]dt \\ &+ g(t)\acute{f}\big(w(t)\big)dw(t), \end{aligned}$$

i.e., for any $0 \leq t_1 < t_2 \leq T$,

$$G(w(t_2), t_2) - G(w(t_1), t_1)$$

$$= \int_{t_1}^{t_2} \left[G_t(w(t), t) + \frac{1}{2} G_{xx}(w(t), t)\right] dt$$

$$+ \int_{t_1}^{t_2} G_x(w(t), t) dw(t). \qquad (b.3)$$

Step 3:

Formula (b.3) remains valid if

$$G(x, t) = \sum_{i=1}^{m} f_i(x) g_i(t)$$

Where $f_i(x)$ are polynomials and $g_i(t)$ are continuously differentiable. Now let $G_n(x, t)$ be polynomials in x and t such that

$$G_{\boldsymbol{n}}(x, t) \to f(x, t),$$

$$\frac{\partial}{\partial x} G_{\boldsymbol{n}}(x, t) \to f_x(x, t), \qquad \frac{\partial^2}{\partial x^2} G_{\boldsymbol{n}}(x, t) \to f_{xx}(x, t),$$

$$\frac{\partial}{\partial t} G_{\boldsymbol{n}}(x, t) \to f_t(x, t)$$

Uniformly on compact subsets of $(x, t) \in R^1 \times [0, \infty)$, we have

$$G_n(w(t_2), t_2) - G_n(w(t_1), t_1)$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial}{\partial t} G_n(w(t), t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} G_n(w(t), t)\right] dt$$

$$+ \int_{t_1}^{t_2} \frac{\partial}{\partial x} G_n(w(t), t) dw(t). \qquad (b.4)$$

It is clear that

$$\int_{t_1}^{t_2}\left[\frac{\partial}{\partial t}G_n(w(t),t)+\frac{1}{2}\frac{\partial^2}{\partial x^2}G_n(w(t),t)\right]dt$$

$$\rightarrow \int_{t_1}^{t_2}\left[f_t(w(t),t)-\frac{1}{2}f_{xx}(w(t),t)\right]dt \qquad a.s.,$$

$$\int_{t_1}^{t_2}\left|\frac{\partial}{\partial x}G_n(w(t),t)-f_x(w(t),t)\right|^2 dt \rightarrow 0 \quad a.s.$$

Hence, taking $n \rightarrow \infty$ in (b.4), we get the relation

$$f(w(t_2),t_2)-f(w(t_1),t_1)$$

$$=\int_{t_1}^{t_2}\left[f_t(w(t),t)-\frac{1}{2}f_{xx}(w(t),t)\right]dt$$

$$+\int_{t_1}^{t_2} f_x(w(t),t)dw(t) \qquad (b.5)$$

***Step 4*:**

Formula (b.5**)** extends to the process

$$\Phi(w(t),t)=f(\xi_1+a_1t+b_1w(t),t)$$

Where ξ_1, a_1, b_1 are random variables measurable with respect to $\mathcal{F}_{t_1}$, i.e.,

$$\Phi(w(t_2),t_2)-\Phi(w(t_1),t_1)$$

$$=\int_{t_2}^{t_2}[f_t(\tilde{\xi}(t),t)+f_x(\tilde{\xi}(t),t)a_1+\frac{1}{2}f_{xx}(\tilde{\xi}(t),t)b_1^2]dt$$

$$+\int_{t_1}^{t_2} f_x(\tilde{\xi}(t),t)b_1dw(t) \qquad (b.6)$$

Where $\tilde{\xi}(t) = \xi_1 + a_1 t + b_1 w(t)$.

The proof of (b.6) is a repetition of the proof of (b.5) with obvious changes resulting from the formula

$$d\left(\tilde{\xi}(t)\right)^m = m\left(\tilde{\xi}(t)\right)^{m-1}[a_1 dt + b_1 dw(t)] + \frac{1}{2}m(m-1)\left(\tilde{\xi}(t)\right)^{m-2} b_1^2 dt, \qquad (b.7)$$

Which replaces by (b.1).

Step 5:

if $a(t), b(t)$ are step functions, then

$$f(\xi(t_2), t_2) - f(\xi(t_2), t_2) = \int_{t_2}^{t_2} [f_t(\xi(t), t) + f_x(\xi(t), t)a(t) + \frac{1}{2} f_{xx}(\xi(t), t)b^2(t)]dt + \int_{t_1}^{t_2} f_x(\xi(t), t)b(t)dw(t) \qquad (b.8)$$

Indeed, denoted by $I_1, \dots, I_k$ the successive intervals in $[t_1, t_2]$ in which a, b are constant. If we apply (b.6) with t_1, t_2 replaced by the end points of I_l, and some over l, the formula (b.8) follows.

Step 6:

Let a_i, b_i be nonanticipative step functions such that

$$\int_0^T |a_i(t) - a(t)|dt \to 0 \quad a.s. \qquad (b.9)$$

$$\int_0^T |b_i(t) - b(t)|^2 dt \xrightarrow{P} 0, \qquad (b.10)$$

And let

$$\xi_i(t) = \xi(0) + \int_0^t a_i(s)ds + \int_0^t b_i(s)dw(s).$$

Then

$$sup_{0\leq t\leq T}|\xi_i(t) - \xi(t)| \overset{P}{\rightarrow} 0.$$

Hence, for a subsequence $\{\acute{\iota}\}$,

$$sup_{0\leq t\leq T}|\xi_i(t) - \xi(t)| \rightarrow 0 \qquad a.s.$$

if

$$i = \acute{\iota} \rightarrow \infty. \qquad (b.11)$$

This and (b.10) imply that

$$\int_0^T |f_x(\xi_i(t),t)b_i(t) - f_x(\xi_i(t),t)b(t)|^2 dt \overset{P}{\rightarrow} 0$$

If $i = \acute{\iota} \rightarrow \infty$.

It follows that

$$\int_{t_1}^{t_2} f_x(\xi_i(t),t)b_i(t)dw(t) \overset{P}{\rightarrow} \int_{t_1}^{t_2} f_x(\xi(t),t)b(t)dw(t)$$

If $i = \acute{\iota} \rightarrow \infty$.

It is clear from (b.9)−(b.11) that also

$$\int_{t_1}^{t_2} \left[f_t(\xi_i(t),t) + f_x(\xi_i(t),t)a_i(t) + \frac{1}{2}f_{xx}(\xi_i(t),t)\big(b_i(t)\big)^2\right] dt$$

$$\overset{P}{\rightarrow} \int_{t_1}^{t_2} \left[f_t(\xi(t),t) + f_x(\xi(t),t)a(t) + \frac{1}{2}f_{xx}(\xi(t),t)b^2(t)\right] dt$$

If $i = \acute{\iota} \rightarrow \infty$.

Writing (b.8) for $a = a_i$, $b = b_i$, $\xi = \xi_i$ and taking $i = \acute{\imath} \rightarrow \infty$, the formula (b.8) follows for general a, b. This complete the proof of the theorem.

المستخلص

تعتبر المعادلات التفاضلية التصادفية واحدة من أهم الحقول في نظرية المتغير التصادفي (theory of Stochastic Processes) وتطبيقاتها في حقول الرياضيات.

أخذت بنظر الاعتبار بعض الانظمة الدينامية العشوائية من نوع (Ito) والمشتقة بواسطة المتغير براونين (Browian Motion) مستندا على نظام ديناميكي مخمن.

ناقشنا و طورَنا مسيطر حصين (robust) مسيطر مثالي (optimal control) لضمان نوع من الاستقرارية ضمن الاحتمالية (in probability) لانظمة السيطرة التصادفية غير الخطية . لقد طورنا و برهنا النظريات الاساسية الضامنة لهكذا نوع من الاستقرارية وتبنينا أسلوبية دالة ليابانوف التصادفية لتسويق وتبرير البرهان الرياضي للنظريات المقترحة.

تم كذالك عرض و تطوير نوع من الاستقرارية المثلى المعكوسة (inverse optimal stabilization) ضمن الاحتمال مع دالة هدف ملائمة مدعمة بالمتطلبات الرياضية الضرورية.

لقد تم عرض ملاحظات أستنتاجية, خطة عمل مستقبلية, خوارزميات عددية مستندة على النظريات مدعمة بأمثلة تطبيقية.

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قابلية الاستقرارية لنظام سيطرة غير خطي متغير العشوائية مع الزمن بوساطة مسيطر استرجاعي لمخرجات النظام

رسالة

مقدمة إلى كلية العلوم - جامعة النهرين وهي جزء من متطلبات نيل درجة ماجستير

في علوم الرياضيات

مِنْ قِبَل

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